Spectral sequences: examples in algebra and algebraic geometry

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Abstract

These are the notes for my talk in the ANAGRAMS seminar on spectral sequences, November 27, 2014. The goal is to give some examples of spectral sequences, and some example computations, in the context of algebra and algebraic geometry. It is by no means exhaustive, written down in full generality, or other things you might care about.

Contents

1 Grothendieck spectral sequence

Slogan The chain rule for derived functors: how to express $\mathrm{R}^{p+q}(G \circ F)$ in terms of $\mathrm{R}^p G$ and $\mathrm{R}^q F$.

Theorem 1. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories, such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and $\mathcal{C}$ is cocomplete (i.e. all colimits exist, or equivalently all coproducts and fibered coproducts exist). Consider the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, "G \circ F"'] & \mathcal{B} \arrow[d, "G"] \\ & \mathcal{C} \end{tikzcd} \tag{1}$$

where F and G are additive, and F sends injective objects to G-acyclic objects. Then there exists a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p G\,(\mathrm{R}^q F(A)) \Rightarrow \mathrm{R}^{p+q}(G \circ F)(A). \tag{2}$$

Remark 2. The Grothendieck spectral sequence is a tool for actually computing something for the composition of true derived functors, where we have $\mathbf{R}(G \circ F) \cong \mathbf{R}G \circ \mathbf{R}F$. The latter approach is the best choice when trying to prove general statements, but often hands-on computations with derived functors boil down to spectral sequences. Of course, when the Grothendieck spectral sequence was introduced in [2] the notion of derived categories wasn't around yet, so it is only in hindsight that one can say this.

Observe that the Grothendieck spectral sequence covers many interesting instances of spectral sequences in day-to-day use (but not all!). In these notes we will cover the following instances of Grothendieck spectral sequences:

1. Čech-to-derived spectral sequence, section 2;
2. Tor and Ext spectral sequences, section 4;
3. Leray spectral sequence, section 3;

but many other exist.

2 The Čech-to-derived functor spectral sequence

Slogan Compute sheaf cohomology on X via appropriately chosen covers of X.

Let X be a topological space. Let $\mathcal{F}$ be a sheaf on X. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X. Denote by $\mathcal{H}^q(X, \mathcal{F})$ the presheaf which takes an open set $U \subseteq X$ to $\mathrm{H}^q(U, \mathcal{F})$. Denote for any presheaf (!) $\mathcal{P}$ by $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{P})$ the p-th Čech cohomology of $\mathcal{P}$ with respect to $\mathcal{U}$, i.e. we consider the complex

$$\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{P}) := \prod_{i_0 \in I} \mathcal{P}(U_{i_0}) \to \prod_{i_0 < i_1 \in I} \mathcal{P}(U_{i_0, i_1}) \to \dots \tag{3}$$

by choosing some ordering for I, and where $U_{i_0, \dots, i_k} := U_{i_0} \cap \dots \cap U_{i_k}$. Then

$$\mathrm{E}_2^{p,q} = \check{\mathrm{H}}^p\left(\mathcal{U}, \mathcal{H}^q(X, \mathcal{F})\right) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{F}). \tag{4}$$

This is a Grothendieck spectral sequence, by taking (let's switch to the context of ringed spaces)

categories $\mathcal{A} = \mathcal{O}_X$-Mod, $\mathcal{B} = \mathcal{O}_X$-PMod and $\mathcal{C} = \mathcal{O}_X(X)$-Mod;

functors $F = i$ and $G = \check{\mathrm{H}}^0(\mathcal{U}, -)$.

For more information and context, see [SP, tag 006P], [SP, tag 01EO].

Example 3 (Mayer–Vietoris)**.** If $\mathcal{U} = \{U, V\}$ consists of two open subsets, the spectral sequence degenerates at the E_2-page because there are only two columns and yields the long exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(U, \mathcal{F}) \oplus \mathrm{H}^0(V, \mathcal{F}) \to \mathrm{H}^0(U \cap V, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F}) \to \dots \tag{5}$$

See also [1, §II.5.6], [SP, tag 01E9].

