The restriction of a homotopy-injective complex to a Zariski open subset is not necessarily homotopy-injective

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October 2, 2016

Abstract

On page 10 in Leonid Positselski’s manuscript Contraherent cosheaves [2] one reads

[...] the restriction of a homotopy-injective complex of quasicoherent sheaves to such a subscheme may no longer be homotopy-injective.

In a mail to the author from October 1, 2014 Leonid Positselski explained the construction of an example, which goes along the lines of Amnon Neeman’s [1, example 6.5]. This note is written in order to put it in \( \text{LaTeX} \) and flesh out some details, and is made public with the permission of Leonid Positselski.

1 Introduction

Acknowledgements All mathematical ideas here are due to Leonid Positselski and Amnon Neeman, and I would like to thank the first for outlining the example in an email and allowing me to make this public. All mistakes are due to the author.

2 The example

Situation The setup is as in [1, example 6.5] and the notation is chosen to reflect the construction there (to some extent). The main difference is that we compute the functor \( f^! = \text{RHom}_S(R, -) \) via a homotopy-injective resolution in the second variable, whereas in the article a projective resolution of the first variable is used. But to get to the conclusion we again reduce to the fact that \( i^* \circ f^! \neq g^! \circ j^* \) on the unbounded level, as in the example of loc. cit.

Let \( R \) be any sufficiently general commutative noetherian ring (e.g. \( \mathbb{Z} \) or \( k[x] \) would do). Let \( r \in R \) be a non-invertible and non-nilpotent element. Then we set

\[
S := R[e]/(e^2),
\]

\[
A := R[r^{-1}],
\]

\[
B := S[r^{-1}] = R[r^{-1}, e]/(e^2).
\]
The geometric picture corresponding to this choice of rings is

\[
\begin{array}{ccc}
U := \text{Spec} A & \xrightarrow{i} & X := \text{Spec} R \\
\downarrow f & & \downarrow f \\
V := \text{Spec} B & \xrightarrow{j} & Y := \text{Spec} S
\end{array}
\]

where \( f \) and \( g \) are proper morphisms of finite type, whilst \( i \) and \( j \) are open immersions. Remark that the non-reducedness of the rings doesn’t play an essential role (as far as I can tell): we are looking for the easiest proper morphism available, hence we use a proper affine morphism, but these are necessarily finite.

Because \( f \) (resp. \( g \)) are affine we have already on the underived level an adjunction \( f_* \dashv f^* \) (resp. \( g_* \dashv g^* \)), which reduces to the adjunction

\[
\text{Hom}_S(M, N) \cong \text{Hom}_R(M, \text{Hom}_S(R, N))
\]

for \( M \) an \( R \)-module and \( N \) an \( S \)-module, with \( f_* \) the transport of structure along \( f \) and \( f^* = \text{Hom}_S(R, -) \). If go to the derived setting we get (together with a possible confusing notation: usually \( f^* \) is unambiguously on the derived level but in this case there is already an underived incarnation which we denote in the same way) that \( f^* = \mathbb{R}\text{Hom}_S(R, -) \) as hinted before (likewise for \( g \)).

**Construction of a homotopy-injective complex on \( Y \)** We first construct the homotopy-injective complex whose restriction will no longer be homotopy-injective.

We will denote by

\[
C^*_S := \cdots \to 0 \to S \to 0 \to \cdots
\]

a complex on \( Y \). This is not yet homotopy-injective, as homotopy-injective implies degreewise injective (and \( S \) is not self-injective).

Pick any injective resolution \( I^*_S \) of \( S \) as a module over itself.

Now set

\[
J^*_S := \prod_{n \in \mathbb{Z}} \Sigma^n I^*_S.
\]

This is a homotopy-injective complex because \( \Sigma^n I^*_S \) as a bounded below complex of injectives is homotopy-injective and infinite products of homotopy-injective complexes are homotopy-injective. The complex \( J^*_S \) is quasi-isomorphic to \( C^*_S \) via the obvious morphism (i.e. the product of the injective augmentation maps).

**Remark 1.** The complex \( \bigoplus_{n \in \mathbb{Z}} \Sigma^n I^*_S \) is also quasi-isomorphic to \( C^*_S \), but it is not necessarily homotopy-injective: the Hom-functor commutes with limits in the second variable, not colimits. However, as in [1, example 6.5] we use this complex to show that \( j^* \) commutes with the particular infinite product that we are using here.

**Restriction of the homotopy-injective complex on \( Y \) to \( V \)** The restriction of \( J^*_S \) to \( V \) is given by \( j^*(J^*_S) = J^*_S[r^{-1}] \). It is our goal to show that this complex is not homotopy-injective.
Construction of a homotopy-injective complex on \( V \) We then construct a homotopy-injective complex on the open subset \( V \) in order to compare it to the restriction of the homotopy-injective complex. The construction goes along the same lines as the construction of the first homotopy-injective complex.

We will denote by

\[
C^\bullet_B := \cdots \to 0 \to B \to 0 \to \cdots
\]

a complex on \( V \). This is not yet homotopy-injective, as homotopy-injective implies degreewise injective (and \( B \) is not self-injective).

