The $(\infty, 1)$-topos of derived stacks

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Description

These are the notes for my talk at the Winter school on Derived algebraic geometry (see `http://www.math.ethz.ch/u/calaqued/DAG-school`). Below the official abstract is given, more information on the contents of my talk can be found in the introduction on page 2.

Abstract

The étale $(\infty, 1)$-site of derived affine schemes, derived stacks, truncations and relations with underived stacks. Basic examples:

1. classifying stacks,
2. derived fibered products,
3. derived mapping stacks

References: [HAG-II, §2.2.1, 2.2.2 & 2.2.4].

Contents

Introduction

The goal of this talk (and these notes) is to tie together the homotopy theory of commutative dg algebras (cdga's from now on, always concentrated in non-positive cohomological degree) and the abstract notion of a model topos (or $(\infty, 1)$-topos in Lurie's sense) to obtain a good definition of "derived algebraic geometry" (and show how we can obtain the usual algebraic geometry too). Hence talks 4–6 (by Claudio Sibilia, Simon Häberli; Georg Oberdieck, Jon Skowera and Damien Lejay) are used to get two concrete interpretations of talk 7 (by David Carchedi). In §1 some of the definitions David has discussed are repeated, and the way to apply them is explained.

Another important aspect of this talk is to show how this "derived algebraic geometry" ties in with "classical algebraic geometry", where classical no longer means Italian but rather Grothendieckian geometry from the sixties (or rather: higher geometry, as developed by Carlos Simpson). This is done in §3.

In §4 we give examples of objects in derived algebraic geometry. These examples will help to motivate the construction of derived algebraic geometry, especially the case of derived fibered products is worked out in detail in §4.2.

We have tried to give statements in terms of dg algebras whenever possible, but the first part ties in with the highly abstract formalism of homotopical algebraic geometry so sometimes things will be expressed in simplicial language.

We also take the model topos point of view: $(\infty, 1)$ toposes were covered in the previous talk, and these notes are mostly in the spirit of the formalism of Toën–Vezzosi. The reader is encouraged to read Lurie's approach, and think about the relationships.

Acknowledgements I would like to thank the organisers (Damien Calaque and Claudia Scheimbauer) and the mentors (Bertrand Toën, Michel Vaquié and Gabriele Vezzosi) of the winter school, for creating this wonderful occasion to talk and learn about derived algebraic geometry. I also wish to thank Mauro Porta for many interesting discussions on the subject in preparation of this winter school, and introducing me to the subject in the first place.

1 Preliminaries on homotopical algebraic geometry

1.1 Homotopical algebra contexts

In this section we quickly review the "local" aspect of homotopical algebraic geometry. It corresponds to [HAG-II, §1.1] (the development of the notions in §1.2 of op. cit. is not repeated here, but it can be interesting to read this as a motivation, especially §1.2.13 op. cit.).

Definition 1. A *homotopical algebra context* is a triple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ where

1. $\mathcal{C}$ (our big category of modules) is a symmetric monoidal model category;
2. $\mathcal{C}_0$ (the non-positively graded modules) is a full subcategory of $\mathcal{C}$ closed under weak equivalences;
3. $\mathcal{A}$ (the ring-like objects in our setup) is a full subcategory of $\mathbf{com}\,\mathbf{Mon}(\mathcal{C})$, the category of commutative monoids in $\mathcal{C}$;

such that

1. $\mathcal{C}$ is proper, pointed, the natural morphisms

   $$\mathrm{Q}(X) \amalg \mathrm{Q}(Y) \to X \amalg Y \to \mathrm{R}(X) \times \mathrm{R}(Y) \tag{1}$$

   are weak equivalences and $\mathrm{Ho}(\mathcal{C})$ is an additive category;
2. let A be a commutative monoid object in $\mathcal{C}$, then modules over A induce a combinatorial proper model category which is also a symmetric monoidal model category;
3. tensoring with a cofibrant object preserves weak equivalences in the module categories;
4. comma categories of (non-unital) commutative monoids in $\mathcal{C}$ are combinatorial proper model categories and base change with a cofibrant object preserves weak equivalences in the module categories;
5. the category $\mathcal{C}_0$ is closed under weak equivalences and homotopy colimits, while its homotopy category is closed under tensor products;
6. for every $A \in \mathcal{A}$ we have that the restricted Yoneda embedding

   $$\mathbf{R}\mathrm{h}_0^- : \mathrm{Ho}(A\text{-}\mathbf{Mod}_0^{\mathrm{op}}) \to \mathrm{Ho}(A\text{-}\mathbf{Mod}_0^{\mathrm{op},\wedge}) : M \mapsto \mathrm{Map}(M, -)_* \tag{2}$$

   is fully faithful[1].

These additional requirements can be motivated as follows.

1. The model category $\mathcal{C}$ is "additive" in some sense: finite homotopy coproducts are finite homotopy products, Hom-sets in the homotopy category are abelian groups, …

[1] The codomain is the left Bousfield localisation of the category of simplicial presheaves along weak equivalences, which yields the category of simplicial prestacks, see [HAG-I, definition 4.1.4].

2. Module categories are like classical module categories, and the homotopy category of a module category has a natural symmetric monoidal structure with $\mathbf{R}\mathcal{H}\mathrm{om}$ and $\otimes^{\mathbf{L}}_A$.

3. Base change behaves nice with respect to cofibrant objects.

4. Another base change property: if B and B' are commutative A-algebras over a commutative monoid A in $\mathcal{C}$ then

$$B \amalg^{\mathbf{L}}_A B' \to B \otimes^{\mathbf{L}}_A B' \tag{3}$$

is an isomorphism in $\mathrm{Ho}(A\text{-}\mathbf{Mod})$, i.e. the homotopy coproducts are really given by the monoidal structure, as desired. The non-unital statement is to make cotangent complexes work.

5. The t-structure defined by $\mathcal{C}_0$, which describes the objects that are non-positively graded, is compatible with the monoidal model structure. Different choices of t-structures will yield different notions of formal smoothness, hence different notions of geometricity of stacks.

6. The non-positively graded objects are homotopically dense, hence each object in A-**Mod** is the homotopy limit of non-positively graded objects.

Requirements 1, 2 and 3 are there to make statements easier, in order to avoid too much (co)fibrant replacements in statements. Requirement 4 relates coproducts to tensor products. Requirement 5 is a compatibility statement, and requirement 6 says that the non-positively graded objects $\mathcal{C}_0$ can see all of $\mathcal{C}$ by using homotopy limits and shifts.

Examples of HA contexts will be given in §2.2, but first we have to introduce HAG contexts, which are a globalisation of the previous results to obtain a "homotopical algebraic geometry".