Example 4 (Affine open cover of a scheme)**.** Let X be a quasicompact and separated scheme. Let $\mathcal{U} = (U_i)_{i \in I}$ be an finite affine open cover of X. Let $\mathcal{F}$ be a quasicoherent sheaf on X. By the assumptions on X we have that $U_{i_0, \dots, i_k}$ is again affine. By a theorem of Serre we have that affine schemes have no higher cohomology, hence $\mathrm{H}^i(U_{i_0, \dots, i_k}, \mathcal{F}) = 0$ for $i \geq 1$.

But this means that the spectral sequence (4) degenerates at the E_2 page because $\mathrm{E}_2^{p,q} = \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F}))$ is zero if $q \neq 0$. Hence we get that

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \cong \mathrm{H}^p(X, \mathcal{F}). \tag{6}$$

Observe the example certainly isn't phrased in its full generality for the sake of clarity. For more generality one can look at e.g. [SP, tag 01ET].

The upshot is that Čech cohomology can be seen as an *algorithm* to compute abstractly defined sheaf cohomology. The usual definition of Čech cohomology requires taking a direct limit over all covers, but in the situation described here it suffices to look at a particular cover.

3 Leray spectral sequence

Slogan Compute sheaf cohomology on X via sheaf cohomology on Y.

Let $f : X \to Y$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Then

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(Y, \mathrm{R}^q f_*(\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{F}). \tag{7}$$

This is a Grothendieck spectral sequence, by taking

categories $\mathcal{A} = \mathrm{Ab}(X)$, $\mathcal{B} = \mathrm{Ab}(Y)$ and $\mathcal{C} = \mathrm{Abgp}$;

functors $F = f_*$ and $G = \Gamma(X, -)$.

For more information and context, see [SP, tag 01EY]. Or search for “Leray” in the Stacks project, you’ll get some interesting applications but I won’t delve into them here.

4 Base change for Tor and Ext spectral sequence

Slogan Approximate Tor or Ext on B via Tor or Ext on A

Let $f : A \to B$ be a morphism of rings. Let M be an A-module, N be a B-module. Then we have spectral sequences

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^B_p\left(\mathrm{Tor}^A_q(M,B),N\right) \Rightarrow \mathrm{Tor}^A_{p+q}(M,N) \tag{8}$$

and

$$\mathrm{E}_2^{p,q} = \mathrm{Ext}^p_B\left(M,\mathrm{Ext}^q_A(B,N)\right) \Rightarrow \mathrm{Ext}^{p+q}_A(M,N). \tag{9}$$

For a reference, see [4, §5.6 and §5.8] (watch out for the typos) or [SP, tag 0620].

These are again examples of Grothendieck spectral sequences, by taking

categories $\mathcal{A} = \mathrm{Mod}/A$, $\mathcal{B} = \mathrm{Mod}/B$ and $\mathcal{C} = \mathrm{Abgp}$,

functors $F = - \otimes_A B$ and $G = - \otimes_B M$;

and

categories $\mathcal{A} = \mathrm{Mod}/A$, $\mathcal{B} = \mathrm{Mod}/B$ and $\mathcal{C} = \mathrm{Abgp}$,

functors $F = \mathrm{Hom}_A(B,-)$ and $G = \mathrm{Hom}_B(M,-)$.

Example 5. This is just a silly example to indicate how one could use the base change spectral sequence for Tor. Let us assume that B is flat as an A-module (a situation which is common in algebraic geometry). Then we have $\mathrm{Tor}^A_q(M,B) = 0$ for $q \geq 1$, hence the $\mathrm{E}^2_{p,q}$-page of our spectral sequence contains only non-zero entries on the bottom row, i.e. it collapses (or degenerates, whichever terminology you like more). But then we immediately get that

$$\mathrm{Tor}^B_p(M \otimes_A B, N) \cong \mathrm{Tor}^A_p(M,N). \tag{10}$$

5 Hodge-to-de Rham spectral sequence

Unfortunately I don't have the time and place to repeat the required definitions here. Let's assume you are familiar with Hodge theory, (algebraic) de Rham cohomology and Witt vectors (after all, who isn't?)

