Understanding $\mathbf{D}^{\mathrm{b}}(kQ)$ using moduli spaces

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Goals

Bernhard Keller and Sarah Scherotzke, Graded quiver varieties and derived categories, `arXiv:1303.2318v2`:

1. connect $\mathbf{D}^{\mathrm{b}}(kQ)$ to a moduli variety $\mathfrak{M}_0(w)$;
2. describe the moduli variety in terms of $\mathbf{D}^{\mathrm{b}}(kQ)$ and vice versa.

My feeble goal:

3. generalise $\mathbf{D}^{\mathrm{b}}(kQ)$ to a derived moduli stack $\mathbf{R}\mathrm{Perf}_Q$;
4. describe the derived moduli stack using the moduli variety.

Context

It generalises

1. Hiraku Nakajima, Quiver varieties and finite-dimensional representations of quantum affine algebras, `arXiv:math/9912158`
2. Hiraku Nakajima, Quiver varieties and cluster algebras, `arXiv:0905.0002v5`
3. Yoshiyuki Kimura and Fan Qin, Graded quiver varieties, quantum cluster algebras and dual canonical bases, `arXiv:1205.2066v2`
4. Bernard Leclerc and Pierre-Guy Plamondon, Nakajima varieties and repetitive algebras, `1208.3910v2`

Conventions

1. k algebraically closed
2. Q a finite acyclic quiver
3. take Q connected for ease of statements

A reminder on derived categories

Construction

1. A a ring;
2. Mod-A abelian category of A-modules;
3. Ch(Mod-A) abelian category of chain complexes of A-modules;

(4.) $\mathbf{K}$(Mod-A) triangulated category of chain complexes up to homotopy;

5. $\mathbf{D}$(Mod-A) triangulated category of chain complexes with quasi-isomorphisms inverted.

Motivation

Natural location to do homological algebra.

A reminder on moduli spaces

Philosophy

A moduli "space" is an geometric object parametrising "families of objects".

- A "space" could be: topological space, manifold, variety, scheme, stack, derived stack, ...
- A "family of objects" could be: curves, algebra structures, modules, sheaves, subvarieties in a given variety, ... Then the geometric structure of the space determines which objects "look a like".

1. moduli space of curves (= Riemann surfaces) $\mathcal{M}_g$, $\dim \mathcal{M}_g = 3g - 3$
2. moduli space of algebra structures on finite-dimensional vectorspace Alg_r

Repetition quivers

We need a (technical) construction...

Definition

The *repetition quiver* $\mathbb{Z}Q$ has as vertices

$$Q_0 \times \mathbb{Z} = \{(i,p) \mid i \in Q_0, p \in \mathbb{Z}\}$$

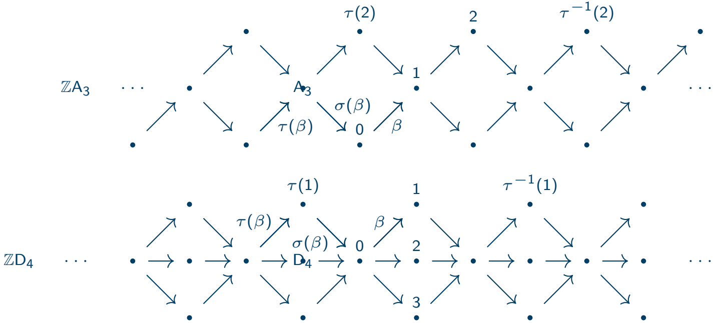
and edges

$$\bigcup_{\alpha\colon i \to j} \{(\alpha,p)\colon (i,p) \to (j,p); \sigma(\alpha,p)\colon (j,p-1) \to (i,p)\}\,.$$

Translations in repetition quivers

1. in the definition: $\sigma\colon \mathbb{Z}Q_1 \to \mathbb{Z}Q_1$;
2. translation to the left: τ, both on $\mathbb{Z}Q_0$ and $\mathbb{Z}Q_1$;
3. we have $\sigma^2 = \tau$.