1.2 Homotopical algebraic geometry contexts

In this section we quickly review the "global" aspect of homotopical algebraic geometry. It corresponds to [HAG-II, §1.3]. The most important definitions will be covered in the talk by David Carchedi.

This is [HAG-II, definition 1.3.1.1].

Definition 2. Let $\mathcal{M}$ be a model category. A *model (pre)topology* τ on $\mathcal{M}$ is the datum of a set $\mathrm{Cov}_\tau(x)$ for every $x \in \mathrm{Obj}(\mathcal{M})$ of subsets of objects in $\mathrm{Ho}(\mathcal{M})/x$ which we will call *τ-covering families* of x, such that

stability for all $x \in \mathcal{M}$ and for all isomorphisms $y \to x$ in $\mathrm{Ho}(\mathcal{M})$ we have

$$\{y \to x\} \in \mathrm{Cov}_\tau(x); \tag{4}$$

composition if $\{u_i \to x\}_{i\in I} \in \mathrm{Cov}_\tau(x)$, and we have $\{u_{i,j} \to u_i\}_{j\in J_i} \in \mathrm{Cov}_\tau(u_i)$ for all $i \in I$ then the family of compositions

$$\{u_{i,j} \to x\}_{i\in I, j\in J_i} \in \mathrm{Cov}_\tau(x); \tag{5}$$

homotopy base change assuming that the stability and composition condition are satisfied, then for $\{u_i \to x\}_{i\in I} \in \mathrm{Cov}_\tau(x)$ and $y \to x$ in $\mathrm{Ho}(\mathcal{M})$ we have

$$\left\{u_i \times_x^{\mathrm{h}} y \to y\right\}_{i\in I} \in \mathrm{Cov}_\tau(y). \tag{6}$$

The pair $(\mathcal{M}, \tau)$ is a *model site*.

By [HAG-I, proposition 4.3.5] a model (pre)topology τ on the model category $\mathcal{M}$ induces a Grothendieck (pre)topology τ on its homotopy category $\mathrm{Ho}(\mathcal{M})$.

We will now make a specific choice for $\mathcal{M}$ in the following definition, which will yield the abstract setup for a homotopic algebraic geometry.

Definition 3. A *homotopical algebraic geometry context* is a quintuple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P})$ such that $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ is a homotopical algebra context, τ is a model pretopology on $\mathbf{Aff}_\mathcal{C} := \mathbf{com\,Mon}(\mathcal{C})^{\mathrm{op}}$ such that

1. the topology on $\mathrm{Ho}(\mathbf{Aff}_\mathcal{C})$ is quasicompact;
2. a finite family $(X_i)_{i\in I}$ in $\mathbf{Aff}_\mathcal{C}$ induces a covering $(X_i \to \coprod^{\mathbf{L}}_{j\in I} X_j)_{i\in I}$ of the homotopy coproduct $\coprod^{\mathbf{L}}_{i\in I} X_i$, i.e. the coproduct of coverings is a covering of the coproduct;
3. if $X_* \to Y$ is an augmented simplicial object in $\mathbf{Aff}_\mathcal{C}$ such that each morphism

   $$X_n \to X_*^{\mathbf{R}\partial\Delta^n} \times^{\mathrm{h}}_{Y^{\mathbf{R}\partial\Delta^n}} Y \tag{7}$$

   forms a one-element τ-covering family, then $X_* \to Y$ satisfies the descent condition [HAG-II, definition 1.2.12.1(2)][2];

and $\mathbf{P}$ is a class of morphisms in $\mathbf{Aff}_\mathcal{C}$ such that

4. coverings for τ are morphisms in $\mathbf{P}$;
5. morphisms in $\mathbf{P}$ are stable by composition, weak equivalences and homotopy pullbacks;

[2] This technical condition yields a way of checking when a simplicial presheaf F on $\mathbf{com\,Mon}(\mathcal{C})$ is a stack [HAG-II, corollary 1.3.2.4], namely if and only if

(a) a weak equivalence $A \xrightarrow{\sim} B$ in $\mathbf{com\,Mon}(\mathcal{C})$ is sent to a weak equivalence $F(A)_* \to F(B)_*$ in $\mathbf{s\,Set}$;

(b) for every family $(A_i)_{i\in I}$ we have that

$$F\left(\prod^{\mathrm{h}}_{i\in I} A_i\right)_* \to \prod_{i\in I} F(A_i)_* \tag{8}$$

is an isomorphism in $\mathrm{Ho}(\mathbf{s\,Set})$;

(c) for every τ-hypercover $\mathrm{Spec}\, B_* \to \mathrm{Spec}\, A$ in $\mathbf{Aff}_\mathcal{C}$ is the induced

$$F(A)_* \to \mathrm{holim}_{[n]\in\Delta} F(B_n)_* \tag{9}$$

an isomorphism in $\mathrm{Ho}(\mathbf{s\,Set})$.

Moreover, it will give us [HAG-II, corollary 1.3.2.5]: the model pretopology on $\mathbf{Aff}_\mathcal{C}$ is subcanonical, representable presheaves are stacks, we get a fully faithful functor $\mathbf{R}\mathrm{h}_- \colon \mathrm{Ho}(\mathbf{Aff}_\mathcal{C}) \to \mathbf{St}(\mathcal{C}, \tau)$.

6. being in **P** is local for the pretopology τ;
7. the natural morphisms $X \to X \amalg^{\mathbf{L}} Y$ and $Y \to X \amalg^{\mathbf{L}} Y$ are in **P**.

Hence we will have to pick elements of the quintuple and see how they yield (derived) algebraic geometry. Remark that the choice of **P** (which yields to the concept of geometricity) is not discussed in detail here, as it will be covered in later talks. Some words on it later in this text are required though, to make statements in §3 possible.

1.3 Stacks

We quickly review the abstract definition of a stack with respect to a model site, as explained in [HAG-II, §1.3.1 and 1.3.2].

1. Consider the category of *simplicial presheaves* $\mathrm{s\,Pr}(\mathbf{Aff}_{\mathcal{C}})$, i.e. its objects are functors $\mathbf{Aff}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{s\,Set}$, and this category is equipped with its projective model category structure (i.e. weak equivalences and fibrations are taken objectwise).
2. Next, consider the category of *prestacks* $\mathbf{Aff}_{\mathcal{C}}^{\wedge}$, which we obtain as the left Bousfield localisation of $\mathrm{s\,Pr}(\mathbf{Aff}_{\mathcal{C}})$ with respect to the class of morphisms $\{\mathrm{h}_f \mid f \text{ a weak equivalence}\}$ where

   (10) $\mathrm{h}_- \colon \mathbf{Aff}_{\mathcal{C}} \to \mathrm{Pr}(\mathbf{Aff}_{\mathcal{C}}) \hookrightarrow \mathrm{s\,Pr}(\mathbf{Aff}_{\mathcal{C}})$

   is the constant Yoneda embedding. Hence we have enlarged our class of weak equivalences.

