# Connections between commutative and noncommutative algebraic geometry

**Thesis committee:**
prof. dr. Alexey Bondal, Steklov Mathematical Institute
prof. dr. Colin Ingalls, University of New Brunswick
prof. dr. Boris Shoikhet, University of Antwerp
prof. dr. Alain Verschoren, University of Antwerp
prof. dr. Michael Wemyss, University of Glasgow

prof. dr. Wendy Lowen, University of Antwerp
prof. dr. Michel Van den Bergh, University of Hasselt

# Connections between commutative and noncommutative algebraic geometry

Verbanden tussen commutatieve en
niet-commutatieve algebraïsche meetkunde

thesis for the degree of doctor of science: mathematics
at the University of Antwerp

thesis for the degree of doctor of science: mathematics
at the University of Hasselt

to be defended by

Pieter Belmans

Antwerp, 2017

supervised by

prof. dr. Wendy Lowen
prof. dr. Michel Van den Bergh

*Dedicated to the memory of Martine Bosschaerts (1961–2010)*

# Contents

# List of tables

# List of figures

# List of publications

The chapters in this thesis correspond to the following publications and preprints.

**chapter 2** Pieter Belmans and Theo Raedschelders. *Embeddings of quivers in derived categories of surfaces*. Accepted for publication in Proceedings of the American Mathematical Society. 2015. arXiv: `1501.04197 [math.AG]`

**chapter 3** Pieter Belmans and Theo Raedschelders. *Noncommutative quadrics and Hilbert schemes of points*. 2016. arXiv: `1605.02795 [math.AG]`

**chapter 4** Pieter Belmans. *Hochschild cohomology of noncommutative planes and quadrics*. arXiv: `1705.06098 [math.AG]`

**chapter 5** Pieter Belmans and Dennis Presotto. *Construction of noncommutative surfaces with exceptional collections of length 4*. arXiv: `1705.06943 [math.AG]`

**chapter 6** Pieter Belmans, Dennis Presotto, and Michel Van den Bergh. "Comparing two constructions of noncommutative del Pezzo surfaces of rank 4"

**chapter 7** Pieter Belmans, Kevin De Laet, and Lieven Le Bruyn. "The point variety of quantum polynomial rings". In: *Journal of Algebra* 463 (2016), pp. 10–22. arXiv: `1509.07312 [math.RA]`

**chapter 8** Pieter Belmans and Sebastian Klein. *Relative tensor triangular Chow groups for coherent algebras*. Accepted for publication in Journal of Algebra. 2016. arXiv: `1607.03423 [math.AG]`

# Acknowledgements

First and foremost I want to thank Wendy and Michel, for being such knowledgeable advisers from whom I have learnt so much. Thank you for the questions when I needed questions, for the suggestions when I needed suggestions and for the nudges in the right direction whenever I was going astray. Thank you for letting me fool around with mathematics, even when at first it wasn't clear whether what I was doing made any sense at all. I am also grateful for the support I have received from the Research Foundation—Flanders (FWO), which allowed me to travel and visit many interesting conferences.

I want to thank the external members of the jury for accepting their role as a jury member, the interesting conversations we have had and the relevant suggestions they have made.

I want to thank my co-authors Kevin, Sebastian, Lieven, Dennis and Theo. I have learnt many things during our discussions, and I hope that if you decide to stay in academia we can work together at some point in the future on other exciting mathematics. A special thank you goes out to Lieven, whose writings on Mumford's picture of $\operatorname{Spec}\mathbb{Z}[x]$ inspired me to study algebraic geometry in the first place. And I would also like to thank all the people who gave comments on the preprints and papers which led to the various chapters of this thesis.

I am grateful to my colleagues at the university for the many mathematical and non-mathematical conversations, and in particular Julia who had to share an office with me for 4 years. Lo siento por mi mal español. Likewise I am grateful for all the interesting and fun times I have had at conferences, and the interesting conversations with other PhD students, postdocs and professors.

I am not sure whether reading this thesis will help in understanding what I have been doing for the past 4 (or 7) years, but I would like to thank my father, Sofie and Abderrahim for their support and for keeping me in touch with the real world. Thank you Astrid, Miguel and Sarah, for being such good friends. And thank you Lynn. For your love and patience.

