The restriction of a homotopy-injective complex to a Zariski open subset is not necessarily homotopy-injective

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Abstract

On page 10 in Leonid Positselski's manuscript Contraherent cosheaves [2] one reads

> [...] the restriction of a homotopy-injective complex of quasicoherent sheaves to such a subscheme may no longer be homotopy-injective.

In a mail to the author from October 1, 2014 Leonid Positselski explained the construction of an example, which goes along the lines of Amnon Neeman's [1, example 6.5]. This note is written in order to put it in LaTeX and flesh out some details, and is made public with the permission of Leonid Positselski.

1 Introduction

Acknowledgements All mathematical ideas here are due to Leonid Positselski and Amnon Neeman, and I would like to thank the first for outlining the example in an email and allowing me to make this public. All mistakes are due to the author.

2 The example

Situation The setup is as in [1, example 6.5] and the notation is chosen to reflect the construction there (to some extent). The main difference is that we compute the functor $f^! = \mathbf{R}\mathrm{Hom}_S(R,-)$ via a homotopy-injective resolution in the second variable, whereas in the article a projective resolution of the first variable is used. But to get to the conclusion we again reduce to the fact that $i^* \circ f^! \neq g^! \circ j^*$ on the unbounded level, as in the example of loc. cit.

Let R be any sufficiently general commutative noetherian ring (e.g. $\mathbb{Z}$ or $k[x]$ would do). Let $r \in R$ be a non-invertible and non-nilpotent element. Then we set

$$\begin{aligned} &S := R[\epsilon]/(\epsilon^2), \\ (1)\quad &A := R[r^{-1}], \\ &B := S[r^{-1}] = R[r^{-1},\epsilon]/(\epsilon^2). \end{aligned}$$

The geometric picture corresponding to this choice of rings is

$$\begin{array}{ccc} U := \operatorname{Spec} A & \overset{i}{\hookrightarrow} & X := \operatorname{Spec} R \\ \downarrow g & & \downarrow f \\ V := \operatorname{Spec} B & \overset{j}{\hookrightarrow} & Y := \operatorname{Spec} S \end{array} \tag{2}$$

where f and g are proper morphisms of finite type, whilst i and j are open immersions. Remark that the non-reducedness of the rings doesn't play an essential role (as far as I can tell): we are looking for the easiest proper morphism available, hence we use a proper affine morphism, but these are necessarily finite.

Because f (resp. g) are affine we have already on the underived level an adjunction $f_* \dashv f^!$ (resp. $g_* \dashv g^!$), which reduces to the adjunction

$$\operatorname{Hom}_S(M,N) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_S(R,N)) \tag{3}$$

for M an R-module and N an S-module, with f_* the transport of structure along f and $f^! = \operatorname{Hom}_S(R,-)$. If go to the derived setting we get (together with a possible confusing notation: usually $f^!$ is unambiguously on the derived level but in this case there is already an underived incarnation which we denote in the same way) that $f^! = \mathbf{R}\operatorname{Hom}_S(R,-)$ as hinted before (likewise for g).

Construction of a homotopy-injective complex on Y We first construct the homotopy-injective complex whose restriction will no longer be homotopy-injective.

We will denote by

$$C_S^\bullet := \cdots \xrightarrow{0} S \xrightarrow{0} S \xrightarrow{0} \cdots \tag{4}$$

a complex on Y. This is not yet homotopy-injective, as homotopy-injective implies degreewise injective (and S is not self-injective).

Pick any injective resolution $I_S^\bullet$ of S as a module over itself.

Now set

$$J_S^\bullet := \prod_{n\in\mathbb{Z}} \Sigma^n I_S^\bullet. \tag{5}$$

This is a homotopy-injective complex because $\Sigma^n I_S^\bullet$ as a bounded below complex of injectives is homotopy-injective and infinite products of homotopy-injective complexes are homotopy-injective. The complex $J_S^\bullet$ is quasi-isomorphic to $C_S^\bullet$ via the obvious morphism (i.e. the product of the injective augmentation maps).

Remark 1. The complex $\bigoplus_{n\in\mathbb{Z}} \Sigma^n I_S^\bullet$ is also quasi-isomorphic to $C_S^\bullet$, but it is not necessarily homotopy-injective: the Hom-functor commutes with limits in the second variable, not colimits. However, as in [1, example 6.5] we use this complex to show that j^* commutes with the particular infinite product that we are using here.

Restriction of the homotopy-injective complex on Y to V The restriction of $J_S^\bullet$ to V is given by $j^*(J_S^\bullet) = J_S^\bullet[r^{-1}]$. It is our goal to show that this complex is *not homotopy-injective*.

Construction of a homotopy-injective complex on V We then construct a homotopy-injective complex on the open subset V in order to compare it to the restriction of the homotopy-injective complex. The construction goes along the same lines as the construction of the first homotopy-injective complex.

We will denote by

$$C_B^\bullet := \cdots \xrightarrow{0} B \xrightarrow{0} B \xrightarrow{0} \cdots \tag{6}$$

a complex on V. This is not yet homotopy-injective, as homotopy-injective implies degreewise injective (and B is not self-injective).

Consider the complex $I_B^\bullet := I_S^\bullet[r^{-1}]$, as we are in the noetherian setting this is an injective (and not just flasque) resolution of B.