   Remark that we have not used our topology yet.
3. Finally, consider the category of *stacks* $\mathbf{Aff}_{\mathcal{C},\tau}^{\sim}$ on $(\mathbf{Aff}_{\mathcal{C}}, \tau)$, obtained as the left Bousfield localisation of $\mathbf{Aff}_{\mathcal{C}}^{\wedge}$ with respect to the homotopy τ-hypercovers [HAG-I, definition 4.4.4][3]. This is where our topology comes into play.

This category of stacks $\mathbf{Aff}_{\mathcal{C},\tau}^{\sim}$ is a *model topos* in the sense of [HAG-I, §3.8], and we will see it again in §2.3.

By the general machinery of left Bousfield localisations we obtain an adjoint pair

(11) $\mathrm{id} \colon \mathbf{Aff}_{\mathcal{C}}^{\wedge} \rightleftarrows \mathbf{Aff}_{\mathcal{C},\tau}^{\sim} : \mathrm{id}$

which gives an adjunction on the homotopy categories

(12) $\mathrm{a} := \mathbf{L}\,\mathrm{id} \colon \mathrm{Ho}(\mathbf{Aff}_{\mathcal{C}}^{\wedge}) \rightleftarrows \mathrm{Ho}(\mathbf{Aff}_{\mathcal{C},\tau}^{\sim}) : \mathbf{R}\,\mathrm{id} =: \jmath.$

The functor a corresponds to *stackification*, $\jmath$ is the inclusion of stacks into prestacks. This allows for the following definition.

Definition 4. A *stack* on the model site $(\mathbf{Aff}_{\mathcal{C}}, \tau)$ is a simplicial presheaf $\mathcal{F}$ on $\mathbf{Aff}_{\mathcal{C}}$ such that its image in $\mathrm{Ho}(\mathbf{Aff}_{\mathcal{C}}^{\wedge})$ is in the essential image of the inclusion functor $\jmath$.

The homotopy category of stacks $\mathrm{Ho}(\mathbf{Aff}_{\mathcal{C},\tau}^{\sim})$ will be denoted $\mathbf{St}(\mathcal{C}, \tau)$.

[3] For more information on this technical aspect, see the MathOverflow question `http://mathoverflow.net/questions/87427`, which was answered by David Carchedi, who also attended the winter school.

2 The topos of derived stacks

The goal of this section is to make sense of the étale site of derived affine schemes, and the topos of derived stacks. To a great extent this chapter is just a way of unraveling the myriads of definitions in [HAG-I; HAG-II], and discuss the classical notions as given in [$\mathrm{SGA4}_1$; $\mathrm{SGA4}_2$; Stacks].

2.1 Reminder on sites

To review the literature on sites and toposes here would be impossible. But we will discuss some notions that caused some confusion when the work on these notes started. It concerns small and big sites, a concept which is well known in classical algebraic geometry, but the terminology seems to be less popular in derived algebraic geometry and distinguishing them is left to the reader.

Let S be a base scheme. For the étale topology there are two sites: the *big étale site on* S and the *small étale site on* S. Let's first discuss the latter, because the intuition is closer to the notion of a topological space.

The *small* étale site is obtained by putting a Grothendieck topology on the category of S-schemes which are étale over S, i.e. the structure morphism $X \to S$ itself is required to be étale. Hence this category corresponds to the "open sets" in the étale topology on S. This moreover implies that any morphism $X \to Y$ over S is étale.

To define the *big* étale site, one considers the category of *all* schemes over S (not just the ones with étale structure morphisms as before). This category is equipped with the étale topology by asking morphisms $X \to Y$ over S to be étale, but it contains more than just the étale open subsets of the base scheme S.

Conclusion We will discuss the *big* étale site in most of this work. This is mentioned incorrectly on a preliminary program. Some statements refer to the small site, and this is indicated.

2.2 The étale site of (derived) affine schemes

To emphasise the geometric view on things, we will first define

$$\begin{aligned} \mathbf{Aff}(k) &:= \mathbf{c\,Alg}(k)^{\mathrm{op}} \\ \mathbf{d\,Aff}(k) &:= \mathbf{cdg\,Alg}^{\leq 0}(k)^{\mathrm{op}} \end{aligned} \tag{13}$$

and Spec A (resp. Spec $A^\bullet$) is the object corresponding to a commutative k-algebra A (resp. a non-positively graded commutative dg algebra $A^\bullet$). Remark that we will (ab)use the same notation for the image of Spec A (resp. Spec $A^\bullet$) under the (model) Yoneda embedding in the category of stacks!

So we wish to consider the model sites $(\mathbf{Aff}(k), \text{ét})$ and $(\mathbf{d\,Aff}(k), \text{ét})$. We first have to make sense of the notation by explaining to which HAG contexts they correspond.

Algebraic geometry To obtain the "classical" notion of higher algebraic geometry we take

$$(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P}) := (k\text{-}\mathbf{Mod}, k\text{-}\mathbf{Mod}, \mathbf{c\,Alg}(k), \text{ét}, \text{smooth}). \tag{14}$$

The base category $\mathcal{C}$ is equipped with the trivial model structure: weak equivalences are isomorphisms, every morphism is a fibration and cofibration. The étale covering families are taken in their classical sense [Stacks, tag 0215]. This way (i.e. also incorporating our choice of **P**) we obtain an interpretation of (Simpson's higher) schemes, algebraic spaces, Deligne–Mumford stacks and Artin stacks.

Remark that by taking **P** to be the smooth morphisms we obtain Artin stacks [Art74], with Deligne–Mumford stacks as a special case [HAG-II, proposition 2.1.2.1]. If we'd restricted ourselves to étale morphisms for **P** we get Deligne–Mumford stacks as our main object [DM69].

From this point on we will not use the choice of a class **P** until we get to §2.4.

Derived algebraic geometry To obtain derived algebraic geometry we take

$$(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P}) := (k\text{-}\mathbf{dg\,Mod}, k\text{-}\mathbf{dg\,Mod}, \mathbf{cdg\,Alg}^{\leq 0}(k), \text{ét}, \text{smooth}) \tag{15}$$

where étale and smooth have to be interpreted in the correct sense. For completeness' sake we repeat their reinterpretations [HAG-II, proposition 2.2.2.5 and theorem 2.2.2.6] *but give them as definitions*. These reinterpretations explain why we can take classical algebraic geometry as a special case of derived algebraic geometry.