Finally, Mom, I hope you would be proud of me.

# Introduction

Whenever people ask me what I do, the answer is usually some mumbling involving noncommutative algebraic geometry. But the term *noncommutative algebraic geometry* is considered to be a somewhat problematic name for a diverse field. Different people have different reasons to dislike it:

1. some results can be applied uniformly to commutative and noncommutative objects, suggesting the need for a rather clumsy name such as "not necessarily commutative algebraic geometry";

2. other results are of a purely algebraic nature for noncommutative rings, but the tools required to obtain (or even state them) require algebraic geometry, but does this warrant the presence of the word geometry?

3. or it could be that most of the geometry is abstracted away (maybe only at first sight), leaving very little for a usual algebraic geometer to recognise.

I do not wish to let this introduction act as an unsolicited noncommutative algebraic geometer's apology for his or her autonym. Rather I will explain which of the possible interpretations of this umbrella term exist and how they will appear in this thesis, and what is meant by those connections that appear in this thesis's title. If so, the reader may decide for him- or herself whether this should be read as a motivation for the blanket term noncommutative algebraic geometry.

## Noncommutative algebraic geometry?

The phrase "noncommutative algebraic geometry" at least suggests that there is some algebraic geometry happening, and that unlike usual algebraic geometry there is a level of noncommutativity involved. But this noncommutativity can enter at various stages and in different degrees. Victor Ginzburg identifies two scales of noncommutative algebraic geometry in his monograph [88].

### Two scales

Based on how usual algebraic geometry is treated in your theory of noncommutative algebraic geometry, Ginzburg suggests that there is:

1. noncommutative algebraic geometry *in the small*,
2. noncommutative algebraic geometry *in the large*.

By the former he means a *generalisation* of commutative algebraic geometry. In particular he thinks of deformations and quantisations. This is featured in this thesis as the description of the point schemes for skew polynomial rings and the description of tensor triangulated Chow groups. Both are machines into which it is also possible to feed commutative objects, and get back the original results.

On the other hand, by the latter he means a *replacement* of commutative algebraic geometry, developing analogous notions that do not necessarily give back the properties when restricted to commutative objects. The main example in this thesis is the replacement of varieties and schemes (and their isomorphisms) with their derived categories (and their equivalences). In this setting there is a notion of smoothness for dg categories, which agrees with the usual one for $\mathbf{D}^{\mathrm{perf}}(X)$, but $\mathbf{D}^{\mathrm{b}}(X)$ is smooth regardless of the smoothness of $X$.

Another point is that in this setting it is usually not possible to go back to geometry without losing information, but often one does not lose too much in the process. Kontsevich uses the word *non-geometry*[1] for a closely related interpretation [219], and we will get back to this interpretation later in the introduction.

But the distinction between small and large feels somewhat arbitrary in the context of this thesis. For instance the study of graded algebras as a form of noncommutative projective geometry is usually considered to be "in the small". Yet what happens if you consider their derived category and use it to construct fully faithful functors or compute its Hochschild cohomology?

Or in the context of tensor triangulated geometry we have Balmer's reconstruction theorem that tells us it is possible to completely reconstruct a scheme from its derived category together with the derived tensor product. So it turns out that we can completely recover things, but the other types of tensor triangulated categories that we can feed into this machine are certainly not deformations or quantisations in the sense of noncommutative algebraic geometry in the small.

The results in this thesis are probably a form of *in between*, studying examples and applying techniques from both philosophies. But it turns out that often in the questions studied in this thesis there is a derived category involved, which are the central objects in Kontsevich's non-geometry, so it is a good idea to at least explain how they appear and why they can be considered to be part of noncommutative algebraic geometry.

### Non-geometry

In topology one can associate the algebra of continuous complex-valued functions to

[1] The author of this thesis feels that using this terminology in the title of his PhD thesis would be dissatisfying.

a compact Hausdorff space. By the Gelfand–Naimark theorem it is possible to recover the space using the set of maximal ideals of this (commutative) algebra. In other words, there exists a fully faithful functor from the category of compact Hausdorff spaces to the category of Banach algebras.

In noncommutative geometry[2], sometimes called à la Connes, one takes the study of Banach algebras as the starting point.

