**Quivers of exceptional collections on smooth projective varieties**

Pieter Belmans*  
Theo Raedschelders†

---

**Introduction**

We study the structure of strong (but not full) exceptional collections on smooth projective varieties, by understanding which quivers (with relations) can (or cannot) arise describing the structure of the exceptional collection. An overview of known results is given, and some new observations are made.

**Standing assumptions**

$k$ an algebraically closed field of characteristic $0$, $X/k$ a smooth projective variety, $A$ a finite-dimensional $k$-algebra of finite global dimension with a presentation $A = kQ/I$ where $Q$ is a finite acyclic quiver.

**Structure of triangulated categories**

Let $C$ be a $k$-linear Hom-finite triangulated category (e.g. $D^b(\operatorname{coh}/X)$ or $D^b(\operatorname{mod}/A)$).

One uses semi-orthogonal decompositions and exceptional collections to describe the structure of $C$.

A strong exceptional collection is described by a quiver with relations:

- **Quiver**
  - Vertices
  - Arrows
  - Relations

- **Exceptional Collection**
  - Objects
  - Compositions law

**Example 1.** The triple $(O_X, O_Y(3), O_Y(2))$ is a strong (but not full) exceptional collection on $\mathbb{P}^3$, whose quiver is

$$
\begin{array}{ccc}
  x_0 & \rightarrow & x_1 \\
  y_0 & \rightarrow & y_1 \\
  x_0 & \rightarrow & y_0 \\
\end{array}
$$

with relations $x_j y_i = x_i y_j$ for all $i, j = 0, 1, 2, 3$.

**Smooth and proper noncommutative schemes**

**Definition 2.** A geometric noncommutative scheme is a $k$-linear dg category $\operatorname{Perf}(A)/E^*$, where $E^*$ is a cohomologically bounded dg $k$-algebra, such that there exists a smooth and projective scheme $X$ and an admissible subcategory $A$ of $\operatorname{Perf}(X) = D^b(\operatorname{coh}/X)$ such that $\operatorname{Perf}(A)/E^*$ is quasi-equivalent to the (unique) enhancement of $A$.

The category of geometric noncommutative schemes has many nice properties, see [1].

**Motivating question**

How small can $X$ be for particular classes of finite-dimensional algebras?

The class of algebras we are most interested in are path algebras, but the general results also hold for more general algebras $A = kQ/I$.

**Remark 3.** A path algebra is hereditary, but global dimension is not a derived invariant. The derived equivalence class of a path algebra contains iterated tilted algebras, all of which are of finite global dimension.

**Additive invariants**

In order to obtain obstructions to embeddings of $D^b(\operatorname{mod}/A)/E$ in $D^b(\operatorname{coh}/X)$ one uses additive invariants, roughly speaking these are functors that send semi-orthogonal decompositions to direct sums [2].

**Grothendieck group**

As $K_0(A)$ is a free $\mathbb{Z}$-module whose rank is the number of vertices, we get

$$
\# Q_0 \leq \operatorname{rk} K_0(X).
$$

Unfortunately the rank of $K_0(X)$ is hard to compute.

**Hochschild homology**

For $A = kQ/I$ one has

$$
\operatorname{HH}(A) = \left\{ \begin{array}{c}
  k^{\# Q_i} & i = 0 \\
  0 & i \neq 0
\end{array} \right\},
$$

i.e. determined completely by the number of vertices (this is not a coincidence).

Similarly for $X$ one has $\dim \operatorname{HH}(X) < +\infty$, and the following decomposition result:

**Theorem 4 (Hochschild–Kostant–Rosenberg).** For $i = 0, \ldots, 2 \dim X$ one has

$$
\operatorname{HH}(X) = \bigoplus_{p+i} \operatorname{HH}^p(X, \Omega^i_X).
$$

**Corollary 5.** For an admissible embedding $D^b(\operatorname{mod}/A) \to D^b(\operatorname{coh}/X)$ one has

$$
\# Q_i \leq 2 + \sum_{j=1}^{\dim X-1} h^{i,j}.
$$

Noncommutative motives

The use of additive invariants hints at noncommutative motives, but the noncommutative motive of $A$ only depends on $\# Q_i$, which therefore not something more: motives don’t see the gluing!