Definition 5. Let $f\colon A^\bullet \to B^\bullet$ be a morphism in $\mathbf{cdg\,Alg}^{\leq 0}(k)$. We say that

1. f is a *Zariski open immersion* if

   $$\mathrm{H}^\bullet(A^\bullet) \otimes_{\mathrm{H}^0(A^\bullet)} \mathrm{H}^0(B^\bullet) \to \mathrm{H}^\bullet(B^\bullet) \tag{16}$$

   is an isomorphism and $\mathrm{H}^0(A^\bullet) \to \mathrm{H}^0(B^\bullet)$ is a Zariski open immersion (i.e. as morphisms in $\mathbf{c\,Alg}(k)$);

2. f is *étale* if

   $$\mathrm{H}^\bullet(A^\bullet) \otimes_{\mathrm{H}^0(A^\bullet)} \mathrm{H}^0(B^\bullet) \to \mathrm{H}^\bullet(B^\bullet) \tag{17}$$

   is an isomorphism and $\mathrm{H}^0(A^\bullet) \to \mathrm{H}^0(B^\bullet)$ is étale (i.e. as morphisms in $\mathbf{c\,Alg}(k)$);

3. f is *smooth* if

   $$\mathrm{H}^\bullet(A^\bullet) \otimes_{\mathrm{H}^0(A^\bullet)} \mathrm{H}^0(B^\bullet) \to \mathrm{H}^\bullet(B^\bullet) \tag{18}$$

   is an isomorphism and $\mathrm{H}^0(A^\bullet) \to \mathrm{H}^0(B^\bullet)$ is smooth (i.e. as morphisms in $\mathbf{c\,Alg}(k)$);

We then define our étale covering families in the obvious sense.

Definition 6. Let $(f_i\colon \operatorname{Spec} A_i \to \operatorname{Spec} A)_{i\in I}$ be a family of morphisms in model category $\mathbf{d\,Aff}(k)$. We say it is an *étale covering family* if

1. each $A \to A_i$ is étale (in the derived sense);
2. there exists a finite subset $J \subseteq I$ such that the family $(f_i\colon A \to A_i)$ in $\mathbf{cdg\,Alg}^{\leq 0}(k)$ induces a *formal covering*, i.e. the family of pullback functors $(\mathbf{L}f_i^*\colon \operatorname{Ho}(A\text{-}\mathbf{Mod}) \to \operatorname{Ho}(B_i\text{-}\mathbf{Mod}))_{i\in J}$ is conservative.

By [HAG-II, lemma 2.2.2.13] we can make the following definition, which (finally) introduces the main player of these notes.

Definition 7. The *étale site of derived affine schemes* is the model site $(\mathbf{d\,Aff}(k), \text{ét})$.

Further comparison between underived and derived geometry is delayed until §3.

2.3 The topos of derived stacks

By the construction outlined in §1.3 we have that $\mathbf{d\,Aff}(k)^{\sim}_{\text{ét}}$ is a topos.

Definition 8. The topos $\mathbf{d\,Aff}(k)^{\sim}_{\text{ét}}$ (or its homotopy category) is the *topos of derived stacks*, denoted $\mathbf{d\,St}(k)$.

Similarly, we define

Definition 9. The topos $\mathbf{Aff}(k)^{\sim}_{\text{ét}}$ (or its homotopy category) is the *topos of (higher) stacks*, denoted $\mathbf{St}(k)$.

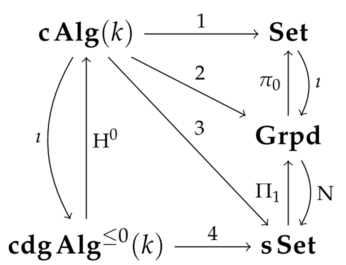
So if we consider $\operatorname{Spec} A^\bullet$, which a priori lives in $\mathbf{d\,Aff}$, we will consider it (by abuse of notation) to be in $\mathbf{d\,Aff}^{\sim}_{\text{ét}}$, and then $\mathbf{R}\operatorname{Spec} A^\bullet$ is the associated stack, because the topos of derived stacks is the homotopy category. For the underived case, we have by the remark after [HAG-II, lemma 2.1.1.1] that (because all the model structures are trivial anyway)

(19) $\operatorname{Spec} A \cong \mathbf{R}\operatorname{Spec} A$

where $\operatorname{Spec} A$ is used in both its interpretations.

The following diagram should be in these notes somewhere anyway, so we will show it here. It displays the situation of classical, higher and derived algebraic geometry.

(20)



The numbers in this diagram correspond to

1. classical scheme theory;

2. classical stack theory;
3. higher stack theory;
4. derived stack theory.

So summarising the construction of §1.3: derived stacks are simplicial presheaves on $\mathbf{cdg\,Alg}^{\leq 0}(k)$ (or rather on its opposite category $\mathbf{d\,Aff}(k)$), which

1. send quasi-isomorphisms in $\mathbf{cdg\,Alg}^{\leq 0}$ to weak equivalences in $\mathbf{s\,Set}$;
2. satisfy descent with respect to étale hypercoverings (whatever that may be in explicit terms, just think about Čech covers).

2.4 Geometric stacks

Morally we can say that *geometric* stacks, with respect to $(\tau, \mathbf{P})$ are obtained by gluing affine schemes along (iterated) equivalence relations in $\mathbf{P}$ (or phrased in another way: the structural morphisms of the groupoid objects). This mimicks the case of Deligne–Mumford and Artin (or sometimes algebraic) stacks, in which we glue along étale resp. smooth morphisms. The terminology is unfortunately not fixed, so we hope no confusion arises.

So, just like in the classical theory of stacks, we will only be interested in algebraic stacks. To implement this in the case of derived algebraic geometry, we use [HAG-II, §2.2.3]. By proposition 2.2.3.2 op. cit. we can say that the smooth morphisms can be used to define geometricity.

From now on we will assume that the notion of n-geometric derived stack is known[4].

So we can add a third point to the description of derived stacks in §2.3 to describe *geometric* stacks:

3. there exists an atlas, i.e. a map $f\colon \coprod_{i\in I}^{\mathbf{L}} \mathbf{R}\mathrm{Spec}\, A_i^\bullet \to \mathcal{F}$ (satisfying some conditions) such that if f is smooth (resp. étale, resp. Zariski) we get a derived Artin stack (resp. derived Deligne–Mumford stack, resp. derived scheme), hence we start using the class of morphisms $\mathbf{P}$ (see also §2.5).