Grothendieck defined the category of affine schemes as the opposite of the category of commutative rings, and uses these as local models for scheme theory. In some sense this takes the Gelfand–Naimark theorem as the input for the starting definition. But it is not possible to completely mimick the approach taken in topology to get a suitable noncommutative version of algebraic geometry, the Zariski topology being such a coarse topology:

1. there are only a few continuous functions, e.g. on the smooth and proper varieties which feature so prominently in this thesis we only have the constant functions;

2. replacing (global) functions by the structure sheaf we still have obstructions to gluing local data, which is encoded in the higher cohomology of the structure sheaf.

Hence there is no obvious geometric candidate for an algebraic object to encode a (non-affine) geometric object.

The correct algebraic object turns out to be a *category*. Even if there is a shortage of continuous functions, there is a wealth of vector bundles or (quasi)coherent sheaves and morphisms between them. Gabriel and Rosenberg have shown that the *abelian category* of quasicoherent sheaves determines the scheme up to isomorphism. So in this sense replacing schemes by their abelian categories of schemes seems like a viable approach. Moreover, $\mathcal{O}_X$ is an object in this category, and the continuous functions given by $\mathrm{H}^0(X, \mathcal{O}_X)$ are precisely the endomorphisms of this object.

Regarding the idea that abelian categories could be just as good to study algebraic geometry as the scheme, Manin remarks the following [159, page 83]:

> *In short, as Alexander Grothendieck taught us, to do geometry you really don't need a space, all you need is a category of sheaves on this would-be space.*

On the other hand, using abelian categories does not yet address the higher cohomology of the structure sheaf, and depending on your tastes this abelian category is not quite an algebraic object: it lacks a projective generator. Also they are not flexible enough to develop a noncommutative algebraic geometry out of them. For this we consider the *triangulated category* obtained as the derived category of the category of (quasi)coherent sheaves. This derived category was originally introduced as a tool to study functors [97], but since the eighties its structure itself has become the object of study [27, 49, 115, 117, 177].

[2] This field has less issues about its name, and the interpretation outlined here is the accepted description.

Also, by Bondal–Van den Bergh we know that "schemes are derived affine": the derived category is compactly generated by a single object, hence it is equivalent to the derived category of a dg algebra [51]. Therefore studying invariants of the derived category becomes equivalent to studying invariants of a dg algebra.

Moreover, on the level of dg categories we can witness a unification of various branches of mathematics. All of the following are interesting examples of triangulated or dg categories, which can be treated uniformly in this language:

1. derived categories of algebras;
2. stable module categories;
3. categories of matrix factorisations;
4. derived categories of schemes;
5. singularity categories;
6. Fukaya categories;
7. cluster categories;
8. …

Taken all this into consideration it might seem like a good idea to use dg algebras and categories as central objects in our noncommutative algebraic geometry, as we are now capable of treating many objects from algebraic geometry and more noncommutative origin on a single footing.

But the question is of course: *how much geometry do we still have?* Closely related is the question: *how do morphisms behave?* By considering $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1)$ and its full and strong exceptional collection $(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))$ we see that the notion of points and open subschemes does not behave very well.

Points of $\mathbb{P}^1$ being morphisms $\operatorname{Spec} k \to \mathbb{P}^1$ we could take functors from $\mathbf{D}^{\mathrm{b}}(k)$ towards $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1)$, but these correspond to objects of this category. It is possible to make sense of such a space [218] but the answer is too complicated to work with in general.

Regarding open subschemes we run into another problem: if we are to consider the complement of the point $\operatorname{Spec} k \to \mathbf{D}^{\mathrm{b}}(\mathbb{P}^1)$ as an open subscheme on the categorical level, we are led to consider the quotient of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1)$ by the subcategory generated by the object $\mathcal{O}_{\mathbb{P}^1}$, but this is the subcategory generated by $\mathcal{O}_{\mathbb{P}^1}(1)$, which is equivalent to $\mathbf{D}^{\mathrm{b}}(k)$. So $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1)$ contains an "open subscheme" which is just a point.

So it turns out that we lose many concepts from algebraic geometry. But this is not the end of the story, because we gain other things. As mentioned before, we have gained the flexibility to treat many objects from other fields in a uniform language. But what can we learn from understanding triangulated categories associated to these objects?

