

The triangulated structure on the category of projectives

Pieter Belmans

28 August 2013

Description

These are the preliminary notes for my lecture at the Summer school on Preprojective algebras (see <http://www.math.uni-bielefeld.de/~plinke/meetings/birepschool2013/>).

Their goal is to provide a reasonably self-contained manuscript on which I could base my lecture. Certain parts in the proof of the main theorem are written more explicitly than in the original article, and some background is provided wherever appropriate. Explicit examples are provided to enlighten the abstract theory.

For exact categories we refer to [Buh10], for notions not defined in the context of quivers, we refer to the general literature. And we are aware that they are not perfect when it comes to the general context, but the exposition of the main theorem is as optimal as the author can make it. Some of the (supposedly) easy lemmas have not been written down.

I would like to thank Adam-Christiaan van Roosmalen, Johan Steen, Greg Stevenson for the interesting conversations which helped me improve the exposition, and the organisers of the summer school for this opportunity.

Abstract

Auslander's algebraic McKay correspondence explained in a previous talk may be phrased as an additive equivalence between the Frobenius category of maximal Cohen-Macaulay R -modules over a Kleinian singularity R and the category of projective modules over the corresponding skew-group algebra, which we know is Morita equivalent to a preprojective algebra of an affine Dynkin diagram. By Happel's work the stable category of MCM R -modules is naturally triangulated.

Now, the additive equivalence mentioned above induces an equivalence to the category of projective modules over the corresponding preprojective algebra of Dynkin type (obtained by dividing out by the ideal generated by the idempotent corresponding to the extension vertex). Therefore, the latter category admits a triangulated structure. This motivates Amiot's construction.

Her construction is quite technical, so it might be helpful to explain many of the definitions by means of accessible examples, e.g. for Dynkin type A_2 or A_3 .

Contents

1	Introduction	3
1.1	Statements	3
1.2	Context	4
2	The triangulated structure on the category of projectives	5
2.1	Definitions	5
2.2	Properties of the functors Z^0 and T	6
2.3	Proof of the main theorem	12
3	Application: Deformed preprojective algebras of generalized Dynkin type	20

Chapter 1

Introduction

1.1 Statements

Let k be an algebraically closed field. Let \mathcal{P} be a k -linear karoubian category (i.e. idempotents split). The category $\mathcal{P}\text{-mod}$ of contravariant finitely presented k -linear functors from \mathcal{P} to $k\text{-mod}$ is exact.

Lemma 1. Let \mathcal{P} be a triangulated category whose idempotents split. Then the projective objects in $\mathcal{P}\text{-mod}$ are the representable functors.

Proof. See [Nee01, lemma 5.1.11]. □

Now let \mathcal{P} be a k -linear karoubian Frobenius category. Its associated stable category $\underline{\mathcal{P}\text{-mod}}$ is a triangulated category in which we denote the suspension functor by Σ .

Let S be an auto-equivalence of \mathcal{P} . It extends to an exact endofunctor on $\mathcal{P}\text{-mod}$ and hence to a triangle endofunctor on $\underline{\mathcal{P}\text{-mod}}$. We wish to find a necessary condition on S such that S is the suspension functor of a triangulated structure on \mathcal{P} . The following theorem provides such a condition.

Theorem 2 (Main theorem). Let \mathcal{P} be as before. Assume the existence of an exact sequence of exact endofunctors on $\mathcal{P}\text{-mod}$

$$(1.1) \quad 0 \rightarrow \text{id}_{\mathcal{P}\text{-mod}} \rightarrow X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow S \rightarrow 0$$

such that $X_i(\mathcal{P}\text{-mod}) \subseteq \mathcal{P}\text{-proj}$ for $i = 0, 1, 2$. Then we can equip \mathcal{P} with a triangulated structure for which S is the suspension functor.

The goal is to apply this to the case of modules over a (deformed) preprojective algebra of generalised Dynkin type, i.e. we take $\mathcal{P} = \text{proj } \mathcal{P}^f(\Delta)$, because the projectives coincide with the representables we obtain the triangulated structure on $\text{proj } \mathcal{P}^f(\Delta)$ itself. The notation and terminology will be introduced in chapter 3, the statement is as follows.

Corollary 3. Let $\mathcal{P}^f(\Delta)$ be a deformed preprojective algebra of generalised Dynkin type. Its category $\mathcal{P}^f(\Delta)\text{-proj}$ of finite-dimensional projective modules is triangulated, its suspension functor is the Nakayama functor.

Using these results it will be possible in later talks to relate this extra triangulated structure to the triangulated structure on the other side of the equivalence, obtain results on Hochschild cohomology, e.g. the 6-periodicity (see talk 15).

1.2 Context

In previous talks (especially 7, 8 and 9) we have seen instances of McKay correspondence and the structural results of some of the categories that are involved. Namely we have seen that the Frobenius category of maximal Cohen-Macaulay modules over a Kleinian singularity is equivalent to the category of projective modules over a preprojective algebra on an *extended* Dynkin quiver. Taking the stable category corresponds to modding out the extension vertex, which yields a preprojective algebra on a Dynkin quiver.

So algebraic McKay correspondence yields by structure transport a triangulated structure on the category $\text{proj } P^f(\Delta)$, but one could ask the question whether this structure can be described intrinsically. The answer is yes. Strengthening a result by Heller [Hel68] Amiot [Ami08] proves that given a certain resolution of an auto-equivalence one desires to be the shift functor, one can actually equip the category with a triangulated structure.

Chapter 2

The triangulated structure on the category of projectives

2.1 Definitions

The goal is to define two functors, T and Z^0 relating the categories \mathcal{P} , $\mathcal{P}\text{-mod}$ and the category of cochain complexes of \mathcal{P} -modules. This way we can work in the more flexible setting of cochain complexes to prove the existence of a triangulated structure on the category \mathcal{P} .

