Cheat sheet on recollements

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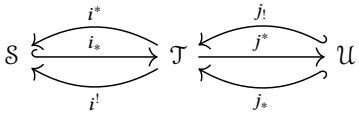
Summary

A recollement looks like this:

(1) 

If you can remember the shape of (1), you only need to remember:

1. each pair of adjacent arrows is an adjoint pair;
2. the composition from left to right satisfies $j^* \circ i_* = 0$;
3. functors to $\mathcal{T}$ are fully faithful;
4. the four remaining (co)unit transformations are used to decompose objects of $\mathcal{T}$.

With more details: each arrow is left adjoint to the arrow below, i.e. we are given 4 adjunctions:

$$\begin{array}{ll} i^* \dashv i_* & i_* \dashv i^! \\ j_! \dashv j^* & j^* \dashv j_*, \end{array} \tag{2}$$

such that

1. i_*, $j_!$ and j_* are fully faithful, which tells us that we have the following four isomorphisms for the appropriate (co)units:

$$\begin{aligned} \epsilon^{i^* \dashv i_*} &: i^* \circ i_* \Rightarrow \mathrm{id}_{\mathcal{S}} \\ \eta^{i_* \dashv i^!} &: \mathrm{id}_{\mathcal{S}} \Rightarrow i^! \circ i_* \\ \eta^{j_! \dashv j^*} &: \mathrm{id}_{\mathcal{U}} \Rightarrow j^* \circ j_! \\ \epsilon^{j^* \dashv j_*} &: j^* \circ j_* \Rightarrow \mathrm{id}_{\mathcal{T}} \end{aligned} \tag{3}$$

2. the (only) composition from left to right satisfies

   $$j^* \circ i_* = 0, \tag{4}$$

   which tells us by adjunction that

   $$\begin{aligned} i^* \circ j_* &= 0 \\ i^! \circ j_* &= 0. \end{aligned} \tag{5}$$

3. the four remaining (co)unit transformations are related to eachother by the existence of distinguished triangles decomposing every object in a piece coming from $\mathcal{S}$ and $\mathcal{T}$:

$$\begin{aligned} & i_* \circ i^!(T) \xrightarrow{\epsilon_T^{i_* \dashv i^!}} T \xrightarrow{\eta_T^{j^* \dashv j_*}} j_* \circ j^*(T) \longrightarrow \\ & j_! \circ j^*(T) \xrightarrow{\epsilon_T^{j_! \dashv j^*}} T \xrightarrow{\eta_T^{i^* \dashv i_*}} i_* \circ i^*(T) \longrightarrow . \end{aligned} \tag{6}$$