Positselski's main lemma: a criterion for shifted vector bundles in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$

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Abstract

We redo the proof of the main lemma in [2]. It says that, if for an object $\mathcal{E}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$ we have $\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)^{\leq 0}$, then $\mathcal{E}^\bullet$ is the shift of a vector bundle.

The proof is completely the same, just spelled out with a little more details.

Standing assumptions Let k be a field. Let X be a smooth projective variety.

The following criterion is a way to check that a coherent sheaf is actually a vector bundle: if you dualise it (in the derived category) and it remains pure in degree 0 it must be a vector bundle.

Lemma 1. Let $\mathcal{E}$ be an object in coh/X, or equivalently a pure sheaf concentrated in degree 0 (i.e. as an object of $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$). If $\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}, \mathcal{O}_X)$ is a pure sheaf concentrated in degree 0, then $\mathcal{E}$ is a vector bundle.

Proof. Take a locally free resolution

(1) $$0 \to \mathcal{P}_k \to \ldots \to \mathcal{P}_0 \to 0$$

of $\mathcal{E}$. If $\mathcal{E}$ is not already a vector bundle, we see that the morphism

(2) $$\mathcal{H}\mathrm{om}(\mathcal{P}_{k-1}, \mathcal{O}_X) \to \mathcal{H}\mathrm{om}(\mathcal{P}_k, \mathcal{O}_X)$$

is a surjection, as $\mathcal{E}\mathrm{xt}^k(\mathcal{E}, \mathcal{O}_X) = 0$ for $k \geq 1$ by the assumption. To see this, remark that the cohomology of $\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}^\bullet, \mathcal{O}_X)$ is computed from the cohomology of the cochain complex

(3) $$0 \to \mathcal{H}\mathrm{om}(\mathcal{P}_0, \mathcal{O}_X) \to \ldots \to \mathcal{H}\mathrm{om}(\mathcal{P}_k, \mathcal{O}_X) \to 0.$$

Hence the condition on $\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}, \mathcal{O}_X)$ gives us that there shouldn't be cohomology in this particular degree.

Therefore *locally* the inclusion $\mathcal{P}_k \to \mathcal{P}_{k-1}$ is split (as we can construct the splitting from the surjection on the dual vector bundles locally), and the quotient $\mathcal{P}_{k-1}/\mathcal{P}_k$ is again a vector bundle. But then we can replace our locally free resolution by a shorter one, hence $\mathcal{E}$ must be a vector bundle. □

The following criterion is a way to check disjointness of supports of cohomology sheaves of two objects in the bounded derived category, based on where their derived tensor product lives in the t-structure.

Lemma 2. Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be objects in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$. If $\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{F}^\bullet$ is in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)^{\leq 0}$ then for all $i+j \geq 0$ we have

$$\operatorname{supp}\mathcal{H}^i(\mathcal{E}^\bullet) \cap \operatorname{supp}\mathcal{H}^j(\mathcal{F}^\bullet) = \emptyset. \tag{4}$$

Proof. Consider the Künneth spectral sequence

$$\mathrm{E}_2^{p,q} = \bigoplus_{i+j=q} \mathrm{Tor}_{-p}\left(\mathcal{H}^i(\mathcal{E}^\bullet), \mathcal{H}^j(\mathcal{F}^\bullet)\right) \Rightarrow \mathcal{H}^{p+q}(\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{F}^\bullet) = \mathrm{E}_\infty^{p,q}. \tag{5}$$

We apply a descending induction on $i+j$, as for $i+j \gg 0$ the statement is true. Assume that for some $i+j \geq 0$ the intersection of the supports of the cohomology sheaves is nonempty. Then $\mathcal{H}^i(\mathcal{E}^\bullet) \otimes \mathcal{H}^j(\mathcal{F}^\bullet) \neq 0$ (as a sheaf: just consider its stalks, then sheafify). This implies $\mathrm{E}_2^{0,q} \neq 0$.