Definition 4. Let \mathcal{P} be as before. Let M be an object of $\mathcal{P}\text{-mod}$. Then we denote

$$(2.1) \quad T_M : X_0(M) \rightarrow X_1(M) \rightarrow X_2(M) \rightarrow S \circ X_0(M)$$

and we say that T_M is a *standard triangle*.

A *triangle* of \mathcal{P} will then be a sequence

$$(2.2) \quad P \rightarrow Q \rightarrow R \rightarrow S(P)$$

which is isomorphic to a standard triangle in $\mathcal{P}\text{-mod}$.

Definition 5. Let \mathcal{P} be as before. Denote the category of acyclic cochain complexes with projective components with values in $\mathcal{P}\text{-mod}$ by $\text{Ch}_{\text{ac,proj}}(\mathcal{P}\text{-mod})$. We have a functor

$$(2.3) \quad Z^0 : \text{Ch}_{\text{ac,proj}}(\mathcal{P}\text{-mod}) \rightarrow \mathcal{P}\text{-mod}$$

sending a cochain complex to the kernel of d^0 .

Lemma 6. The category $\text{Ch}_{\text{ac,proj}}(\mathcal{P}\text{-mod})$ is a Frobenius category whose projective-injective objects are the contractible complexes.

Proof. □

Lemma 7. The functor Z^0 sends projective-injective objects to projective-injective objects, and therefore induces a triangle equivalence between the stable categories $\underline{\text{Ch}_{\text{ac,proj}}(\mathcal{P}\text{-mod})}$ and $\underline{\mathcal{P}\text{-mod}}$.

Proof. □

Definition 8. If an object of $\text{Ch}_{\text{ac,proj}}(\mathcal{P}\text{-mod})$ is S -periodic, i.e. it has the form

$$(2.4) \quad \cdots \rightarrow P \xrightarrow{u} Q \rightarrow R \rightarrow S(P) \xrightarrow{S(u)} S(Q) \rightarrow \cdots$$

then it is called an S -complex. The non-full subcategory of S -complexes with the S -periodic morphisms is denoted $S\text{-comp}$.

Lemma 9. The category $S\text{-comp}$ is a Frobenius category whose projective-injective objects are the S -contractible complexes.

Proof. □

Lemma 10. The functor Z^0 induces an exact functor $S\text{-comp} \rightarrow \mathcal{P}\text{-mod}$ and a triangle functor

$$(2.5) \quad \underline{Z}^0: \underline{S\text{-comp}} \rightarrow \underline{\mathcal{P}\text{-mod}}.$$

Proof. □

Definition 11. Consider an exact sequence of exact functors as in (1.1). For each object M in $\underline{\mathcal{P}\text{-mod}}$ it induces a functorial isomorphism

$$(2.6) \quad \Phi_M: \Sigma^3(M) \rightarrow S(M).$$

Consider an S -complex

$$(2.7) \quad \cdots \rightarrow P \xrightarrow{u} Q \rightarrow R \rightarrow S(P) \xrightarrow{S(u)} S(Q) \rightarrow \cdots$$

This induces an isomorphism $\theta: \Sigma^3(\ker(u)) \rightarrow S(\ker(u))$. If $\theta = \Phi_{\ker(u)}$ then the S -complex will be called a Φ - S -complex. The full (with respect to $S\text{-comp}$) subcategory of Φ - S -complexes with the S -periodic morphisms is denoted Φ - $S\text{-comp}$.

2.2 Properties of the functors Z^0 and T

Lemma 12. Let M be an object of $\mathcal{P}\text{-mod}$. Then T_M is a Φ - S -complex.

Proof. By its very definition it is S -periodic. To see that $\theta = \Phi_M$ it suffices to see that $M = \ker(u)$, if we denote $u: X_0(M) \rightarrow X_1(M)$ in the notation of definition 11. □

Lemma 13. We have the relation

$$(2.8) \quad Z^0 \circ T \cong \text{id}_{\mathcal{P}\text{-mod}}.$$

Proof. Let M be an object of $\mathcal{P}\text{-mod}$. The morphism between the objects in degree 0 and 1 of T_M is the map

$$(2.9) \quad X_0(M) \longrightarrow X_1(M)$$

induced by (1.1). By the exactness of (1.1) we observe that applying Z^0 to T_M yields a functor equivalent to the identity. □

Lemma 14. The functor T sends projective-injective objects to projective-injective objects, and induces a triangle functor

$$(2.10) \quad \underline{T}: \underline{\mathcal{P}\text{-mod}} \rightarrow \underline{S\text{-comp}}.$$

Proof. □

The following is [Ami08, lemma 2.8.1].

Lemma 15. An S -complex which is homotopy equivalent to a Φ - S -complex is itself a Φ - S -complex.

Proof. Let

$$(2.11) \quad P \xrightarrow{u} Q \longrightarrow R \longrightarrow S(P)$$

be an S -complex which is homotopy equivalent to the Φ - S -complex

$$(2.12) \quad P' \xrightarrow{u'} Q' \longrightarrow R' \longrightarrow S(P').$$

There exists an S -periodic (we work in the non-full subcategory) homotopy f between these complexes, and it induces a morphism $g = Z^0(f): \ker(u) \rightarrow \ker(u')$. Because f becomes an isomorphism in the stable category of $S\text{-comp}$ we have that g is an isomorphism in the stable category of $\mathcal{P}\text{-mod}$.

Now consider the commutative diagram (2.13). Because g is an isomorphism in the stable category we see that $\Sigma^3(g)$ and $S(g)$ are isomorphisms too. This yields the following equality

$$(2.14) \quad \theta = ((S(g))^{-1} \circ \Phi_{\ker(u')} \circ \Sigma^3(g)) = \Phi_{\ker(u)}$$

as the three isomorphisms are functorial, hence the first complex is a Φ - S -complex as well. □

The following is [Ami08, lemma 2.8.2].