Answering this question in general for each of the examples would take us way too long, but we can do this for the smooth projective varieties we hold so dear from algebraic geometry. By Gabriel and Rosenberg we know that the abelian category of coherent sheaves determines the scheme up to isomorphism [84, 187]. This is no longer the case for the derived category [163]. But the derived category *can* tell us about the dimension, Kodaira dimension, canonical ring, sums of Hodge numbers, … [109]. Also, if the (anti)canonical bundle is ample, then derived equivalent varieties are necessarily isomorphic [48], so in this case nothing is lost. The list goes on, with new derived invariants being described on a regular basis. And the new isomorphisms we have gained are an interesting feature of the theory, with close connections to birational geometry, Hodge theory, moduli spaces, …

Associated to a triangulated category (or an enhancement) we can consider invariants, such as algebraic K-theory, Hochschild (co)homology, cyclic homology, … Even though it is possible to give a geometric definition of these, it turns out that they are actually invariants of the derived category, so they are defined for any of the examples listed above.

Besides understanding the categories themselves, we can try to understand the functors between them. An important type of functor in the context of invariants of dg categories are fully faithful functors, as for most derived invariants it turns out that this exhibits the invariant of the smaller category as a direct summand of the invariant of the larger. For Hochschild cohomology, which features prominently in this thesis, we only get a comparison morphism, for which this thesis suggests there could be an interesting interaction between the deformation theories of both categories.

Another interesting question, raised by Orlov [178], is understanding the extent to which it is possible to embed the derived category of a smooth and proper dg algebra in the derived category of a smooth and projective variety, thereby reducing certain questions about additive invariants to a geometric situation where more tools are available.

For more examples in the context of algebraic geometry and representation theory one is referred to chapter 1.

**Motivation**

Besides the abstract motivation involving the natural question of generalising well-known constructions, there are also external reasons to consider the study of noncommutative algebraic geometry.

By studying various flavours of noncommutative algebraic geometry we can try to understand in which sense usual algebraic geometry is special, in the sense that certain types of behaviour cannot occur. Or we can consider the opposite question, and see in which sense behaviour in algebraic geometry is exemplary for behaviour in noncommutative algebraic geometry. In this thesis, this question is mostly studied in the context of fully faithful functors between triangulated categories, and their invariants, as suggested by Orlov's comparison between smooth and proper (dg) algebras, and smooth and projective varieties.

Whenever a non-mathematician is especially insistent on *why* to study noncommutative algebraic geometry, and how useful all this abstract nonsense can be for his or her day-to-day life, the go-to answer to satisfy them is mentioning physics.

Because the answerer is ignorant in this field, and usually the asker is too, this settles the question, albeit in an unsatisfactory way. This introduction is a good place to do some extra effort.

A famous example of physics inspiring mathematics is in the study of graded algebras as a noncommutative form of projective geometry, where Sklyanin algebras were introduced whilst solving the Yang–Baxter equation from statistical mechanics [193, 194]. For a mathematician these turn out to be the generic noncommutative plane, and important (but no longer generic) objects in higher dimensions. In this way, physics often suggests interesting objects to study in mathematics.

More generally quantisation in physics is a source of noncommutative algebra and geometry. Quantisation is the transition from classical mechanics to quantum mechanics, and the asymmetry in quantum mechanics can be interpreted as a form of noncommutativity. One way of performing this is by Kontsevich's deformation quantisation [127, 128]. His celebrated result shows that whenever you are given a Poisson bracket (which you can obtain from the classical geometry of a physical system) it is possible to quantise the system giving an associative algebra to describe it, and the quantum mechanical properties of the system should be governed by this quantisation procedure.

Another important example of physics suggesting objects to study is (homological) mirror symmetry. This phenomenon originates in string theory, and predicts a close connection between two different kinds of invariants (one from complex geometry, the other from symplectic geometry) associated to a so called mirror pair of geometric objects. On a mathematical level, Kontsevich started the homological mirror symmetry program, suggesting that the connection between the two geometric objects is actually an equivalence of the derived category on one side and the Fukaya category on the other [129]. Verifying this statement is hard, but understanding the structure of the derived category (or Fukaya) category is an important tool in tackling it [18].