Now set

$$J_B^\bullet := \prod_{n\in\mathbb{Z}} \Sigma^n I_B^\bullet = \prod_{n\in\mathbb{Z}} \Sigma^n I_S^\bullet[r^{-1}]. \tag{7}$$

This is a homotopy-injective complex because $\Sigma^n I_B^\bullet$ as a bounded below complex of injectives is homotopy-injective and infinite products of homotopy-injective complexes are homotopy-injective. The complex $J_B^\bullet$ is quasi-isomorphic to $C_B^\bullet$ via the obvious morphism (i.e. the product of the injective augmentation maps).

Comparison of the complexes on V: quasi-isomorphism We have the obvious morphism

$$J_S^\bullet[r^{-1}] = \left(\prod_{n\in\mathbb{Z}} \Sigma^n I_S^\bullet\right)[r^{-1}] \to J_B^\bullet = \prod_{n\in\mathbb{Z}} \Sigma^n I_S^\bullet[r^{-1}] \tag{8}$$

which is *not an isomorphism* because localisation does not preserve infinite products (the same argument is used in [1, example 6.5], all the terms contribute to the same degree whereas in remark 1 we split things in all degrees).

It is nevertheless a *quasi-isomorphism*, because localisation and the direct product are exact functors (for the direct product it is important that we are working affine).

Computing $f^!(I_S^\bullet)$ The argument requires knowledge about $f^!(I_S^\bullet)$, just as in [1, example 6.5]. This reduces to knowing $f^!(S)$, and hence

$$f^!(S) = \mathbf{R}\mathrm{Hom}_S(R,S) = \prod_{m\geq 0} \Sigma^{-m} R \tag{9}$$

as in loc. cit.

Comparison of the complexes on V: applying a left exact functor We wish to show that $J_S^\bullet[r^{-1}]$ is *not homotopy-injective*. We do this by applying a left exact functor F to Mod/B, which defines a right derived functor $\mathbf{R}F$ on $\mathbf{D}(\mathrm{Mod}/B)$ by applying F degreewise to a homotopy-injective resolution. The answer should be the same for each homotopy-injective resolution, hence if $J_S^\bullet[r^{-1}]$ were to be homotopy-injective the result should be the same as for $J_B^\bullet$, these complexes being quasi-isomorphic, and $J_B^\bullet$ homotopy-injective by construction.

Consider the functor $g^!\colon \mathrm{Mod}/B \to \mathrm{Mod}/A$, which is already defined on the underived level, and left exact as discussed before. It corresponds to taking the maximal submodule that is annihilated by the action of ϵ.

We then compute, as in [1, example 6.5]

$$
\begin{aligned}
g^!(J_B^\bullet) &= g^!\left(\prod_{n\in\mathbb{Z}} \Sigma^n I_S^\bullet[r^{-1}]\right) \\
&= \prod_{n\in\mathbb{Z}} \Sigma^n g^!(I_S^\bullet[r^{-1}]) \\
&= \prod_{n\in\mathbb{Z}} \Sigma^n f^!(I_S^\bullet)[r^{-1}] \\
&= \prod_{n\in\mathbb{Z}} \Sigma^n \left(\prod_{m\geq 0} \Sigma^{-m} R\right)[r^{-1}]
\end{aligned}
\tag{10}
$$

where the first step is just unwinding the definition, the second is because $g^!$ as a right adjoint commutes with products, and the third step is an application of the base-change formula for bounded below complexes (with a forgetful functor thrown in, or one applies the argument of loc. cit. using remark 1) and the last step is filling in the computation of $f^!(I_S^\bullet)$.

In cohomology this gives, going straight for H^0

$$
\mathrm{H}^0\left(g^!(J_B^\bullet)\right) = \prod_{n\in\mathbb{Z}} R[r^{-1}]. \tag{11}
$$

On the other hand we have

$$
\begin{aligned}
g^!(J_S^\bullet[r^{-1}]) &= g^! \circ j^*(J_S^\bullet) \\
&= i^* \circ f^!(J_S^\bullet) \\
&= f^!(J_S^\bullet)[r^{-1}] \\
&= f^!\left(\prod_{n\in\mathbb{Z}} \Sigma^n I_S^\bullet\right)[r^{-1}] \\
&= \left(\prod_{n\in\mathbb{Z}} \Sigma^n f^!(I_S^\bullet)\right)[r^{-1}] \\
&= \left(\prod_{n\in\mathbb{Z}} \Sigma^n \prod_{m\geq 0} \Sigma^{-m} R\right)[r^{-1}]
\end{aligned}
\tag{12}
$$

where the first step is just unwinding the definition, the second step is the base change formula which we can apply because we are computing things termwise (in other words: $g^!$ (underived) commutes with localisation), and then we proceed as before.

In cohomology this gives

$$
\mathrm{H}^0\left(g^!(J_S^\bullet[r^{-1}])\right) = \left(\prod_{m\geq 0} R\right)[r^{-1}] \tag{13}
$$

Conclusion By the choice of r and the argument as in [1, example 6.5] we have that the restriction $J_S^\bullet[r^{-1}]$ cannot be homotopy-injective, as the cohomology of the complexes differs.

References

[1] Amnon Neeman. "The Grothendieck duality theorem via Bousfield's techniques and Brown representability". In: *Journal of the American Mathematical Society* 9.1 (1996), pp. 205–236. arXiv: `9412022v1 [math.AG]`.

[2] Leonid Positselski. "Contraherent cosheaves". In: (2014). arXiv: `1209.2995 [math.CT]`.