Lemma 16. Consider two Φ - S -complexes

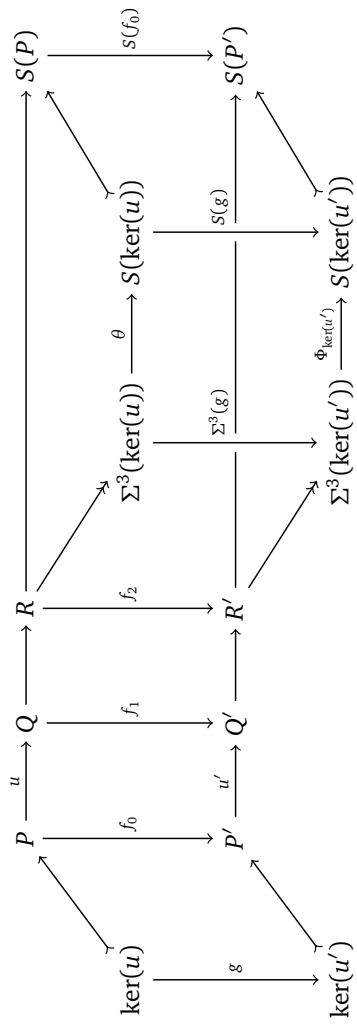
$$(2.15) \quad P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} S(P)$$

and

$$(2.16) \quad P' \xrightarrow{u'} Q' \xrightarrow{v'} R' \xrightarrow{w'} S(P')$$

such that we have a commutative square

$$(2.17) \quad \begin{array}{ccc} P & \xrightarrow{u} & Q \\ \downarrow f_0 & & \downarrow f_1 \\ P' & \xrightarrow{u'} & Q' \end{array}$$



∞ (2.13)

Then there exists a morphism $f_2: R \rightarrow R'$ such that we have an S -periodic morphism

$$(2.18) \quad \begin{array}{ccccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & S(P) \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow S(f_0) \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & S(P') \end{array}$$

of Φ - S -complexes.

Proof. The commutative square (2.17) induces a morphism $f: \ker(u) \rightarrow \ker(u')$. Because R and R' are projective-injective (recall that we have assumed that $\mathcal{P}\text{-mod}$ is Frobenius) we obtain a commutative square

$$(2.19) \quad \begin{array}{ccc} Q & \xrightarrow{v} & R \\ \downarrow f_1 & & \downarrow g_2 \\ Q' & \xrightarrow{v'} & R' \end{array}$$

The morphism g_2 induces a morphism $g: S(\ker(u)) \rightarrow S(\ker(u'))$, which in turn by functoriality induces a commutative square

$$(2.20) \quad \begin{array}{ccc} \Sigma^3(\ker(u)) & \xrightarrow{\Phi_{\ker(u)}} & S(\ker(u)) \\ \downarrow \Sigma^3(f) & & \downarrow g \\ \Sigma^3(\ker(u)) & \xrightarrow{\Phi_{\ker(u')}} & S(\ker(u')) \end{array}$$

in the stable category of $\mathcal{P}\text{-mod}$. Hence $S(f)$ and g are equal in the stable category, so we can find a projective-injective object I of $\mathcal{P}\text{-mod}$, together with morphisms $\alpha: S(\ker(u)) \rightarrow I$ and $\beta: I \rightarrow S(\ker(u'))$ such that

$$(2.21) \quad g - S(f) = \beta \circ \alpha.$$

Denote by p (resp. p') the epimorphism $R \rightarrow S(\ker(u))$ (resp. $R' \rightarrow S(\ker(u'))$). By the projectivity of I we can factor β through p' , i.e. we obtain the diagram

$$(2.22) \quad \begin{array}{ccccc} Q & \xrightarrow{v} & R & \xrightarrow{w} & S(P) \\ \downarrow f_1 & & \downarrow g_2 & & \downarrow S(f_0) \\ Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & S(P') \\ & & \downarrow p' & & \downarrow p' \\ & & S(\ker(u')) & & \end{array}$$

$\begin{array}{c} \begin{array}{ccc} & \xrightarrow{p} & S(\ker(u)) \\ & \searrow & \downarrow \alpha \\ & & I \\ & \swarrow & \downarrow \beta \\ & & S(\ker(u')) \end{array} \\ \begin{array}{ccc} & \xrightarrow{g} & S(\ker(u)) \\ & \searrow & \downarrow \alpha \\ & & I \\ & \swarrow & \downarrow \beta \\ & & S(\ker(u')) \end{array} \\ \begin{array}{ccc} & \xrightarrow{S(f)} & S(\ker(u)) \\ & \searrow & \downarrow \alpha \\ & & I \\ & \swarrow & \downarrow \beta \\ & & S(\ker(u')) \end{array} \end{array}$

We take

$$(2.23) \quad f_2 := g_2 - \gamma \circ \alpha \circ p.$$

This yields the equalities

$$(2.24) \quad \begin{aligned} f_2 \circ v &= g_2 \circ v - \gamma \circ \alpha \circ p \circ v && \text{definition of } f_2 \\ &= g_2 \circ v && p \circ v = 0 \\ &= v' \circ f_1 \end{aligned}$$

and

$$(2.25) \quad \begin{aligned} w' \circ f_2 &= w' \circ g_2 - w' \circ \gamma \circ \alpha \circ p && \text{definition of } f_2 \\ &= i' \circ p' \circ g_2 - i' \circ p' \circ \gamma \circ \alpha \circ p && w' = i' \circ p' \\ &= i' \circ p' \circ g_2 - i' \circ \beta \circ \alpha \circ p && \beta = p' \circ \gamma \\ &= i' \circ p' \circ g_2 - i' \circ g \circ p + i' \circ S(f) \circ p && \beta \circ \alpha = g - S(f) \\ &= i' \circ S(f) \circ p && p' \circ g_2 = g \circ p \\ &= S(f_0) \circ w \end{aligned}$$

hence (f_0, f_1, f_2) extends to a morphism in the non-full subcategory $S\text{-comp}$. \square

The following is [Ami08, proposition 2.8.1].