But for me personally, the main reason to study the subjects in this thesis are because I find the mathematics that is involved beautiful, in particular the way so many things interconnect. Which brings us to the promised overview of the contents of this thesis.

## Connections?

There are different interpretations of the word *connections* in the title of this thesis. I will list the ones which are important in the context of this thesis, and explain how they are related to the various chapters.

**Fully faithful functors and equivalences** The first and foremost type of functors between triangulated (or dg) categories to consider are fully faithful functors, and equivalences. In Ginzburg's philosophy of noncommutative algebraic geometry *in the large* they correspond to the isomorphisms and subspaces of a suitable notion of "noncommutative spaces".

In this thesis, we restrict ourselves to studying these for smooth and proper dg categories, where fully faithful functors are closely related to admissible subcategories and semiorthogonal decompositions. Because this is an important and common theme for multiple chapters of this thesis, chapter 1 is dedicated to giving an overview.

The results in this thesis pertaining to fully faithful functors are the following.

1. In chapter 2 we study when it is possible to embed the derived category of a finite-dimensional algebra in the derived category of a smooth projective surface. In other words we describe which algebras are "subspaces" of smooth projective surfaces.

2. In chapter 3 we describe fully faithful functors between the derived category of a noncommutative quadric and the derived category of a deformation of the Hilbert scheme of 2 points on the quadric surface. Such an embedding was shown to exist in the commutative setting for many surfaces, and it is yet another instance of the close connection between the derived category of a variety and the derived category of the appropriate moduli space of sheaves for this variety. Other instances of such a connection can be found for K3 surfaces or abelian varieties [163, 165] (where one even obtains equivalences) or vector bundles on curves [82]. These are all examples where a variety is a "subspace" of another variety (possibly twisted), but not interpreted in the usual sense of the word.

   It is also an important open question raised by Orlov whether every smooth and proper dg category is a "subspace" of a smooth projective variety [178]. If this were indeed the case, it would be possible to prove properties of arbitrary smooth and proper dg categories by showing them for smooth projective varieties, provided of course these properties are compatible with semiorthogonal decompositions.

The use of fully faithful functors (and equivalences) in this setting is to compare invariants, and study the geometry associated to both objects. The invariants used in this thesis are Hochschild (co)homology and algebraic K-theory, and they are introduced in chapter 1.

**Deformation theory** Another way of getting noncommutative objects out of commutative objects is deformation theory, or quantisation as mentioned earlier. The deformation theory of associative algebras is studied using Hochschild cohomology, and it turns out that the natural setting for Hochschild cohomology are abelian and dg categories [155, 156]. Then deformation theory is a natural bridge between commutative objects and noncommutative objects.

1. In chapter 3 we study how a fully faithful functor between derived categories of commutative varieties behaves under deformation, using the deformation theory of noncommutative quadrics and at the same time describing (a large chunk of) the deformations of the Hilbert scheme of 2 points on $\mathbb{P}^1 \times \mathbb{P}^1$.

By limited functoriality this moreover suggests a strong connection between (the Hochschild–Kostant–Rosenberg components of) the Hochschild cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Hilb}^2 \mathbb{P}^1 \times \mathbb{P}^1$. This observation together with the results of chapter 4 suggests a new interplay between algebraic geometry and noncommutative algebraic geometry, and is in the eyes of the author the most important contribution of this thesis.

2. In chapter 4 we study the infinitesimal deformation theory of noncommutative planes and quadrics, by describing their Hochschild cohomology. Again we see the phenomenon that there is a generic behaviour, whilst the commutative case is in some sense the most special. The approach taken here is also an interesting example of how representation theory of finite-dimensional algebras can be used to study derived categories by using tilting theory.

   As a side result to chapter 4 we have also computed the Gerstenhaber algebra structure of $\mathrm{HH}^\bullet(\mathbb{P}^n)$ in appendix A, showing how the geometry of (partial) flag varieties and the representation theory of algebraic groups and Lie algebras can be used to describe Hochschild cohomology.

3. In chapter 7 we study how a special class of deformations of the commutative polynomial algebra behaves regarding their point schemes, and for which values of the deformation parameters there is special behaviour.

