Quivers of exceptional collections on smooth projective varieties

Pieter Belmans* Theo Raedschelders†

*University of Antwerp, pieter.belmans@uantwerpen.be, Ph.D. fellowship of the Research Foundation—Flanders

†University of Brussels, theo.raedschelders@vub.ac.be, Ph.D. fellowship of the Research Foundation—Flanders

Introduction

We study the structure of strong (but not full) exceptional collections on smooth projective varieties, by understanding which quivers (with relations) can (or cannot) arise describing the structure of the exceptional collection. An overview of known results is given, and some new observations are made.

Standing assumptions k an algebraically closed field of characteristic 0, X/k a smooth projective variety, A a finite-dimensional k-algebra of finite global dimension with a presentation $A = kQ/I$ where Q is a finite acyclic quiver

Structure of triangulated categories

Let $\mathcal{C}$ be a k-linear Hom-finite triangulated category (e.g. $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$ or $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A)$). One uses semi-orthogonal decompositions and exceptional collections to describe the structure of $\mathcal{C}$.

A strong exceptional collection is described by a quiver with relations:

quiver	exceptional collection
vertices	objects
arrows	bases of the Hom-spaces
relations	composition law

Example 1. The triple $(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(2))$ is a strong (but not full) exceptional collection on $\mathbb{P}^3$, whose quiver is

(1)
$$\begin{tikzcd} \circ \arrow[r, bend left=40, "x_0"] \arrow[r, shift left=1, "x_1"] \arrow[r, shift right=1, "x_2"'] \arrow[r, bend right=40, "x_3"'] & \circ \arrow[r, bend left=40, "y_0"] \arrow[r, shift left=1, "y_1"] \arrow[r, shift right=1, "y_2"'] \arrow[r, bend right=40, "y_3"'] & \circ \end{tikzcd}$$

with relations $x_i y_j = x_j y_i$ for all $i, j = 0, 1, 2, 3$.

Smooth and proper noncommutative schemes

Definition 2. A *geometric noncommutative scheme* is a k-linear dg category $\mathrm{Perf}_{\mathrm{dg}}/E^\bullet$, where $E^\bullet$ is a cohomologically bounded dg k-algebra, such that there exists a smooth and projective scheme X and an admissible subcategory $\mathcal{A}$ of $\mathrm{Perf}/X = \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$ such that $\mathrm{Perf}_{\mathrm{dg}}/E^\bullet$ is quasi-equivalent to the (unique) enhancement of $\mathcal{A}$.

The category of geometric noncommutative schemes has many nice properties, see [1]. It contains all finite-dimensional algebras of finite global dimension, by constructing some *huge* X such that we can embed $\mathrm{Perf}/A = \mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A)$ into $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$.

Motivating question *How small can X be for particular classes of finite-dimensional algebras?*

The class of algebras we are most interested in are path algebras, but the general results also hold for more general algebras $A = kQ/I$.

Remark 3. A path algebra is hereditary, but global dimension *is not a derived invariant*. The derived equivalence class of a path algebra contains *iterated tilted algebras*, all of which are of finite global dimension.

Additive invariants

In order to obtain *obstructions* to embeddings of $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A)$ in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$ one uses *additive invariants*, roughly speaking these are functors that send semi-orthogonal decompositions to direct sums [2].

Grothendieck group As $\mathrm{K}_0(A)$ is a free $\mathbb{Z}$-module whose rank is the number of vertices, we get

(2)
$$\# Q_0 \leq \operatorname{rk} \mathrm{K}_0(X).$$

Unfortunately the rank of $\mathrm{K}_0(X)$ is hard to compute...

Hochschild homology For $A = kQ/I$ one has

(3)
$$\mathrm{HH}_i(A) = \begin{cases} k^{\# Q_0} & i = 0 \\ 0 & i \neq 0, \end{cases}$$

i.e. determined completely by the number of vertices (this is not a coincidence).

Similarly for X one has $\dim \mathrm{HH}_\bullet(X) < +\infty$, and the following decomposition result:

Theorem 4 (Hochschild–Kostant–Rosenberg). For $i = 0, \ldots, 2\dim X$ one has

(4)
$$\mathrm{HH}_i(X) = \bigoplus_{q-p=i} \mathrm{H}^q(X, \Omega_X^p).$$

Corollary 5. For an admissible embedding $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}/A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$ one has

(5)
$$\# Q_0 \leq 2 + \sum_{i=1}^{\dim X - 1} \mathrm{h}^{i,i}.$$

Noncommutative motives The use of additive invariants hints at noncommutative motives, but the noncommutative motive of A only depends on $\# Q_0$ (which explains the previous results) hence this cannot tell us anything more: *motives don't see the gluing*!