Examples of repetition quivers

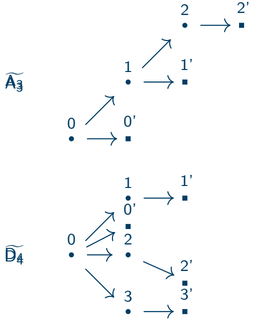


Framed quivers

Definition

The *framed quiver* $\widetilde{Q}$ of Q has vertices Q_0 and $Q_0' = \{i' \mid i \in Q_0\}$, and edges Q_1 and $\{i \to i' \mid i \in Q_0\}$. The vertices i' are the *frozen vertices*.

Examples of framed quivers



Mesh categories

Definition

The *mesh category* $k(\mathbb{Z}Q)$ is the k-linear category with $\mathrm{Obj}(k(\mathbb{Z}Q)) = \mathbb{Z}Q_0$ and

$$\mathrm{Hom}_{k(\mathbb{Z}Q)}(a,b) = \langle \text{paths from } a \text{ to } b \text{ in } \mathbb{Z}Q \rangle / (u\mathrm{r}_x v \mid x \in \mathbb{Z}Q_0)$$

where r_x is the *mesh relator* associated to x, given by

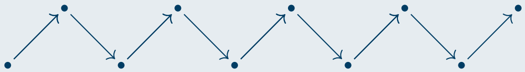
$$\mathrm{r}_x = \sum_{\beta\colon y \to x} \sigma(\beta)\beta\colon \quad \begin{array}{ccccc} & \overset{\sigma(\beta_1)}{\nearrow} & y_1 & \overset{\beta_1}{\searrow} & \\ \tau(x) & & \vdots & & x \\ & \underset{\sigma(\beta_n)}{\searrow} & y_n & \underset{\beta_n}{\nearrow} & \end{array}$$

Remarks on mesh categories

This construction finds its origins in Auslander-Reiten theory.

Example

In the mesh category $k(A_2)$ all paths of length 2 or more are identified with 0.



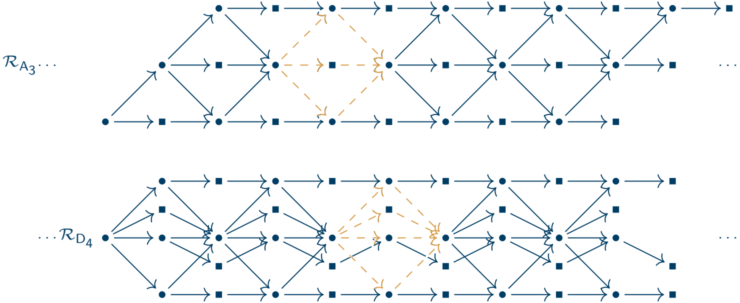
More interesting examples: see next, when we've introduced Nakajima categories.

Regular Nakajima categories

Definition

The *regular Nakajima category* $\mathcal{R}_Q$ (or just $\mathcal{R}$) is the mesh category on the framed quiver, where we only impose the mesh relators on the non-frozen vertices.

Examples of regular Nakajima categories



Singular Nakajima categories

Definition

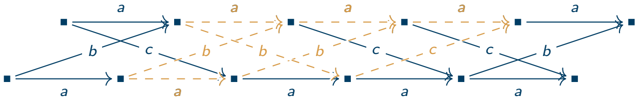
The *singular Nakajima category* $\mathcal{S}_Q$ (or just $\mathcal{S}$) is the full subcategory of the regular Nakajima category $\mathcal{R}$ on the frozen vertices.

We have regular versus singular because of the related moduli varieties: one is regular, the other can be singular.

Examples of singular Nakajima categories

These categories become really hard to draw, see Keller–Scherotzke for the case D_4 which is next to impossible to reproduce.