**Antisymmetric Euler form on the Grothendieck group**

Hence the additive invariants themselves don’t suffice to make strong conclusions, we have to incorporate extra data!

The Grothendieck group $K_0(C)$ comes equipped with an Euler form:

$$
\chi(E, F) := \sum_{i=0}^{\operatorname{dim} C} (-1)^i \dim \operatorname{Ext}^i(E, F).
$$

Its antisymmetrisation $\chi^-$ is given by $\chi(E, F) - \chi(F, E)$.

The following result is the main technique in studying exceptional collections.

**Theorem 6.** Let $S$ be a smooth projective surface. Then $\operatorname{rk} K_0^-(S) = 2$.

One easily computes the rank of $\chi^-_S$. We do this for the Dynkin quivers, and the generalised Kroncker quivers

$$
\begin{array}{c}
  K_4: \circ \rightarrow \circ \\
  K_3: \circ \rightarrow \circ \\
\end{array}
$$

We obtain

$$
\begin{array}{c}
  Q \to A_n \to D_n (n \geq 4) \to \cdots \to E_8, E_6, E_7, K_0
\end{array}
$$

Hence we can try to embed those quivers for which $\operatorname{rk} K_0^- = 2$.

**Indecomposability of derived categories**

The reason to focus on surfaces is because one has the following result due to Okawa.

**Theorem 7.** Let $C$ be a smooth projective curve of genus $\geq 1$. Then $D^b(\operatorname{coh}/C)$ does not admit a semi-orthogonal decomposition.

Moreover one can rule out Calabi–Yau varieties, or varieties whose canonical bundle is globally generated.

**Positive results**

The use of additive invariants rules out embeddings of large quivers in surfaces for which the invariants are known (and small). The use of the Euler form rules out embeddings of certain quivers into surfaces.

**Question** Can we explicitly construct embeddings of $kQ^n$ into the derived category of a surface, for some of the $Q$ such that $\operatorname{rk} K_0^- = 2$?

**Theorem 8.** The only Dynkin quivers that can be realised on a smooth projective surface are $A_1, A_2, A_3, D_4$.

Outline of proof.

- For $A_1$ just take any exceptional object, which can be done on Spec $k$ or $\mathbb{P}^1$.
- For $A_2$ take an exceptional line bundle $L$ on a surface $X$ (e.g. $O_{\mathbb{P}^2}[1]$) and consider $\pi_1: B_1 X \to X$. The quiver is then realised by the collection $(\pi_1^* L, O_X)$.
- For $A_3$, first observe that the derived category is independent of the orientation. Then take an exceptional line bundle $L$ on a surface $X$ and consider $\pi_2: B_1 X \to X$, the exceptional collection is given by $(\pi_2^* L)$ and the structure sheaves of the exceptional divisors.
- For $D_4$ the same procedure, with $\pi_3: B_1 X \to X$.

All other Dynkin quivers have $\operatorname{rk} K_0^- \geq 4$.

**Remark 9.** This shows we should consider these Dynkin 4 quivers as the start of an infinite family of star quivers

$$
\begin{array}{cc}
  A_1: \circ \\
  A_2: \circ \rightarrow \circ \\
  A_3: \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \\
  D_4: \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ \\
\end{array}
$$

that allow an embedding into a surface.

**Theorem 10.** All quivers $K_n$ can be realised on a smooth projective surface.

**Open questions**

**Question** The rank criterion applied to a quiver with 3 vertices cannot rule out an embedding into a surface. We don’t have a general (optimal) construction yet.

**Question** Are there higher-dimensional analogues of the obstructions?

**Bibliography**