Proposition 17. The functor \underline{Z}^0 is full and essentially surjective. Its kernel is a square-zero ideal.

Proof. Because $\underline{Z}^0 \circ \underline{T} = \text{id}_{\mathcal{P}\text{-mod}}$ by lemma 13 we have that \underline{Z}^0 is *essentially surjective*. To see that it is *full*, consider two Φ - S -complexes

$$(2.26) \quad P \xrightarrow{u} Q \longrightarrow R \longrightarrow S(P)$$

and

$$(2.27) \quad P' \xrightarrow{u'} Q' \longrightarrow R' \longrightarrow S(P').$$

Let $f : \ker(u) \rightarrow \ker(u')$ be a morphism in $\mathcal{P}\text{-mod}$. It fits in a diagram

$$(2.28) \quad \begin{array}{ccccc} \ker(u) & \hookrightarrow & P & \xrightarrow{u} & Q \\ \downarrow f & & \downarrow f_0 & & \downarrow f_1 \\ \ker(u') & \hookrightarrow & P' & \xrightarrow{u'} & Q' \end{array}$$

where f_0 and f_1 are obtained by lifting f through the projective-injective objects P , Q , P' and Q' . By applying lemma 16 we obtain a morphism $f_2 : R \rightarrow R'$ such that (f_0, f_1, f_2) is an S -periodic morphism of Φ - S -complexes, hence \underline{Z}^0 is surjective on the level of the morphism spaces.

Consider a morphism \underline{f} in the kernel of \underline{Z}^0 . Then we can assume that (up to homotopy) the morphism \underline{f} can be written as

$$(2.29) \quad \begin{array}{ccccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & S(P) \\ \downarrow 0 & & \downarrow 0 & & \downarrow f_2 & & \downarrow 0 \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & S(P'). \end{array}$$

Because $w' \circ f_2 = 0 \circ w = 0$ and Q' is projective(-injective) we can factor f_2 through v' by a morphism h_2 . Similarly, because $w \circ v = 0$ we can factor f_2 through w by a morphism h_3 .

Now consider composable morphisms \underline{f} and \underline{f}' in the kernel of \underline{Z}^0 . This yields the diagram

$$(2.30) \quad \begin{array}{ccccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & S(P) \\ \downarrow 0 & & \downarrow 0 & \swarrow h_2 & \downarrow f_2 & & \downarrow 0 \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & S(P') \\ \downarrow 0 & & \downarrow 0 & & \downarrow f'_2 & \swarrow h'_3 & \downarrow 0 \\ P'' & \xrightarrow{u''} & Q'' & \xrightarrow{v''} & R'' & \xrightarrow{w''} & S(P''). \end{array}$$

□

The following is [Ami08, lemma 2.8.3].

Lemma 18. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a full functor between additive categories. If $\ker(F)$ is a square-zero ideal then F reflects isomorphisms.

Proof. Let $f \in \text{Hom}_{\mathcal{C}}(A, B)$ be a morphism such that $F(f)$ is an isomorphism in \mathcal{D} . Because F is assumed to be full we can find a morphism $g \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $F(g) = F(f)^{-1}$. Denote

$$(2.31) \quad h := f \circ g - \text{id}_B.$$

We see that $F(h) = 0$, hence $h \in \ker(F)$ and therefore $h \circ h = 0$ by assumption. We then see that

$$(2.32) \quad f \circ g \circ (\text{id}_B - h) = (\text{id}_B + h) \circ (\text{id}_B - h) = \text{id}_B$$

hence $g \circ (\text{id}_B - h)$ is a right inverse for f .

Similarly, by taking $h' := g \circ f - \text{id}_A$ and $(h' - \text{id}_A) \circ g$ we obtain a left inverse for f , hence it is an isomorphism. □

The following is [Ami08, proposition 2.8.2].

Proposition 19. The category Φ - S -comp is equivalent to the subcategory of S -comp of complexes homotopy equivalent to standard triangles.

Proof. The statement can be rephrased as “the functor T is essentially surjective on the level of stable categories”, and will be proved as such.

By lemma 12 we have that standard triangles are Φ - S -complexes. Hence by lemma 15 we have that an S -complex that is homotopy equivalent to a standard triangle is also a Φ - S -complex.

So in order to prove the essential surjectivity of \underline{T} , let

$$(2.33) \quad X^\bullet: P \xrightarrow{u} Q \longrightarrow R \longrightarrow S(P)$$

be a Φ - S -complex. We obtain morphisms $f_0: P \rightarrow X_0(\ker(u))$ and $f_1: Q \rightarrow X_1(\ker(u))$ such that

$$(2.34) \quad \begin{array}{ccccc} \ker(u) & \hookrightarrow & P & \xrightarrow{u} & Q \\ \parallel & & \downarrow f_0 & & \downarrow f_1 \\ \ker(u) & \hookrightarrow & X_0(\ker(u)) & \longrightarrow & X_1(\ker(u)) \end{array}$$

commutes. By lemma 16 we can complete the f_0 and f_1 into an S -periodic morphism $F = (f_0, f_1, f_2)$ from the Φ - S -complex X^\bullet to $T_{\ker(u)}$. Because $Z^0(F) = \text{id}_{\ker(u)}$ we see that $\underline{Z}^0(T_{\ker(u)}) = Z^0(X^\bullet)$ in $\underline{\mathcal{P}\text{-mod}}$.