Moreover, once we have introduced truncation and extension, we will see that these definitions will agree with the classical definitions of these notions.

If we consider the notion of n-geometric to be known[5], we have the following properties

$$\mathbf{d\,St}_\tau^{\mathbf{P}}(k) = \bigcup_{n\geq -1} \mathbf{d\,St}_\tau^{n,\mathbf{P}}(k) \tag{21}$$

and

$$\mathbf{d\,St}_\tau^{-1,\mathbf{P}}(k) \cong \mathrm{Ho}(\mathbf{d\,Aff}(k)) \tag{22}$$

which is independent of τ and $\mathbf{P}$ as these are exactly the affine schemes: no gluing has been performed.

In the next section some other choices of geometricity conditions are explained.

[4] It will be explained in the talk by Mauro Porta.

[5] It's not explained in the notes, but it measures how many times we've iterated the construction with a groupoid object. If n is low we get "easier" objects, corresponding to the classical scheme theory, or algebraic spaces, or Deligne–Mumford stacks, or Artin stacks.

2.5 What about other topologies?

The only topology that is considered in [HAG-II, §2.2] is the étale topology. There are nevertheless (a myriad of) others in algebraic geometry. In the study of algebraic stacks, the other main player is the fppf topology (see e.g. [Stacks, tag 021L] for definitions).

The geometric derived stacks as discussed up to now have been obtained by taking the étale topology, and equivalence relations are smooth morphisms. In the notation of [HAG-II] we have $(\tau, \mathbf{P}) = (\text{ét}, \text{smooth})$. These two notions generalised to the derived setting in a nice way, and in the next section we will show some more properties of the étale topos.

On the other hand, to define fppf covers one needs a good notion of finite presentation. But the finite presentation that one obtains in the category of cdga's is *stronger* than the classical notion, as explained in [HAG-II, §2.2.1]. In other words, the extension functor does not preserve being of finite presentation. One has to work around this, by introducing another notion of finite presentation [Toe11].

Definition 10. Let $f\colon A^\bullet \to B^\bullet$ be a morphism of k-cdga's. Then f is *of almost finite presentation* if $\mathrm{H}^0(f)\colon \mathrm{H}^0(A^\bullet) \to \mathrm{H}^0(B^\bullet)$ is of finite presentation as a map between k-algebras.

It is obvious that it is a generalisation of the classical notion. One then constructs a notion of geometric derived stacks by taking the topology defined by the faithfully flat morphisms of almost finite presentation (shorthand: fpppf[6]) and geometricity is obtained by considering the flat morphisms. Hence we take

(23) $(\tau, \mathbf{P}) = (\text{fpppf}, \text{flat})$.

In the classical case one has Artin's theorem (see [Art74, theorem 6.1]) comparing $(\text{ét}, \text{smooth})$ and $(\text{fppf}, \text{flat})$: in the language used in that paper we have that giving a groupoid in the fppf topology with flat structural morphisms is equivalent to giving a groupoid in the étale topology with smooth structural morphisms.

In the derived case Toën has proved that the same is true [Toe11, théorème 2.1], comparing $(\text{ét}, \text{smooth})$ and $(\text{fpppf}, \text{flat})$: the functor

(24) $\mathbf{d\,St}_{\text{ét}}^{n,\text{smooth}} \to \mathbf{d\,St}_{\text{fpppf}}^{n,\text{flat}}$

(which is induced by sheafification with respect to the finer fpppf topology) is an equivalence of categories[7] for every n.

Remark 11. Lurie also considers the Nisnevich topology (in spectral algebraic geometry), which classically is a topology in between Zariski and étale. For more information, see [DAG-XI]. I am not capable of saying interesting things about the Nisnevich topology in derived algebraic geometry at the moment.

[6] Unfortunately this looks rather silly.

[7] In the spirit of the winter school the equivalence is given relative to a base k containing Q. But the result is true in its greatest generality by working with simplicial commutative rings.

3 Truncation and extension

3.1 Definition

We are now ready to define the two main players of this talk: *truncation* and *extension*. We have an adjunction

(25) $\mathrm{H}^0 \colon \mathbf{d\,Aff}(k) \rightleftarrows \mathbf{Aff}(k) : \imath$

where $\imath$ is the natural inclusion functor of a commutative k-algebra as a cdga in degree 0. The pair $(\mathrm{H}^0, \imath)$ is a Quillen adjunction between the two HAG contexts introduced in the previous section.
Remark that this is the dg version of the adjunction

(26) $\pi_0 \colon \mathbf{d\,Aff}(k) \rightleftarrows \mathbf{Aff}(k) : \imath$

where the inclusion functor realises a commutative k-algebra as a discrete simplicial k-algebra.
Because both functors preserves weak equivalences they induce a Quillen adjunction on the model categories of prestacks (these categories being defined by the left Bousfield localisation along the Yoneda embedding of weak equivalences), i.e. we get

(27) $\imath_! \colon \mathrm{s\,Pr}(\mathbf{Aff}(k)) \rightleftarrows \mathbf{d\,Aff}(k)^\wedge : \imath^*$.

Because the inclusion $\imath$ is continuous [HAG-I, §4.8] we obtain that

(28) $\mathbf{R}\imath^* \colon \mathrm{Ho}(\mathrm{s\,Pr}(\mathbf{d\,Aff}(k))) \to \mathrm{Ho}(\mathrm{s\,Pr}(\mathbf{Aff}(k)))$

preserves the subcategories of stacks. Hence by the properties of left Bousfield localisations we obtain a Quillen adjunction

(29) $\imath_! \colon \mathbf{Aff}(k)^\sim_{\text{ét}} \rightleftarrows \mathbf{d\,Aff}(k)^\sim_{\text{ét}} : \imath^*$

on the model categories of stacks, which in turn induces a derived adjunction

(30) $\mathbf{L}\imath_! \colon \mathbf{St}(k) \rightleftarrows \mathbf{d\,St}(k) : \mathbf{R}\imath^*$

on the homotopy categories of stacks. This relation allows us to define the functors that will relate underived stacks to derived stacks, and vice versa.
Definition 12. We define the *truncation functor* t_0 to be

(31) $\mathrm{t}_0 \coloneqq \mathbf{R}\imath^* \colon \mathbf{d\,St}(k) \to \mathbf{St}(k)$.

We define[8] the *extension functor* $\imath$ to be

(32) $\imath \coloneqq \mathbf{L}\imath_! \colon \mathbf{St}(k) \to \mathbf{d\,St}(k)$.