Consider the complex \( I^\bullet := I^\bullet_S[r^{-1}] \), as we are in the noetherian setting this is an injective (and not just flasque) resolution of \( B \).

Now set

\[
J^\bullet_B := \prod_{n \in \mathbb{Z}} \Sigma^n I^\bullet_B = \prod_{n \in \mathbb{Z}} \Sigma^n I^\bullet_S[r^{-1}].
\]

This is a homotopy-injective complex because \( \Sigma^n I^\bullet_B \) as a bounded below complex of injectives is homotopy-injective and infinite products of homotopy-injective complexes are homotopy-injective. The complex \( J^\bullet_B \) is quasi-isomorphic to \( C^\bullet_B \) via the obvious morphism (i.e. the product of the injective augmentation maps).

Comparison of the complexes on \( V \): quasi-isomorphism We have the obvious morphism

\[
J^\bullet_S[r^{-1}] = \left( \prod_{n \in \mathbb{Z}} \Sigma^n I^\bullet_S \right)[r^{-1}] \to J^\bullet_B = \prod_{n \in \mathbb{Z}} \Sigma^n I^\bullet_S[r^{-1}]
\]

which is not an isomorphism because localisation does not preserve infinite products (the same argument is used in [1, example 6.5], all the terms contribute to the same degree whereas in remark 1 we split things in all degrees).

It is nevertheless a quasi-isomorphism, because localisation and the direct product are exact functors (for the direct product it is important that we are working affine).

Computing \( f^!(I^\bullet_S) \) The argument requires knowledge about \( f^!(I^\bullet_S) \), just as in [1, example 6.5]. This reduces to knowing \( f^!(S) \), and hence

\[
(9) \quad f^!(S) = \mathbb{R}\text{Hom}_S(R, S) = \prod_{m \geq 0} \Sigma^{-m} R
\]

as in loc. cit.

Comparison of the complexes on \( V \): applying a left exact functor We wish to show that \( J^\bullet_S[r^{-1}] \) is not homotopy-injective. We do this by applying a left exact functor \( F \) to \( \text{Mod}/B \), which defines a right derived functor \( \mathbb{R}F \) on \( \mathbb{D}(\text{Mod}/B) \) by applying \( F \) degreewise to a homotopy-injective resolution. The answer should be the
same for each homotopy-injective resolution, hence if $J_+^* [r^{-1}]$ were to be homotopy-injective the result should be the same as for $J_+^*$, these complexes being quasi-isomorphic, and $J_+^*$ homotopy-injective by construction.

Consider the functor $g^! : \text{Mod}/B \to \text{Mod}/A$, which is already defined on the underived level, and left exact as discussed before. It corresponds to taking the maximal submodule that is annihilated by the action of $\epsilon$.

We then compute, as in [1, example 6.5]

$$(10) \quad g^!(J^* B) = g^! \left( \prod_{n \in \mathbb{Z}} \Sigma^n I_+^* [r^{-1}] \right)$$

$$= \prod_{n \in \mathbb{Z}} \Sigma^n g^!(I_+^* [r^{-1}])$$

$$= \prod_{n \in \mathbb{Z}} \Sigma^n f^!(I_+^* [r^{-1}])$$

$$= \prod_{n \in \mathbb{Z}} \Sigma^n \left( \prod_{m \geq 0} \Sigma^{-m} R \right) [r^{-1}]$$

where the first step is just unwinding the definition, the second is because $g^!$ as a right adjoint commutes with products, and the third step is an application of the base-change formula for bounded below complexes (with a forgetful functor thrown in, or one applies the argument of loc. cit. using remark 1) and the last step is filling in the computation of $f^!(I_+^*)$.

In cohomology this gives, going straight for $\text{H}^0$

$$(11) \quad \text{H}^0 \left( g^!(J^* B) \right) = \prod_{n \in \mathbb{Z}} R[r^{-1}]$$

On the other hand we have

$$g^! (J_+^* [r^{-1}]) = g^! \circ j^!(J_+^*)$$

$$= i^* \circ f^!(J_+^*)$$

$$= f^!(I_+^* [r^{-1}])$$

$$= \prod_{n \in \mathbb{Z}} \Sigma^n f^!(I_+^* [r^{-1}])$$

$$= \prod_{n \in \mathbb{Z}} \Sigma^n \left( \prod_{m \geq 0} \Sigma^{-m} R \right) [r^{-1}]$$

where the first step is just unwinding the definition, the second step is the base change formula which we can apply because we are computing things termwise (in other words: $g^!$ (underived) commutes with localisation), and then we proceed as before.

In cohomology this gives

$$(13) \quad \text{H}^0 \left( g^!(J_+^* [r^{-1}]) \right) = \prod_{m \geq 0} R[r^{-1}]$$
Conclusion  By the choice of $r$ and the argument as in [1, example 6.5] we have that the restriction $J_r[r^{-1}]$ cannot be homotopy-injective, as the cohomology of the complexes differs.

References