By lemma 18 and proposition 17 we see that $T_{\ker(u)}$ and X^\bullet are homotopy equivalent, hence we can conclude that \underline{T} is essentially surjective. \square

We have the following situation

$$(2.35) \quad \begin{array}{ccccc} \Phi\text{-}S\text{-comp} & \hookrightarrow & S\text{-comp} & \longrightarrow & \text{Ch}_{\text{ac,proj}}(\mathcal{P}\text{-mod}) \\ & \swarrow T & \updownarrow T, Z^0 & \searrow Z^0 & \\ & & \mathcal{P}\text{-mod} & & \end{array}$$

where

1. the categories $S\text{-comp}$, $\text{Ch}_{\text{ac,proj}}(\mathcal{P}\text{-mod})$ and $\mathcal{P}\text{-mod}$ are Frobenius categories;
2. the top row are inclusions of categories (the first being full, the second not);
3. the functors Z^0 are exact;
4. the functor $T: \mathcal{P}\text{-mod} \rightarrow S\text{-comp}$ is exact.

2.3 Proof of the main theorem

To set the stage we repeat the definition of a triangulated category, with the set of axioms that we will use here. Remark that TR4' is different from the usual TR4 (which is known as the *octahedral axiom*), its equivalence is proved in [Nee01, proposition 1.4.6].

Definition 20. Let \mathcal{C} be an additive category. Let Σ be an auto-equivalence of \mathcal{C} . Let \mathcal{S} be the set of sextuples (X, Y, Z, u, v, w) where $X, Y, Z \in \text{Obj}(\mathcal{C})$ and $u: X \rightarrow Y$, $v: Y \rightarrow Z$ and $w: Z \rightarrow \Sigma(X)$. A morphism of sextuples is a commutative diagram

$$(2.36) \quad \begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma(X) \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow \Sigma(f_0) \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma(X') \end{array}$$

The category \mathcal{C} together with the auto-equivalence Σ is *triangulated* if there exists a subset $\Delta \subseteq \mathcal{S}$ of *distinguished triangles*, such that

TR0 Δ is closed under isomorphisms and

$$(2.37) \quad X \xrightarrow{\text{id}_X} X \longrightarrow 0 \longrightarrow \Sigma(X)$$

is a distinguished triangle;

TR1 for every morphism $u: X \rightarrow Y$ in \mathcal{C} there exists a distinguished triangle

$$(2.38) \quad X \xrightarrow{u} Y \longrightarrow Z \longrightarrow \Sigma(X);$$

TR2 if

$$(2.39) \quad X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$$

is a distinguished triangle, so is

$$(2.40) \quad Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma(u)} \Sigma(Y);$$

TR3 for every commutative diagram

$$(2.41) \quad \begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma(X) \\ \downarrow f_0 & & \downarrow f_1 & & & & \downarrow \Sigma(f_0) \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma(X') \end{array}$$

whose rows are distinguished triangles there exists a (not necessarily unique) morphism $f_2: Z \rightarrow Z'$ such that (f_0, f_1, f_2) is a morphism of triangles;

TR4' for every commutative diagram

$$(2.42) \quad \begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma(X) \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow \Sigma(f_0) \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma(X') \end{array}$$

whose rows are distinguished triangles the morphism f_2 obtained using axiom TR3 is such that the cone

$$(2.43) \quad Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ f_1 & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ f_2 & v' \end{pmatrix}} \Sigma(X) \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma(u) & 0 \\ \Sigma(f_0) & w' \end{pmatrix}} \Sigma(Y) \oplus \Sigma(X')$$

is a distinguished triangle.

Proof of theorem 2. We will show that the set of Φ - S -complexes provides the distinguished triangles for the triangulated structure on \mathcal{P} .

TR0 Let M be an object of \mathcal{P} . The S -complex

$$(2.44) \quad M \xrightarrow{\text{id}_M} M \longrightarrow 0 \longrightarrow S(M)$$

is homotopy equivalent to the zero complex because it is split exact.

As the kernel of the first morphism in the zero complex is zero the morphisms are equal (there is only one morphism), hence the zero complex is a Φ - S -complex. By applying lemma 15 we see that (2.44) is a Φ - S -complex.

That the class of Φ - S -complexes is closed under isomorphisms is immediate from the definition.

TR1 Let $u: P \rightarrow Q$ be a morphism in \mathcal{P} . By using projectiveness-injectiveness and the sequence of (1.1) we find the commutative diagram (2.45). Consider the following pushout diagram defining R

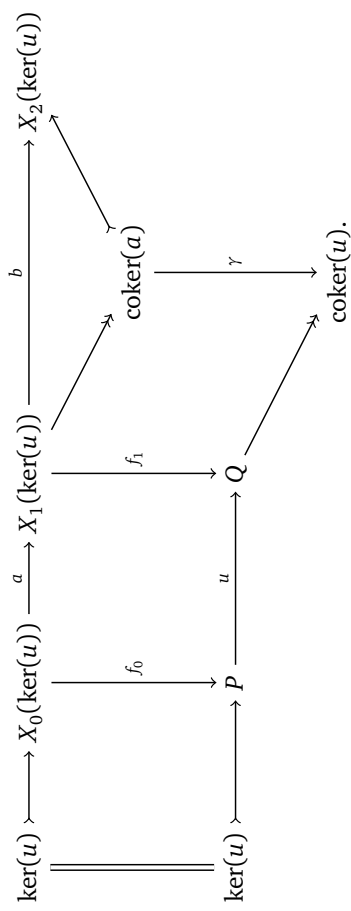
$$(2.46) \quad \begin{array}{ccc} \text{coker}(a) & \xrightarrow{\quad} & X_2(\ker(u)) \\ \downarrow \gamma & & \downarrow \\ \text{coker}(u) & \xrightarrow{\quad} & R. \end{array}$$

It fits in a commutative diagram

$$(2.47) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{coker}(a) & \longrightarrow & X_2(\ker(u)) & \longrightarrow & S(\ker(u)) \longrightarrow 0 \\ & & \downarrow \gamma & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{coker}(u) & \longrightarrow & R & \longrightarrow & S(\ker(u)) \longrightarrow 0 \end{array}$$

because in an exact category the monomorphisms are stable under pushout. By going to the stable category $\underline{\mathcal{P}\text{-mod}}$ we obtain a triangle morphism

$$(2.48) \quad \begin{array}{ccccccc} \text{coker}(a) & \longrightarrow & X_2(\ker(u)) & \longrightarrow & S(\ker(u)) & \longrightarrow & \Sigma(\text{coker}(a)) \\ \downarrow \gamma & & \downarrow & & \parallel & & \downarrow \Sigma(\gamma) \\ \text{coker}(u) & \longrightarrow & R & \longrightarrow & S(\ker(u)) & \longrightarrow & \Sigma(\text{coker}(u)). \end{array}$$