Definition 13. Let $\mathcal{F}$ be a derived stack. Then $\mathcal{F}$ is said to be *truncated* if the counit $\imath \circ \mathrm{t}_0(\mathcal{F}) \to \mathcal{F}$ of the adjunction (29) is an isomorphism in $\mathbf{d\,St}(k)$.

[8] By some abuse of notation which should probably be avoided…

We will see in later talks lots of examples where the relationship between underived and derived stacks will be important. The following definition seems to be appropriate in this light.

Definition 14. Let $\mathcal{F}$ be a (underived) stack. A *derived enhancement* of $\mathcal{F}$ is a derived stack $\widetilde{\mathcal{F}}$ together with an isomorphism $\mathrm{t}_0(\widetilde{\mathcal{F}}) \cong \mathcal{F}$.

Hence it is a derived stack, whose shadow is the original stack. There is the obvious choice of $\imath(\mathcal{F})$, but in general this won't be the "best" derived enhancement. The question as to which of the different choices is the "best" has no clear-cut answer, it depends on the type of application one has in mind.
Some examples of derived enhancements[9]:

1. derived fiber products, as explained in §4.2;
2. derived moduli stacks (relating to the hidden smoothness principle of Kontsevich);
3. the stack of the derived category of complexes of quasicoherent sheaves on a derived Artin stack yields a derived enhancement of the stack of complexes on its truncation, modifying the derived category while keeping its heart unchanged;
4. the virtual structure sheaf on a truncation;
5. inertia stacks;
6. linear derived stacks;
7. derived mapping stacks, as hinted to in §4.3;
8. moduli stacks of objects in a dg category [TV07];
9. …

3.2 Properties

Functorial properties We first discuss some of the functorial properties of extension and truncation.
The following is [HAG-II, lemma 2.2.4.1].

Lemma 15. The extension functor $\imath$ is fully faithful.

Proof. We wish to show that for an underived stack $\mathcal{F}$ the unit of the adjunction

(33) $\mathcal{F} \to \mathrm{t}_0 \circ \imath(\mathcal{F})$

is an isomorphism. Homotopy colimits are computed in the model category of simplicial presheaves, hence they are computed levelwise, and therefore t_0 commutes with homotopy colimits. The functor $\imath$ commutes also with homotopy colimits, because it is the derived functor of a left Quillen functor.
Any stack $\mathcal{F}$ is the homotopy colimit of affine schemes, hence it suffices to take $\mathcal{F} = \operatorname{Spec} A$ for a k-algebra A.

[9] If no reference is given, one is referred to [Toe06].

We have $\imath(\operatorname{Spec} A) \cong \mathbf{R}\operatorname{Spec} A$. Moreover we have that the simplicial set of B-valued points (for B a k-algebra) of $\mathbf{R}\operatorname{Spec} A$ admits the following identifications

$$
\begin{aligned}
(\mathbf{R}\operatorname{Spec} A(B)) &\cong \operatorname{Map}_{\mathbf{cdg\,Alg}^{\leq 0}(k)}(A, B)_* \\
&\cong \operatorname{Hom}_{Alg(k)}(A, B) \\
&\cong (\operatorname{Spec} A)(B)
\end{aligned}
\tag{34}
$$

because A and B have their cohomology concentrated in degree 0. Hence the adjunction morphism

$$\operatorname{Spec} A \to \mathrm{t}_0 \circ \imath(\operatorname{Spec} A) \tag{35}$$

is an isomorphism, and the extension functor is fully faithful. □

The following is [HAG-II, lemma 2.2.4.2].

Lemma 16. The functor $\imath^*$ in the adjunction (29) is both left and right Quillen, hence it preserves equivalences.

Proof. It has the right adjoint $\mathrm{H}^{0,*}\colon \mathbf{Aff}(k)^{\sim}_{\text{ét}} \to \mathbf{d\,Aff}(k)^{\sim}_{\text{ét}}$, obtained by pulling back truncation of cdga's. By [HAG-II, lemma 1.3.2.3(2)] it is right Quillen, hence $\imath^*$ is left Quillen. That it is also right Quillen follows from (29). □

These two lemmas yield the following (important) properties of truncation and extension.

Corollary 17. The truncation functor t_0 (being left and right Quillen) commutes with homotopy limits and homotopy colimits.

Corollary 18. The extension functor $\imath$ (being left Quillen) commutes with homotopy colimits and is fully faithful.

Remark 19. The extension functor $\imath$ *does not* commute with homotopy limits!

The reason for this is that $\imath\colon \mathbf{c\,Alg}(k) \to \mathbf{cdg\,Alg}^{\leq 0}(k)$ does not preserve homotopy pushouts: computing the tensor product of algebras and then extending gives a truncated result, while computing the derived tensor product (or derived fibered product) gives in general a non-truncated result, i.e.

$$\imath(A \otimes_k B) \neq \imath(A) \otimes_k^{\mathbf{L}} \imath(B). \tag{36}$$

But this is sort of the point of the whole machinery, as will be explained in §4.2. See also [DAG-V, warning 4.1.16].

Geometric properties Now we discuss some of the geometric properties of extension and truncation.

The following is [HAG-II, lemma 2.2.2.9]. It yields that a derived stack and its truncation have the same topology (as derived stacks, for now). See also proposition 31.

Corollary 20. Let $A^\bullet$ be a commutative dg k-algebra, and let

$$\imath(\operatorname{Spec} \mathrm{H}^0(A^\bullet)) \to \operatorname{Spec} A^\bullet \tag{37}$$

be the natural morphism. Then the base change functor[10]

$$\text{(38)}\quad \mathrm{Ho}(\mathbf{d\,Aff}(k)/\operatorname{Spec} A^\bullet) \to \mathrm{Ho}(\mathbf{d\,Aff}(k)/\imath(\operatorname{Spec}\mathrm{H}^0(A^\bullet)))$$

induces an equivalence between the full subcategories of étale morphisms (i.e. of the small étale sites!).

The following is [HAG-II, lemma 2.2.2.10]. It specialises the previous corollary.

Corollary 21. The base change functor from the previous corollary induces an equivalence between the full subcategories of Zariski open immersions.

