Understanding $\mathbf{D}^b(kQ)$ using moduli spaces

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Bernhard Keller and Sarah Scherotzke, Graded quiver varieties and derived categories, arXiv:1303.2318v2:

1. connect $D^b(kQ)$ to a moduli variety $M_0(w)$;
2. describe the moduli variety in terms of $D^b(kQ)$ and vice versa.

My feeble goal:

3. generalise $D^b(kQ)$ to a derived moduli stack $R\text{Perf}_Q$;
4. describe the derived moduli stack using the moduli variety.
It generalises

2. Hiraku Nakajima, Quiver varieties and cluster algebras, arXiv:0905.0002v5
4. Bernard Leclerc and Pierre-Guy Plamondon, Nakajima varieties and repetitive algebras, 1208.3910v2
1. $k$ algebraically closed
2. $Q$ a finite acyclic quiver
3. take $Q$ connected for ease of statements
A reminder on derived categories

**Construction**

1. A a ring;
2. Mod-\(A\) abelian category of \(A\)-modules;
3. Ch(Mod-\(A\)) abelian category of chain complexes of \(A\)-modules;
4. \(K\)(Mod-\(A\)) triangulated category of chain complexes up to homotopy;
5. \(D\)(Mod-\(A\)) triangulated category of chain complexes with quasi-isomorphisms inverted.

**Motivation**

Natural location to do homological algebra.
A reminder on moduli spaces

Philosophy

A moduli “space” is an geometric object parametrising “families of objects”.

- A “space” could be: topological space, manifold, variety, scheme, stack, derived stack, . . .
- A “family of objects” could be: curves, algebra structures, modules, sheaves, subvarieties in a given variety, . . . Then the geometric structure of the space determines which objects “look a like”.

1. moduli space of curves (= Riemann surfaces) $\mathcal{M}_g$, $\dim \mathcal{M}_g = 3g - 3$
2. moduli space of algebra structures on finite-dimensional vectorspace $\text{Alg}_r$
We need a (technical) construction... 

**Definition**

The *repetition quiver* $\mathbb{Z}Q$ has as vertices

$$Q_0 \times \mathbb{Z} = \{(i, p) \mid i \in Q_0, p \in \mathbb{Z}\}$$

and edges

$$\bigcup_{\alpha : i \to j} \{(\alpha, p): (i, p) \to (j, p); \sigma(\alpha, p): (j, p - 1) \to (i, p)\}.$$
Translations in repetition quivers

1. in the definition: $\sigma: \mathbb{Z}Q_1 \rightarrow \mathbb{Z}Q_1$;
2. translation to the left: $\tau$, both on $\mathbb{Z}Q_0$ and $\mathbb{Z}Q_1$;
3. we have $\sigma^2 = \tau$. 
Examples of repetition quivers

\[ \begin{array}{c}
\mathbb{Z}A_3 \\
\cdots
\end{array} \]

\[ \begin{array}{c}
\tau(2) \\
\cdots
\end{array} \]

\[ \begin{array}{c}
\tau^{-1}(2)
\end{array} \]

\[ \begin{array}{c}
\mathbb{D}D_4 \\
\cdots
\end{array} \]

\[ \begin{array}{c}
\tau(1) \\
\cdots
\end{array} \]

\[ \begin{array}{c}
\tau^{-1}(1)
\end{array} \]
The **framed quiver** $\tilde{Q}$ of $Q$ has vertices $Q_0$ and $Q'_0 = \{i' \mid i \in Q_0\}$, and edges $Q_1$ and $\{i \to i' \mid i \in Q_0\}$. The vertices $i'$ are the **frozen vertices**.
Examples of framed quivers
Mesh categories

Definition

The mesh category $k(\mathbb{Z}Q)$ is the $k$-linear category with $\text{Obj}(k(\mathbb{Z}Q)) = \mathbb{Z}Q_0$ and

$$\text{Hom}_{k(\mathbb{Z}Q)}(a, b) = \langle \text{paths from } a \text{ to } b \text{ in } \mathbb{Z}Q \rangle / (ur_x \nu \mid x \in \mathbb{Z}Q_0)$$

where $r_x$ is the mesh relator associated to $x$, given by

$$r_x = \sum_{\beta : y \to x} \sigma(\beta) \beta : \tau(x) \to x$$
Remarks on mesh categories

This construction finds its origins in Auslander-Reiten theory.