To make sure that the term $\mathrm{E}_2^{0,q}$ does not contribute a nonzero cohomology sheaf to $\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{F}^\bullet$ (as by assumption $\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)^{\leq 0}$) [TODO: but what if we take $q = 0$?] this term must be killed by some $\mathrm{E}_2^{-r,q+r-1}$, for $r \geq 2$ (as we are already at this step in the convergence, and $\mathrm{E}_2^{0,q}$ gets killed if we find an isomorphism with another term at some point, as $\mathrm{E}_r^{0,q}$ sits at the edge of the nonzero terms in the spectral sequence). But such a term consists of summands for which $i'+j' = q+r-1 \geq q+1 > i+j$, hence the induction hypothesis applies, so the intersection of these supports is empty and we have no Tor: a contradiction. □

The following proposition is "Positselski's main lemma", which is a criterion to check whether an object in the bounded derived category is actually a vector bundle (up to a shift).

Proposition 3 (Main lemma)**.** Let $\mathcal{E}^\bullet$ be an object in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)$. If $\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}^\bullet, \mathcal{E}^\bullet)$ is in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)^{\leq 0}$ then $\mathcal{E}^\bullet$ is (up to a shift) a vector bundle.

Proof. Denote $\mathcal{F}^\bullet := \mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}^\bullet, \mathcal{O}_X)$. Then the tensor-Hom adjunction reads

$$\mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \cong \mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{F}^\bullet. \tag{6}$$

We can freely shift $\mathcal{E}^\bullet$ around, so assume $\mathcal{E}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)^{\leq 0}$ and $\mathcal{H}^0(\mathcal{E}^\bullet) \neq 0$. Then $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)^{\geq 0}$, and $\mathcal{H}^0(\mathcal{F}^\bullet) = \mathcal{H}\mathrm{om}(\mathcal{H}^0(\mathcal{E}^\bullet), \mathcal{O}_X)$.

With this notation we see that $\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)^{\leq 0}$, hence we can apply lemma 2 to $i = 0$ and $j \geq 1$ (whenever we have a collection of cohomology sheaves we consider the union of their supports) and obtain that

$$\operatorname{supp}\mathcal{H}^0(\mathcal{E}^\bullet) \cap \operatorname{supp}\mathcal{H}^{\geq 1}(\mathcal{F}^\bullet) = \emptyset. \tag{7}$$

We wish to show that $\operatorname{supp}\mathcal{H}^0(\mathcal{E}^\bullet) = X$, because this implies $\mathcal{H}^{\geq 1}(\mathcal{F}^\bullet) = 0$, hence $\mathcal{F}^\bullet$ is actually concentrated in degree 0.

We can assume that X is irreducible (i.e. connected, as X is a smooth variety), otherwise we work component per component. So assume that $\operatorname{supp}\mathcal{H}^0(\mathcal{E}^\bullet) \subsetneq X$, then $\mathcal{H}^0(\mathcal{F}^\bullet) = \mathcal{H}\mathrm{om}(\mathcal{H}^0(\mathcal{E}^\bullet), \mathcal{O}_X) = 0$. To see this it suffices to realise that, using the description of the stalk at the generic point (see [EGA III$_1$, proposition 12.3.5]) that it is a torsion sheaf (using [EGA I, proposition 7.4.6]), but that it is also torsion-free (using [1, corollary 1.2 and proposition 1.3]).

But then $\mathcal{F}^\bullet|_{X\setminus\operatorname{supp}\mathcal{H}^{\geq 1}(\mathcal{F}^\bullet)}$ is acyclic, whereas $\mathcal{E}^\bullet|_{X\setminus\operatorname{supp}\mathcal{H}^{\geq 1}(\mathcal{F}^\bullet)}$ is not acyclic. This is impossible, because $\mathbf{R}\mathcal{H}\mathrm{om}^\bullet$ is of a local nature, so we get $\operatorname{supp}\mathcal{H}^0(\mathcal{E}^\bullet) = X$, and $\mathcal{H}^{\geq 1}(\mathcal{F}^\bullet) = 0$, and $\mathcal{F} \in \mathrm{coh}/X$.

Dualising $\mathcal{F}$ yields $\mathcal{E}^\bullet \cong \mathbf{R}\mathcal{H}\mathrm{om}^\bullet(\mathcal{F}, \mathcal{O}_X) \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}/X)^{\geq 0}$, therefore the t-structure implies $\mathcal{E} \in \mathrm{coh}/X$ too, and we can apply lemma 1 to conclude that $\mathcal{E}$ is a vector bundle. □

References

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