(2.45)

First we see that γ is an isomorphism in the stable category, because the projective-injectives vanish (or rather, the morphisms associated to them), so they are both canonically isomorphic to $\Sigma^2(\ker(u))$. Then an application of the five lemma yields that $X_2(\ker(u)) \rightarrow R$ is an isomorphism in the stable category. Hence we can conclude that R (being isomorphic to $X_2(\ker(u))$) is projective-injective, so the complex

$$(2.49) \quad P \xrightarrow{u} Q \longrightarrow R \longrightarrow S(P)$$

is an S -complex.

To see that it is also a Φ - S -complex, denote $\theta: S(\ker(u)) \rightarrow \Sigma^3(\ker(u))$ the morphism induced by this complex. In the previous paragraph we have obtained a (canonical) isomorphism between $\Sigma^2(\ker(u))$ and $\text{coker}(a)$ (resp. $\text{coker}(u)$), and this will be denoted α (resp. β). This yields the commutative diagram (2.50). This yields the equality

$$(2.51) \quad \begin{aligned} \theta &= (\Sigma(\beta))^{-1} \circ \Sigma(\gamma) \circ \Sigma(\alpha) \circ \Phi_{\ker(u)} \\ &= \Phi_{\ker(u)} \end{aligned}$$

in $\mathcal{P}\text{-mod}$ because everything is canonical, hence we have a Φ - S -complex.

TR2 Let

$$(2.52) \quad P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} S(P)$$

be a Φ - S -complex. By proposition 19 is homotopy equivalent to the standard triangle $T_{\ker(u)}$. Hence we get a homotopy between the S -complex

$$(2.53) \quad Q \xrightarrow{-v} R \xrightarrow{-w} S(P) \xrightarrow{-S(u)} S(Q)$$

and $T_{\ker(u)}[1]$.

Because the functor \underline{T} is a triangle functor we observe that the objects $T_{\Sigma(\ker(u))}$ and $T_{\ker(u)}[1]$ are isomorphic in the stable category of $S\text{-comp}$, hence homotopy equivalent in $S\text{-comp}$. By applying lemma 15 we conclude first that $T_{\ker(u)}[1]$ is a Φ - S -complex, and another application yields that (2.52) is a Φ - S -complex.

TR3 Immediately using lemma 16.

TR4' Consider two Φ - S -complexes

$$(2.54) \quad \begin{array}{ccccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R & \xrightarrow{w} & S(P) \\ \downarrow f_0 & & \downarrow f_1 & & & & \downarrow S(f_0) \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' & \xrightarrow{w'} & S(P') \end{array}$$

together with the morphisms f_0 and f_1 between objects of \mathcal{P} .

Let $g: \ker(u) \rightarrow \ker(u')$ be the morphism induced by f_0 . By considering the morphism $\underline{T}(g): T_{\ker(u)} \rightarrow T_{\ker(u')}$ we obtain a morphism $G = (g_0, g_1, g_2)$ between the Φ - S -complexes from (2.54).

$$\begin{array}{ccccccc}
\text{coker}(a) & \longrightarrow & X_2(\ker(u)) & \longrightarrow & S(\ker(u)) & \longrightarrow & \Sigma(\text{coker}(a)) \\
\uparrow \alpha & & \downarrow \gamma & & \parallel & & \uparrow \Sigma(\alpha) \\
\Sigma^2(\ker(u)) & & & & & & \Sigma^3(\ker(u)) \\
\downarrow \beta & & & & & & \downarrow \Sigma(\gamma) \\
\text{coker}(u) & \longrightarrow & R & \longrightarrow & S(\ker(u)) & \longrightarrow & \Sigma(\text{coker}(u)) \\
& & & & \parallel & & \uparrow \Sigma(\beta) \\
& & & & & & \Sigma^3(\ker(u)).
\end{array}$$

(2.50)

We first show the existence of a morphism $f_2: R \rightarrow R'$ such that $F = (f_0, f_1, f_2)$ is an S -complex morphism S -homotopic to G . By construction we have that (g_0, g_1) induce the same morphism $g: \ker(u) \rightarrow \ker(u')$. This yields morphisms h_1 and h_2 in the diagram

$$(2.55) \quad \begin{array}{ccccc} P & \xrightarrow{u} & Q & \xrightarrow{v} & R \\ \downarrow f_0 & \searrow h_1 & \downarrow f_1 & \searrow h_2 & \\ P' & \xrightarrow{u'} & Q' & \xrightarrow{v'} & R' \end{array}$$

such that

$$(2.56) \quad \begin{aligned} f_0 - g_0 &= h_1 \circ u \\ f_1 - g_1 &= u' \circ h_1 + h_2 \circ v. \end{aligned}$$

By setting

$$(2.57) \quad f_2 := g_2 + v' \circ h_2$$

we observe that

$$(2.58) \quad \begin{aligned} f_2 \circ v &= g_2 \circ v + v' \circ h_2 \circ v && \text{definition of } f_2 \\ &= v' \circ (g_1 + h_2 \circ v) && \text{commutative diagram (2.55)} \\ &= v' \circ (f_1 - u' \circ h_1) && \text{property (2.56)} \\ &= v' \circ f_1 && v' \circ u' = 0 \text{ in the complex} \end{aligned}$$

and similarly

$$(2.59) \quad \begin{aligned} w' \circ f_2 &= w' (g_2 + v' \circ h_2) && \text{definition of } f_2 \\ &= w' \circ g_2 && w' \circ v' = 0 \text{ in the complex} \\ &= S(g_0) \circ w && \text{commutative diagram (2.55)} \\ &= (S(f_0 - h_1 \circ u)) \circ w && \text{property (2.56)} \\ &= (S(f_0) - S(h_1) \circ S(u)) \circ w && S \text{ is additive} \\ &= S(f_0) \circ w && S(u) \circ w = 0 \text{ in the complex.} \end{aligned}$$

We can conclude that F is an S -periodic morphism that is S -homotopic to G . Hence the cones $\text{cone}(F)$ and $\text{cone}(G)$ are isomorphic as S -complexes.