Example

In the mesh category \( k(A_2) \) all paths of length 2 or more are identified with 0.

More interesting examples: see next, when we’ve introduced Nakajima categories.
Definition

The \textit{regular Nakajima category} $\mathcal{R}_Q$ (or just $\mathcal{R}$) is the mesh category on the framed quiver, where we only impose the mesh relators on the non-frozen vertices.
Examples of regular Nakajima categories

\[ \mathcal{R}_{A_3} \ldots \]

\[ \mathcal{R}_{D_4} \ldots \]
Singular Nakajima categories

**Definition**

The *singular Nakajima category* $S_Q$ (or just $S$) is the full subcategory of the regular Nakajima category $R$ on the frozen vertices.

We have regular versus singular because of the related moduli varieties: one is regular, the other can be singular.
Examples of singular Nakajima categories

These categories become really hard to draw, see Keller–Scherotzke for the case $D_4$ which is next to impossible to reproduce.

The singular Nakajima category for $A_2$:

\[
\begin{array}{c}
\begin{array}{ccc}
\bullet & \xrightarrow{a} & \bullet \\
\bullet & \xrightarrow{b} & \bullet \\
\bullet & \xrightarrow{c} & \bullet \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\bullet & \xrightarrow{a} & \bullet \\
\bullet & \xrightarrow{b} & \bullet \\
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\begin{array}{c}
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\bullet & \xrightarrow{a} & \bullet \\
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\end{array}
\end{array}
\end{array}
\]

with relations

1. $ab - ba$
2. $a^3 - cb$
Graded affine quiver varieties

Definition

The graded affine quiver variety $\mathcal{M}_0(w)$ for a finitely supported dimension vector $w : \text{Obj}(S) \to \mathbb{N}$ is the variety of $S$-modules $M$, such that $M(x) \cong k^{w(x)}$.

$$\mathcal{M}_0(w) \cong \prod_{x,y \in \text{Obj}(S)} \text{Hom}_k \left( \text{Hom}_S(x, y), k^{w(x)w(y)} \right) / I$$

where $I$ is an ideal of relations: a module $M$ is described by

1. images of the morphisms in $S$;
2. relations that hold in $S$.

Hence, $\mathcal{M}_0(w)$ Zariski closed subset of an affine space!
Structure of $\mathcal{M}_0(w)$

Understanding structure of $S$ implies understanding $\mathcal{M}_0(w)$. We can describe the quiver of $S$, with nodes $\mathbb{Z}\sigma(Q_0)$.

**Theorem (Keller–Scherotzke, 2013)**

We have

$$\#\{\sigma(y) \to \sigma(x)\} = \dim \text{Ext}_S^1(S_{\sigma(x)}, S_{\sigma(y)})$$

and

$$\#\{\text{relations for } \sigma(y) \text{ to } \sigma(x)\} = \dim \text{Ext}_S^2(S_{\sigma(x)}, S_{\sigma(y)}).$$
Theorem (Happel, 1987)

There exists a canonical fully faithful functor

\[ H : k(\mathbb{Z}Q) \to \text{ind}(\mathcal{D}^b(kQ)) \]

such that the vertex \((i, 0)\) is sent to the indecomposable projective module \(P_i\), for \(i \in Q_0\).

It is moreover an equivalence if and only if \(Q\) is a Dynkin quiver.

Hence we get a relationship between the repetition quiver and the derived category!
An isomorphism of Ext’s

**Theorem**

Let $p \geq 1$. For all $x, y \in \mathbb{Z} \mathbb{Q}_0$ we have

$$\text{Ext}^p_S(S_{\sigma(x)}, S_{\sigma(y)}) \cong \text{Hom}_{D^b(kQ)}(H(x), \Sigma^p H(x)).$$

Moreover, if $Q$ is not Dynkin these are zero for $p \geq 2$.

Applying Keller–Scherotzke’s result:

**Corollary**

For $Q$ not Dynkin there are no relations! We have $\mathcal{M}_0(w)$ isomorphic to **affine space**.
## Stability and costability

### Definition

An \( \mathcal{R} \)-module is \textit{stable} if for all \( x \in \mathbb{Z}Q_0 \) non-frozen we have

\[
\text{Hom}_{\mathcal{R}}(S_x, M) = 0.
\]

### Interpretation

\( M \) does not contain a non-zero submodule supported only on non-frozen vertices.