Antisymmetric Euler form on the Grothendieck group Hence the additive invariants themselves don't suffice to make strong conclusions, we have to incorporate extra data!

The Grothendieck group $\mathrm{K}_0(\mathcal{C})$ comes equipped with an *Euler form*:

(6)
$$\chi(E, F) \coloneqq \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathrm{Ext}^i(E, F).$$

Its *antisymmetrisation* χ^- is given by $\chi(E, F) - \chi(F, E)$.

The following result is the main technique in studying exceptional collections.

Theorem 6. Let S be a smooth projective surface. Then $\operatorname{rk} \chi_S^- = 2$.

One easily computes the rank of χ_Q^-. We do this for the Dynkin quivers, and the generalised Kronecker quivers

(7)
$$\mathrm{K}_n\colon \begin{tikzcd} \circ \arrow[r, bend left=40, "x_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "x_n"'] & \circ \end{tikzcd}$$

We obtain

Q	A_n	D_n $(n \geq 4)$	E_6, E_7	E_8	K_n
$\operatorname{rk} \chi_Q^-$	$2\lfloor n/2 \rfloor$	$2(\lceil n/2 \rceil - 1)$	6	8	2

Hence we can try to embed those quivers for which $\operatorname{rk} \chi_Q^- = 2$.

Indecomposability of derived categories

The reason to focus on *surfaces* is because one has the following result due to Okawa.

Theorem 7. Let C be a smooth projective curve of genus ≥ 1. Then $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/C)$ does not admit a semi-orthogonal decomposition.

Moreover one can rule out Calabi–Yau varieties, or varieties whose canonical bundle is globally generated.

Positive results

The use of additive invariants rules out embeddings of large quivers in surfaces for which the invariants are known (and small). The use of the Euler form rules out embeddings of certain quivers into surfaces.

Question *Can we explicitly construct embeddings of kQ into the derived category of a surface, for some of the Q such that $\operatorname{rk} \chi_Q^- = 2$?*

Theorem 8. The only Dynkin quivers that can be realised on a smooth projective surface are $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{D}_4$.

Outline of proof.

- For A_1 just take any exceptional object, which can be done on $\operatorname{Spec} k$ or $\mathbb{P}^1$.
- For A_2 take an exceptional line bundle $\mathcal{L}$ on a surface X (e.g. $\mathcal{O}_{\mathbb{P}^2}$ on $\mathbb{P}^2$) and consider $\pi_1\colon \mathrm{Bl}_1 X \to X$. The quiver is then realised by the collection $(\pi_1^*(\mathcal{L}), \mathcal{O}_E)$.
- For A_3, first observe that the derived category is independent of the orientation. Then take an exceptional line bundle $\mathcal{L}$ on a surface X and consider $\pi_2\colon \mathrm{Bl}_2 X \to X$, the exceptional collection is given by $\pi_2^*(\mathcal{L})$ and the structure sheaves of the exceptional divisors.
- For D_4 the same procedure, with $\pi_3\colon \mathrm{Bl}_3 X \to X$.

All other Dynkin quivers have $\operatorname{rk} \chi_Q^- \geq 4$.

Remark 9. This shows that we should consider these Dynkin 4 quivers as the start of an infinite family of *star quivers*

(8)
$$\mathrm{A}_1\colon \circ, \quad \mathrm{A}_2\colon \circ \to \circ, \quad \mathrm{A}_3\colon \circ \to \circ \to \circ, \quad \mathrm{D}_4\colon \begin{tikzcd}[row sep=small] & \circ \\ \circ & \circ \arrow[l] \arrow[u] \arrow[d] \\ & \circ \end{tikzcd}, \quad \begin{tikzcd}[row sep=small] & \circ & \\ \circ & \circ \arrow[l] \arrow[u] \arrow[d] \arrow[r] & \circ \\ & \circ & \end{tikzcd}, \ldots$$

that allow an embedding into a surface.

Theorem 10. All quivers K_n can be realised on a smooth projective surface.

Ingredients for proof.

Use the Hirzebruch surface $\mathbf{F}_n$ to embed K_n.

Alternatively, use $\mathbb{P}^1 \times \mathbb{P}^1$ for even n and $\mathbf{F}_1$ for odd n.

Open questions

Question *The rank criterion applied to a quiver with 3 vertices cannot rule out an embedding into a surface. We don't have a general (optimal) construction yet.*

Question *Are there higher-dimensional analogues of the obstructions?*

Bibliography

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