Because G is obtained from the composition of $T_g: T_{\ker(u)} \rightarrow T_{\ker(u')}$ and homotopy equivalences we see that $\text{cone}(G)$ and $\text{cone}(T_g)$ are homotopy equivalent.

Consider the triangle

$$(2.60) \quad \ker(u) \xrightarrow{g} \ker(u') \longrightarrow \text{cone}(g) \longrightarrow \Sigma(\ker(u))$$

in $\mathcal{P}\text{-mod}$, which induces a triangle

$$(2.61) \quad T_{\ker(u)} \xrightarrow{T_g} T_{\ker(u')} \longrightarrow T_{\text{cone}(g)} \longrightarrow T_{\Sigma(\ker(u))}$$

in $S\text{-comp}$ because \underline{T} is a triangle functor.

On the other hand we have that

$$(2.62) \quad T_{\ker(u)} \xrightarrow{T_g} T_{\ker(u')} \longrightarrow \text{cone}(T_g) \longrightarrow T_{\ker(u)}[1]$$

is a triangle in $S\text{-comp}$. Hence $\text{cone}(T_g)$ and $T_{\text{cone}(g)}$ are isomorphic in the stable category, i.e. homotopy equivalent in the original category. So $\text{cone}(F)$ is a Φ - S -complex by lemma 15 because it is isomorphic to a complex which is homotopy equivalent to a Φ - S -complex.

□

Chapter 3

Application: Deformed preprojective algebras of generalized Dynkin type

To apply theorem 2 we need to find a sequence like (1.1). The following result from [BES07] generalises the non-deformed case of [ES98a; ES98b].

Theorem 21. Denote Λ the deformed preprojective algebra $P^f(\Delta)$ of generalised Dynkin type. There exists an exact sequence of Λ - Λ -bimodules

$$(3.1) \quad 0 \longrightarrow {}_{\text{id}}\Lambda_{\Phi^{-1}} \longrightarrow P_2 \xrightarrow{R} P_1 \xrightarrow{\delta} P_0 \xrightarrow{u} \Lambda \longrightarrow 0$$

where the P_i are projective Λ - Λ -bimodules, Φ is an automorphism of Λ used to twist the bimodule structure. And for every idempotent e_i of Λ we have $\Phi(e_i) = e_{\nu(i)}$. The modules P_i are defined as follows

$$(3.2) \quad \begin{aligned} P_0 &:= \bigoplus_{i \in \Delta_0} \Lambda(e_i \otimes e_i)\Lambda \\ P_1 &:= \bigoplus_{\alpha \in \Delta_1} \Lambda(e_{s(\alpha)} \otimes e_{t(\alpha)})\Lambda \\ P_2 &:= P_0 \end{aligned}$$

while the morphisms are given by

$$(3.3) \quad \begin{aligned} R(e_i \otimes e_i) &= \sum_{s(\alpha)=i} e_{s(\alpha)} \otimes \bar{\alpha} + \alpha \otimes e_{t(\bar{\alpha})} \\ \delta(e_{s(\alpha)} \otimes e_{t(\alpha)}) &= \alpha \otimes e_{t(\alpha)} - e_{s(\alpha)} \otimes \alpha \\ u &= \text{multiplication.} \end{aligned}$$

This result is originally obtained in the context of computing Hochschild cohomology, and yields many interesting results (periodicity of Hochschild cohomology, explicit descriptions of both additive and multiplicative structure of Hochschild cohomology, ...). As these are mostly covered by other talks we will not delve deeper into these.

Example 22. Let $\Delta = A_2$. The quiver that we will use to define the preprojective algebra $\Lambda = P(A_2)$ is

$$(3.4) \quad 0 \begin{array}{c} \xrightarrow{a_0} \\ \xleftarrow{\bar{a}_0} \end{array} 1.$$

The preprojective algebra is the quotient of the path algebra on this quiver by the ideal $(a_0\bar{a}_0, \bar{a}_0a_0)$. Its Cartan matrix is

$$(3.5) \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

hence a vectorspace basis for this algebra is

$$(3.6) \quad \{e_0, e_1, a_0, \bar{a}_0\}.$$

The projective bimodules are

$$(3.7) \quad \begin{aligned} P_0 = P_2 &= \bigoplus_{i \in Q_0} \Lambda(e_i \otimes e_i)\Lambda \\ &= \Lambda(e_0 \otimes e_0)\Lambda \oplus \Lambda(e_1 \otimes e_1)\Lambda \end{aligned}$$

and

$$(3.8) \quad \begin{aligned} P_1 &= \bigoplus_{a \in Q_1} \Lambda(e_{s(a)} \otimes e_{t(a)})\Lambda \\ &= \Lambda(e_0 \otimes e_1)\Lambda \oplus \Lambda(e_1 \otimes e_0)\Lambda. \end{aligned}$$

As a reality check we can calculate the k -dimensions of these modules, and check whether things work out. We observe that

$$(3.9) \quad \dim_k \Lambda e_i = \dim_k e_i \Lambda = 2$$

hence the exact sequence (3.1) corresponds to the sequence

$$(3.10) \quad 0 \longrightarrow 4 \longrightarrow 8 \longrightarrow 8 \longrightarrow 8 \longrightarrow 4 \longrightarrow 0$$

whose alternating sum equals zero.