### Dual definition for \textit{costable}

\[
\text{Hom}_{\mathcal{R}}(M, S_x) = 0.
\]

### Interpretation

\( M \) does not have a non-zero quotient supported only on non-frozen vertices.
Dimension vectors

We’ll denote \((v, w)\)

\[
v : \text{Obj}(R) \setminus \text{Obj}(S) \rightarrow \mathbb{N}
\]

\[
w : \text{Obj}(S) \rightarrow \mathbb{N}
\]

dimension vectors for the regular Nakajima category.
A related moduli variety

**Definition**

The variety $\tilde{\mathcal{M}}(v, w)$ is a moduli space for the $\mathcal{R}$-modules $M$ such that

1. $M$ is stable;
2. $M(x) \cong k^{v(x)}$;
3. $M(\sigma(x)) \cong k^{w(\sigma(x))}$.

There is moreover a (free) base change action by the group

$$G_v := \prod_{x \in \text{Obj}(\mathcal{R}) \setminus \text{Obj}(S)} \text{GL}_{v(x)}(k)$$

Only on the non-frozen vertices!
Graded quiver varieties

**Definition**

The *graded quiver variety* \( \mathcal{M}(v, w) \) is the quotient \( \tilde{\mathcal{M}}(v, w)/G_v \).

Using GIT this becomes a smooth quasi-projective variety, and the restriction \( \text{res}: \text{Mod-}\mathcal{R} \to \text{Mod-}\mathcal{S} \) becomes a projection map

\[
\pi: \mathcal{M}(v, w) \to \mathcal{M}_0(w)
\]

which is *proper* ("=" inverse images of compacts are compact).
Stratification

Goal
A stratification of $M_0(w)$.

Definition
Denote by $M^{\text{reg}}(v, w)$ the open subset of $M(v, w)$ formed by isomorphism classes of $R$-modules which are also costable.

By varying the vector $v$ ($w$ is fixed) we can stratify $M_0(w)$ by the images of the non-empty $M^{\text{reg}}(v, w)$, and each of these is isomorphic to its image in $M_0(w)$. 
Theorem (Keller–Scherotzke, 2013)

There is a canonical $\delta$-functor

$$\Phi: \text{mod-} S \to D^b(kQ)$$

such that

1. the simple module $S_{\sigma(x)}$ for $x \in \mathbb{Z}Q_0$ is sent to $H(x)$;
2. $M_1, M_2 \in \mathcal{M}_0(w)$ lie in the same stratum if and only if $\Phi(M_1) \cong \Phi(M_2)$ in $D^b(kQ)$. 
1. generalising the following result: Desingularization of quiver Grassmannians for Dynkin quivers, Giovanni Cerulli Irelli, Evgeny Feigin and Markus Reineke, arXiv:1209.3960

2. link with derived algebraic geometry and moduli spaces of derived categories: Moduli of objects in dg categories, Bertrand Toën and Michel Vaquié, arXiv:math/0503269
Derived moduli stacks

\[ i \left( \bigsqcup_w \mathcal{M}_0(w) \right) \to R\text{Perf}_S \]

\[ \downarrow \]

\[ R\text{Perf}_Q \]

All of these objects are “derived”.

**Questions**

1. What are the geometric properties of these morphisms?
2. Do we obtain a smooth atlas for the moduli stacks?
3. Can we strengthen the results on the stratification?
4. Do these stacks have interesting intrinsic structure?
Corollary in NCAG

Claim
The derived moduli stack of vector bundles on a noncommutative curve is $[-1, 0]$-truncated, just like the commutative case.

Context
1. non-derived moduli stack $\text{Vect}_C$ of vector bundles on a commutative curve $C$ is smooth (no need for derivedness);
2. non-derived moduli stack $\text{Vect}_S$ of vector bundles on a commutative surface $S$ is singular (but derived smooth);
3. derived moduli stack of vector bundles (associated to $Q$ non-Dynkin) on a noncommutative curve is as nice as the commutative counterpart.