Remark 22. Using [DAG-V, remark 4.3.4] we obtain that we can even consider the usual small étale site in (38). If we have that $A^\bullet$ is concentrated in degree 0 (or in simplicial terms: it is discrete), and $A^\bullet \to B^\bullet$ is an étale morphism (in the derived sense, i.e. a so called strongly étale morphism, which we took as our definition by virtue of [HAG-II, theorem 2.2.2.6]), we obtain that

$$\text{(39)}\quad \mathrm{H}^\bullet(A^\bullet)\otimes_{\mathrm{H}^0(A^\bullet)}\mathrm{H}^0(B)\cong \mathrm{H}^0(A^\bullet)\otimes_{\mathrm{H}^0(A^\bullet)}\mathrm{H}^0(B^\bullet)\cong \mathrm{H}^\bullet(B^\bullet)$$

hence $\mathrm{H}^\bullet(B^\bullet)$ is also concentrated in degree 0. Hence

> The étale topology is the same in derived and underived algebraic geometry.

or similarly

> The small étale site of derived affine schemes on a derived affine scheme is the same as the small étale site on the truncation of a derived affine scheme.

Remark 23. We have another pleasing way of phrasing the relationship. Let $A^\bullet$ be a commutative k-algebra. We have

$$\text{(40)}\quad \mathrm{t}_0(\mathbf{R}\mathrm{Spec}\,A^\bullet)\cong \operatorname{Spec}\mathrm{H}^0(A^\bullet).$$

Remark 24. The extension functor $\imath$ is characterised by the isomorphism

$$\text{(41)}\quad \imath(\operatorname{Spec} A)\cong \mathbf{R}\mathrm{Spec}\,A$$

for A a k-algebra, and the fact that it commutes with homotopy colimits.

More functorial and geometric properties The following are properties that should rather be covered in the talk by Mauro Porta.

The following is [HAG-II, proposition 2.2.4.4(1–2)].

Proposition 25. The functor truncation functor t_0

1. preserves epimorphisms;
2. sends n-geometric derived stacks to n-geometric stacks;
3. sends flat (resp. smooth, étale) morphisms between derived stacks to flat (resp. smooth, étale) morphisms between stacks.

[10] Minor abuse of notation: $\imath$ in the next equation doesn't live on the level of homotopy categories, it's the underived version.

The following is [HAG-II, proposition 2.2.4.4(3)].

Proposition 26. The functor extension functor $\imath$

1. preserves homotopy pullbacks of n-geometric stacks along flat morphims;
2. sends n-geometric stacks to n-geometric derived stacks;
3. sends flat (resp. smooth, étale) morphisms between n-geometric stacks to flat (resp. smooth, étale) morphisms between n-geometric derived stacks.

The following is [HAG-II, proposition 2.2.4.4(4)].

Proposition 27. Let $\mathcal{F}$ be an n-geometric stack (i.e. underived). Let $\mathcal{F}' \to \imath(\mathcal{F})$ be a flat morphism of n-geometric derived stacks. Then $\mathcal{F}'$ is truncated.

Corollary 28. Let $\mathcal{F}'$ be as in the previous proposition. Then $\mathcal{F}'$ is the image of an n-geometric stack under the extension functor $\imath$.

The following is [HAG-II, proposition 2.2.4.5].

Corollary 29. Let $\mathcal{F}$ be an Artin n-stack. Then the derived stack $\imath(\mathcal{F})$ has an obstruction theory.

The following is [HAG-II, proposition 2.2.4.6].

Corollary 30. Let $\mathcal{F}$ be an n-geometric derived stack. For every commutative dg algebra $A^\bullet$ such that $\mathrm{H}^i(A^\bullet) = 0$ for $i < -k$ we have that $\mathbf{R}\mathcal{F}(A^\bullet)$ is an $(n+k+1)$-truncated simplicial set.

The following is [HAG-II, proposition 2.2.4.7]. It is a manifestation of the mantra that derived algebraic geometry is about "nilpotents on steroids" (see also §4.2, also for the source of this phrase).

Proposition 31. Let $\mathcal{F}$ be an n-geometric derived stack. The counit $\imath \circ \mathrm{t}_0(\mathcal{F}) \to \mathcal{F}$ of the adjunction $\imath \dashv \mathrm{t}_0$ is a representable morphism.

For all $A^\bullet \in \mathbf{cdg\,Alg}^{\leq 0}(k)$ and for every flat morphism $\mathbf{R}\mathrm{Spec}\,A^\bullet \to \mathcal{F}$ we have that the commutative square

$$\begin{array}{ccc} \mathbf{R}\mathrm{Spec}\,\mathrm{H}^0(A^\bullet) & \longrightarrow & \mathbf{R}\mathrm{Spec}\,A^\bullet \\ \downarrow & & \downarrow \\ \imath \circ \mathrm{t}_0(\mathcal{F}) & \longrightarrow & \mathcal{F} \end{array} \tag{42}$$

is homotopy cartesian. This implies that the counit $\imath \circ \mathrm{t}_0(\mathcal{F}) \to \mathcal{F}$ of the adjunction is a closed immersion.

Remark 32. Hence the inclusion $\imath \circ \mathrm{t}_0(\mathcal{F}) \to \mathcal{F}$ is a "formal thickening" of the truncated Artin n-stack.

4 Basic examples

4.1 Classifying stacks

This type of examples will be covered in the talk by Giorgia Fortuna.

4.2 Derived fibered products

For ease of statement we will assume that things have complementary dimension whenever this could be required, and we will work over $\mathbb{C}$.

The goal of intersection theory is to make sense of intersecting subvarieties on a variety, and describe what's going on. In this section we will show how derived algebraic geometry provides the natural place to do intersection theory, by gradually building up the complexity of our (counter)examples to show the deficits in more down-to-earth approaches. Already the case of counting 0-dimensional intersections proves to be interesting, hence we will restrict ourselves to complementary dimension here. Intersection theory is a vastly more rich subject though.

Intersection as sets The first way of making sense of an intersection theory is to naively count the points in an intersection. E.g. consider two conics in $\mathbb{P}^2$ intersection transversely, as indicated in figure 1. Then we can say the intersection multiplicity is 4, as predicted by Bézout's theorem.

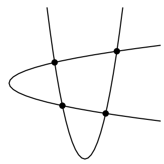


Figure 1: Transverse intersection a conic and a line in $\mathbb{P}^2$

But as Bézout's theorem already indicates: points should be counted with their *multiplicity*[11]. Observe that in this example the multiplicities are all 1, so things work out.

Example 33. Consider the (non-transverse) intersection of a conic with a line in $\mathbb{P}^2$, as indicated in figure 2.

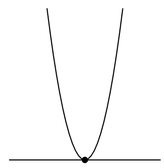


Figure 2: Transverse intersection of two conics in $\mathbb{P}^2$

The set-theoretic intersection is a single point, but if we move the line a bit away from being a tangent line[12] we see two intersection points[13] Bézout's theorem accounts for this discrepancy by saying that the intersection has multiplicity 2.