The singular Nakajima category for A_2:



with relations

1. $ab - ba$
2. $a^3 - cb$

Graded affine quiver varieties

Definition

The *graded affine quiver variety* $\mathfrak{M}_0(w)$ for a finitely supported dimension vector $w\colon \mathrm{Obj}(\mathcal{S}) \to \mathbb{N}$ is the variety of $\mathcal{S}$-modules M, such that $M(x) \cong k^{w(x)}$.

$$\mathfrak{M}_0(w) \cong \prod_{x,y \in \mathrm{Obj}(\mathcal{S})} \mathrm{Hom}_k\left(\mathrm{Hom}_{\mathcal{S}}(x,y), k^{w(x)w(y)}\right)/I$$

where I is an ideal of relations: a module M is described by

1. images of the morphisms in $\mathcal{S}$;
2. relations that hold in $\mathcal{S}$.

Hence, $\mathfrak{M}_0(w)$ Zariski closed subset of an affine space!

Structure of $\mathfrak{M}_0(w)$

Understanding structure of $\mathcal{S}$ implies understanding $\mathfrak{M}_0(w)$. We can describe the quiver of $\mathcal{S}$, with nodes $\mathbb{Z}\sigma(Q_0)$.

Theorem (Keller–Scherotzke, 2013)

We have

$$\#\{\sigma(y) \to \sigma(x)\} = \dim \operatorname{Ext}^1_{\mathcal{S}}(\mathsf{S}_{\sigma(x)}, \mathsf{S}_{\sigma(y)})$$

and

$$\#\{\text{relations for } \sigma(y) \text{ to } \sigma(x)\} = \dim \operatorname{Ext}^2_{\mathcal{S}}(\mathsf{S}_{\sigma(x)}, \mathsf{S}_{\sigma(y)}).$$

Relating $\mathbf{D}^{\mathrm{b}}(kQ)$ to $k(\mathbb{Z}Q)$

Theorem (Happel, 1987)

There exists a canonical fully faithful functor

$$\mathrm{H}\colon k(\mathbb{Z}Q) \to \mathrm{ind}(\mathbf{D}^{\mathrm{b}}(kQ))$$

such that the vertex $(i, 0)$ is sent to the indecomposable projective module P_i, for $i \in Q_0$.
It is moreover an equivalence if and only if Q is a Dynkin quiver.

Hence we get a relationship between the repetition quiver and the derived category!

An isomorphism of Ext's

Theorem

Let $p \geq 1$. For all $x, y \in \mathbb{Z}Q_0$ we have

$$\mathrm{Ext}^p_{\mathcal{S}}(\mathsf{S}_{\sigma(x)}, \mathsf{S}_{\sigma(y)}) \cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(kQ)}(\mathrm{H}(x), \Sigma^p\, \mathrm{H}(x)).$$

Moreover, if Q is not Dynkin these are zero for $p \geq 2$.

Applying Keller–Scherotzke's result:

Corollary

For Q not Dynkin there are no relations! We have $\mathfrak{M}_0(w)$ isomorphic to affine space.

Stability and costability

Definition

An $\mathcal{R}$-module is *stable* if for all $x \in \mathbb{Z}Q_0$ non-frozen we have

$$\mathrm{Hom}_{\mathcal{R}}(\mathsf{S}_x, M) = 0.$$

Interpretation

M does not contain a non-zero submodule supported only on non-frozen vertices.

Dual definition for *costable*: $\mathrm{Hom}_{\mathcal{R}}(M, \mathsf{S}_x) = 0$.

Interpretation

M does not have a non-zero quotient supported only on non-frozen vertices.

Dimension vectors

We'll denote (v, w)

$$v\colon \operatorname{Obj}(\mathcal{R}) \setminus \operatorname{Obj}(\mathcal{S}) \to \mathbb{N}$$
$$w\colon \operatorname{Obj}(\mathcal{S}) \to \mathbb{N}$$

dimension vectors for the regular Nakajima category.