The structure of the triangulated category is

$$(3.11) \quad \begin{array}{c} \nearrow e_1\Lambda = S(e_0\Lambda) \\ e_0\Lambda = S(e_1\Lambda) \nwarrow \end{array}$$

Example 23. Let $\Delta = A_3$. The quiver that we will use to define the preprojective algebra $P(A_3)$ is

$$(3.12) \quad 0 \begin{array}{c} \xrightarrow{a_0} \\ \xleftarrow{\bar{a}_0} \end{array} 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{\bar{a}_1} \end{array} 2$$

The preprojective algebra is the quotient of the path algebra on this quiver by the ideal $(a_0\bar{a}_0, \bar{a}_1a_1, \bar{a}_0a_0 + a_1\bar{a}_1)$. Its Cartan matrix is

$$(3.13) \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

hence a vectorspace basis for this algebra is

$$(3.14) \quad \{e_0, e_1, e_2, a_0, \bar{a}_0, a_1, \bar{a}_1, a_0a_1, \bar{a}_1\bar{a}_0, \bar{a}_0a_0\}.$$

The projective bimodules are

$$(3.15) \quad \begin{aligned} P_0 = P_2 &= \bigoplus_{i \in Q_0} \Lambda(e_i \otimes e_i)\Lambda \\ &= \Lambda(e_0 \otimes e_0)\Lambda \oplus \Lambda(e_1 \otimes e_1)\Lambda \oplus \Lambda(e_2 \otimes e_2)\Lambda \end{aligned}$$

and

$$(3.16) \quad \begin{aligned} P_1 &= \bigoplus_{a \in Q_1} \Lambda(e_{s(a)} \otimes e_{t(a)})\Lambda \\ &= \Lambda(e_0 \otimes e_1)\Lambda \oplus \Lambda(e_1 \otimes e_0)\Lambda \oplus \Lambda(e_1 \otimes e_2)\Lambda \oplus \Lambda(e_2 \otimes e_1)\Lambda. \end{aligned}$$

As a reality check we can calculate the k -dimensions of these modules, and check whether things work out. We observe that

$$(3.17) \quad \dim_k \Lambda e_0 = \dim_k \Lambda e_2 = \dim_k e_0\Lambda = \dim_k e_2\Lambda = 3$$

and

$$(3.18) \quad \dim_k \Lambda e_1 = \dim_k e_1\Lambda = 4$$

hence the exact sequence (3.1) corresponds to the sequence

$$(3.19) \quad 0 \longrightarrow 10 \longrightarrow 34 \longrightarrow 48 \longrightarrow 34 \longrightarrow 10 \longrightarrow 0$$

whose alternating sum equals zero.

$$(3.20) \quad \begin{array}{ccc} & & e_2\Lambda = S(e_0\Lambda) \\ & \nearrow & \nwarrow \\ & e_1\Lambda = S(e_1\Lambda) & \\ \nearrow & & \nwarrow \\ e_0\Lambda = S(e_1\Lambda) & & \end{array}$$

The following example is just for fun, the calculations (= determining the Cartan matrix) are a little cumbersome to do explicitly on the back of an envelope.

Example 24. Let $\Delta = E_6$. The quiver that we will use to define the preprojective algebra $P(E_6)$ is

$$(3.21) \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & & \uparrow \bar{a}_0 & \downarrow a_0 & & \\ & & & 3 & & & \\ 1 & \xleftarrow{a_1} & 2 & \xleftarrow{a_2} & 3 & \xleftarrow{a_3} & 4 & \xleftarrow{a_4} & 5 \\ & \xrightarrow{\bar{a}_1} & & \xrightarrow{\bar{a}_2} & & \xrightarrow{\bar{a}_3} & & \xrightarrow{\bar{a}_4} & \end{array}$$

Its Cartan matrix is

$$(3.22) \quad \begin{pmatrix} 4 & 2 & 4 & 6 & 4 & 2 \\ 2 & 2 & 3 & 4 & 3 & 2 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 6 & 4 & 8 & 12 & 8 & 4 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 2 & 2 & 3 & 4 & 3 & 2 \end{pmatrix}$$

As a reality check we can calculate the k -dimensions of the modules P_i , and check whether things work out. The exact sequence (3.1) corresponds to the sequence

$$(3.23) \quad 0 \longrightarrow 156 \longrightarrow 4560 \longrightarrow 8808 \longrightarrow 4560 \longrightarrow 156 \longrightarrow 0$$

whose alternating sum equals zero.

Remark 25. I have written some SAGE code to determine these numbers. The code can be found at <https://gist.github.com/pbelmans/6380013>. Currently the types A_n , D_n , L_n and E_6 are implemented. This is completely useless, yet fun.

Bibliography

- [Ami08] Claire Amiot. “Sur les petites catégories triangulées”. PhD thesis. Université Paris Diderot – Paris 7, 2008.
- [BES07] Jerzy Białkowski, Karin Erdmann, and Andrzej Skowroński. “Deformed preprojective algebras of generalized Dynkin type”. In: *Transactions of the American Mathematical Society* 359.6 (June 2007), pages 2625–2650.
- [Buh10] Theo Bühler. “Exact categories”. In: *Expositiones Mathematicae* (2010), pages 1–69. arXiv: 0811.1480 [math.HO].
- [ES98a] Karin Erdmann and Nicole Snashall. “On Hochschild cohomology of preprojective algebras, I”. In: *Journal of Algebra* 205.2 (1998), pages 391–412.
- [ES98b] Karin Erdmann and Nicole Snashall. “On Hochschild cohomology of preprojective algebras, II”. In: *Journal of Algebra* 205.2 (1998), pages 413–434.
- [Hel68] Alex Heller. “Stable homotopy categories”. In: *Bulletin of the American Mathematical Society* (1968).
- [Nee01] Amnon Neeman. *Triangulated categories*. Annals of mathematics studies 148. Princeton University Press, 2001.