[11]Remark that the chronology of statements in this section makes little sense as we already used a result which uses multiplicities, but we try to motivate things by going from the nicest possible case to the worst possible case, not by giving a historical overview of the development of the subject.

[12]Hence we get the notion of a moving lemma in intersection theory to make sense of this.

[13]If you move in the *wrong* direction the intersections become imaginary, and you'll get two intersection points too.

Intersection as schemes Bézout's theorem is an incarnation of intersecting varieties as schemes: we incorporate the *non-reduced structure*. Then it becomes a matter of counting lengths of modules, and things work out again. Hence we can take the intersection multiplicity in the point $x \in X$ to be given by

$$\ell_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{X,x}/\mathfrak{p} \otimes \mathcal{O}_{X,x}/\mathfrak{q}\right) = \ell_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{X,x}/(\mathfrak{p}+\mathfrak{q})\right) \tag{43}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are the ideals corresponding to the subvarieties we are intersecting. Unfortunately this doesn't suffice as the next classical example (e.g. [GTM52, Example A.1.1.1]) shows.

Example 34. Consider $X = \mathbb{A}^4 = \operatorname{Spec}\mathbb{C}[x_1, x_2, x_3, x_4]$. Let $S \cong \mathbb{A}^2 \subset \mathbb{A}^4$ be defined by

$$\mathfrak{p} = (x_1 - x_3, x_2 - x_4). \tag{44}$$

Let $T \subset \mathbb{A}^4$ be given by

$$\mathfrak{q} = (x_1, x_2) \cap (x_3, x_4) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4). \tag{45}$$

This is the union of two planes intersecting in the origin, hence it is *singular* and has two non-singular irreducible components[14].

There are two ways of computing the intersection multiplicity of S and T in the origin:

with multiplicities by (43) we are led to consider

$$\dim_{\mathbb{C}}\left(\mathbb{C}[x_1, x_2]/(x_1^2, x_1x_2, x_2^2)\right) = 3; \tag{46}$$

componentwise there are two irreducible components in T, each of which intersects S in exactly 1 point with multiplicity 1, hence we get an intersection multiplicity of 2.

Hence we observe that somehow we overcounted things in the first case, because we certainly want intersection multiplicity to be additive, which is used in the second case.

Remark 35. As indicated in remark 19 and explained for instance in [DAG-V, warning 4.1.16], we have that computing the fiber product in the non-derived sense and then extending it to the derived world yields the truncated part of the derived fiber product. Hence

$$A \otimes_k B \cong \mathrm{H}^0(\imath(A) \otimes^{\mathbf{L}}_{\imath(k)} \imath(B)). \tag{47}$$

If flatness comes into play (as will be explained in the next section) we will get that the strong versions of étale, smooth, etc. behave the way we want. This can be seen in the exposition of [Toe11, §1.2] (but is of course subsumed in [HAG-II]).

[14] This example prevents us from making nice pictures, the reader is encouraged to make up a mental picture herself. The reason for this not-so-intuitive example is that we are looking for a non-Cohen-Macaulay ring, which a not-so-geometric condition.

Intersection as derived schemes The solution to this is Serre's Tor-formula [LNM11, §V.3]. We observe that the tensor product in (43) is not necessarily exact, hence not always well-behaved. This accounts for the discrepancy in the previous example, which is solved by using

$$\sum_{i=0}^{+\infty}(-1)^i\,\ell_{\mathcal{O}_{X,x}}\left(\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{p},\mathcal{O}_{X,x}/\mathfrak{q})\right) \tag{48}$$

in place of (43). In [LNM11] it is then shown that in many interesting cases this formula really gives the correct answer.

The role of derived algebraic geometry is now not to give the correct answer (because we have it) but to explain why it is the correct answer. As remarked in remark 19 we have that fibered products don't commute with the extension functor $\imath$.

By computing the derived fibered product of two non-derived schemes we nevertheless obtain the correct geometric object describing the intersection. The first step we took was considering nilpotents by considering arbitrary schemes, which helped in some of the cases. Now we are considering "nilpotents on steroids" (courtesy of Timo Schürg at `http://mathoverflow.net/a/15697/6263`), to avoid problems if the usual nilpotents don't encode enough as in the previous example. And the object we've obtained is a truly geometric object, whilst Serre's Tor formula is only a way of computing a number. Now we can see that his Tor formula belongs to the realm of derived algebraic geometry.

See also proposition 31 and corollary 20 for the relationship between the underived and the derived situation. In particular we have realized the scheme-theoretic intersection as the truncated part of the derived-scheme-theoretic intersection, as indicated in (36) and remark 35.

Remark 36. There are other interesting ways to motivate derived algebraic geometry from the point of view of intersection theory [DAG-V, Introduction].

4.3 Derived mapping stacks

This section is not finished. It is a rough and incomplete sketch of what could be said about derived mapping stacks, but these will be covered in a better way by other people.

The category $\mathbf{d\,St}(k)$ has internal Homs. Hence we obtain the following adjunction

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{d\,St}(k))}(F,\mathbf{R}\,\mathcal{M}\mathrm{ap}(G,H)) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{d\,St}(k))}(F\times^{\mathbf{L}}G,H) \tag{49}$$

where $\times^{\mathbf{L}}$ denotes the derived fiber product.

This is analogous to the existence of internal Homs for underived (higher) stacks. These stacks are called *derived mapping stacks*. In the underived case they are also called Hom stacks, and we will denote them $\mathcal{H}\mathrm{om}$ in this text. Hence if one wishes to look for properties of mapping stacks in the underived sense, use those keywords.

Remark 37. Just like the case where the extension functor $\imath$ does not preserve homotopy limits, the internal Homs are *not* preserved by $\imath$. This means that

1. looking at a tangent space in the derived sense
2. taking a fiber product in the derived sense

really gives *new* information, that was previously unavailable.
In the case of derived mapping stacks we do get the following relation between the derived and underived cases

(50) $\mathrm{t}_0(\mathbf{R}\,\mathcal{M}\mathrm{ap}(F,G)) \cong \mathcal{H}\mathrm{om}(\mathrm{t}_0(F),\mathrm{t}_0(G)).$

Derived mapping stacks have many applications:

1. *loop spaces*
2. *higher order derived tangent stacks*, extending the cotangent complex, by considering $\mathbf{R}\,\mathcal{M}\mathrm{ap}_{\mathbf{dSt}(k)}(\mathbf{D}_i, X)$ where $\mathbf{D}_i = \mathbf{R}\mathrm{Spec}\,k[\epsilon_i]$ are the dual numbers with ϵ_i living in degree $-i$;
3. derived algebraic de Rham cohomology, derived Dolbeault cohomology, derived non-abelian Hodge theory, ...

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