A related moduli variety

Definition

The variety $\tilde{\mathfrak{M}}(v, w)$ is a moduli space for the $\mathcal{R}$-modules M such that

1. M is stable;
2. $M(x) \cong k^{v(x)}$;
3. $M(\sigma(x)) \cong k^{w(\sigma(x))}$.

There is moreover a (free) base change action by the group

$$\mathsf{G}_v := \prod_{x \in \mathrm{Obj}(\mathcal{R}) \setminus \mathrm{Obj}(\mathcal{S})} \mathsf{GL}_{v(x)}(k)$$

Only on the non-frozen vertices!

Graded quiver varieties

Definition

The *graded quiver variety* $\mathfrak{M}(v, w)$ is the quotient $\tilde{\mathfrak{M}}(v, w)/\mathrm{G}_v$.

Using GIT this becomes a smooth quasi-projective variety, and the restriction res: Mod-$\mathcal{R} \to$ Mod-$\mathcal{S}$ becomes a projection map

$$\pi \colon \mathfrak{M}(v, w) \to \mathfrak{M}_0(w)$$

which is *proper* ("=" inverse images of compacts are compact).

Stratification

Goal

A stratification of $\mathfrak{M}_0(w)$.

Definition

Denote by $\mathfrak{M}^{\text{reg}}(v, w)$ the open subset of $\mathfrak{M}(v, w)$ formed by isomorphism classes of $\mathcal{R}$-modules which are also costable.

By varying the vector v (w is fixed) we can stratify $\mathfrak{M}_0(w)$ by the images of the non-empty $\mathfrak{M}^{\text{reg}}(v, w)$, and each of these is isomorphic to its image in $\mathfrak{M}_0(w)$.

Statement

Theorem (Keller–Scherotzke, 2013)

There is a canonical δ-functor

$$\Phi\colon \mathrm{mod}\text{-}\mathcal{S} \to \mathbf{D}^{\mathrm{b}}(kQ)$$

such that

1. *the simple module* $\mathrm{S}_{\sigma(x)}$ *for* $x \in \mathbb{Z}Q_0$ *is sent to* $\mathrm{H}(x)$*;*
2. $M_1, M_2 \in \mathfrak{M}_0(w)$ *lie in the same stratum if and only if* $\Phi(M_1) \cong \Phi(M_2)$ *in* $\mathbf{D}^{\mathrm{b}}(kQ)$*.*

Applications

1. generalising the following result: Desingularization of quiver Grassmannians for Dynkin quivers, Giovanni Cerulli Irelli, Evgeny Feigin and Markus Reineke, `arXiv:1209.3960`
2. link with derived algebraic geometry and moduli spaces of derived categories: Moduli of objects in dg categories, Bertrand Toën and Michel Vaquié, `arXiv:math/0503269`

Derived moduli stacks

$$\begin{array}{ccc} \mathrm{i}\left(\bigsqcup_w \mathfrak{M}_0(w)\right) & \to & \mathbf{R}\mathrm{Perf}_{\mathcal{S}} \\ & \searrow & \downarrow \\ & & \mathbf{R}\mathrm{Perf}_Q \end{array}$$

All of these objects are "derived".

Questions

1. What are the geometric properties of these morphisms?
2. Do we obtain a smooth atlas for the moduli stacks?
3. Can we strengthen the results on the stratification?
4. Do these stacks have interesting intrinsic structure?

Corollary in NCAG

Claim

The derived moduli stack of vector bundles on a noncommutative curve is $[-1, 0]$-truncated, just like the commutative case.

Context

1. non-derived moduli stack Vect_C of vector bundles on a commutative curve C is *smooth* (no need for derivedness);
2. non-derived moduli stack Vect_S of vector bundles on a commutative surface S is *singular* (but derived smooth);
3. derived moduli stack of vector bundles (associated to Q non-Dynkin) on a noncommutative curve is as nice as the commutative counterpart.