The usual Hodge-to-de Rham spectral sequence for a proper and smooth complex variety reads

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q(X, \Omega^p_{X/\mathbb{C}}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X/\mathbb{C}). \tag{11}$$

On the left we have sheaf cohomology for a particular choice of sheaves, on the right we have de Rham cohomology. The main result about this sequence is that it *degenerates at the* E_1*-page* if X under the conditions above (you can get a spectral sequence for non-compact non-Kähler complex manifolds).

On the other hand it is possible to introduce (algebraic) de Rham cohomology in a more general context over any field k (possibly of positive characteristic). The spectral sequence then reads

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q(X, \Omega^p_{X/k}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X/k). \tag{12}$$

There exist explicit examples in which the spectral sequence doesn't degenerate on the E_1-page, but Deligne and Illusie have proved that if $p < \dim X$ and a certain criterion regarding lifting X to $\mathrm{W}_2(k)$ is satisfied there *is* degeneration. Their proof was the first algebraic proof of a result which up to then was only known using Hodge theory!

It also gives a algebro-geometric proof of the Hodge decomposition in characteristic 0, which uses characteristic p methods! I wish I had the time to say more about this, because there is fascinating mathematics involved.

6 Ext spectral sequence

Slogan Extensions of complexes are approximated by extensions of one complex and cohomology of the other.

Let X be a smooth projective variety. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$. Then

(13) $\mathrm{E}_2^{p,q} = \mathrm{Ext}_X^p(\mathcal{H}^{-q}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$

where by Ext^n in a triangulated category we mean the shifted Hom. A reference for this sequence in various guises can be found in [3, remark 3.7] or [SP, tag 07AA].

Example 6 (Skyscrapers form a spanning class)**.** One application that comes to my mind for this spectral sequence is proving that the skyscraper sheaves form a spanning class in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$ (where X is a smooth projective variety), see e.g. [3, proposition 3.17].

Proof. We wish to prove that "(shifts of) skyscrapers $k(x)[m]$ see all of the derived category", i.e. that for every object $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$ we have the following:

1. if $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)}(k(x)[m], \mathcal{F}^\bullet[i]) = 0$ for all $x \in X$ and $i, m \in \mathbb{Z}$ then $\mathcal{F}^\bullet \cong 0$;
2. if $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)}(\mathcal{F}^\bullet[i], k(x)[m]) = 0$ for all $x \in X$ and $i, m \in \mathbb{Z}$ then $\mathcal{F}^\bullet \cong 0$.

As Serre duality (which is tensoring with a line bundle and shifting) reduces to a shift when applied to a (shift of a) skyscraper sheaf, it suffices to prove the second. I.e. for every non-trivial $\mathcal{F}^\bullet$ we wish to find a point $x \in X$ and an integer m such that $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)}(\mathcal{F}^\bullet, k(x)[m]) \neq 0$ (we take the i and m together and rename it m).

Let m be the highest degree in which $\mathcal{F}^\bullet$ has cohomology. If we plug $\mathcal{F} := \mathcal{H}^m(\mathcal{F}^\bullet)$ and $\mathcal{G} := k(x)$ in (13) we see that all differentials *leaving* $\mathrm{E}_r^{0,-m}$ are zero (as these go down, to a place where no cohomology exists by the choice of m).

Moreover we are looking at extensions between coherent sheaves, which are zero if $p < 0$. In other words: the spectral sequence is concentrated in the right half-plane. Hence differentials *entering* $\mathrm{E}_r^{0,-m}$ are also zero (as they come from a place where no cohomology exists). So $\mathrm{E}_\infty^{0,-m} = \mathrm{E}_2^{0,-m}$.

Now it suffices to take a point in $\mathrm{supp}(\mathcal{H}^m(\mathcal{F}^\bullet)) \neq \emptyset$, because

(14) $\mathrm{E}_\infty^{0,-m} = \mathrm{E}_2^{0,-m} = \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)}(\mathcal{H}^m(\mathcal{F}^\bullet), k(x)) \neq 0$

implies that

(15) $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)}(\mathcal{F}^\bullet, k(x)[-m]) \neq 0.$

$\square$

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