An interesting *non-example* of such a connection in this thesis worth mentioning is the construction of noncommutative surfaces with an exceptional collection of rank 4 as in chapter 5. The examples constructed here *cannot* be obtained by deforming a commutative object, except for 3 special cases in the classification. Nevertheless there is an important role to be played by algebraic geometry for this question, as explained later on.

**Moduli problems associated to noncommutative objects** An important tool in studying noncommutative objects is by associating a moduli space, thereby allowing the use of algebro-geometric techniques.

1. In chapter 3 we study (what ought to be) a noncommutative analogue of the Hilbert scheme of 2 points on $\mathbb{P}^1 \times \mathbb{P}^1$. These are deformations of $\mathrm{Hilb}^2 \mathbb{P}^1 \times \mathbb{P}^1$, and it turns out that it is possible to describe these commutative deformations using the noncommutative deformations of $\mathbb{P}^1 \times \mathbb{P}^1$.

Another type of moduli space associated to a graded algebra is its point scheme. This parametrises graded modules which are the analogues of the usual points on a variety, when this variety is studied through its graded algebra. They have become the main tool in the classification of noncommutative surfaces [12, 13, 50, 226]. In this thesis they arise in the following ways.

2. In chapter 4 the point schemes for noncommutative planes and quadrics are the main tool in computing Hochschild cohomology of their derived categories.

3. In chapter 7 the point schemes for a special class of noncommutative polynomial rings are studied. In higher dimensions the usefulness of point schemes diminishes, as they could very well be empty (and generically they are). But it turns out that for skew polynomial algebras they are in at least 1-dimensional in arbitrary dimension. Moreover they can be described explicitly, and even classified in dimension up to 5.

**Using algebraic geometry to construct and study noncommutative objects** As explained before, one fruitful technique in noncommutative algebraic geometry is to associate a moduli space to a noncommutative object, and study the object and its properties through this (usual) scheme. Often these objects are well-known in other contexts, and they have been the subject of study before. There are two such instances in this thesis.

1. In chapter 4 the classical geometry of plane cubic curves and degree $(2,2)$-divisors on $\mathbb{P}^1 \times \mathbb{P}^1$ is used to compute Hochschild cohomology of noncommutative planes and quadrics. These are the point schemes of the noncommutative plane (resp. quadric), and their automorphisms can be used to determine their Hochschild cohomology.

2. In chapter 6 the classical geometry of nets of conics and nets of binary quartics are the essential ingredient to compare two abelian categories describing a class of noncommutative surfaces with a full exceptional collection of length 4.

In both these cases, the algebraic geometry was (in some form) already known to the Italian school of algebraic geometry in the nineteenth century, but the applications are somewhat more modern.

**Generalisations of tools in algebraic geometry** Another important invariant of a scheme is its Chow group, which describes the structure of its subschemes. Because it is possible to reconstruct a scheme from its derived category and the derived tensor product [22] it is a natural question to describe the Chow group immediately from this data, thereby making it possible to compute it for other examples in tensor triangulated geometry [125].

There is also a relative version of this, with one tensor triangulated category acting on another triangulated category [126]. This is a situation which frequently occurs in algebraic geometry: the action of the center of an associative algebra, and suitable generalisations of this.

In chapter 8 we study the Chow groups in this relative setting. We can show that they agree with classical invariants defined in a down-to-earth way in some special cases, indicating that these relative Chow groups are indeed an interesting invariant to study in more general settings. In this way the relative Chow groups for orders are used in the description of the central Proj of a noncommutative plane finite over its center [9, 229] as used in chapter 5, which is a funny coincidence.

## On the structure of this thesis

This thesis is not a monograph, but rather it is based on a collection of articles. To mitigate this, there is a preliminary chapter about triangulated and differential graded categories. In this chapter I address the main tools and preliminary results used in the majority of the other chapters, explaining more of the connections in detail. This also reduces the duplication present by combining different papers and their preliminary sections. I have also made the chapters themselves more homogeneous by highlighting the connections between them and adding cross-references. The actual mathematical content agrees with that of the published (or arXiv'd) version.

# Nederlandse samenvatting

Dit doctoraat behandelt verschillende verbanden tussen commutatieve en niet-commutatieve algebraïsche meetkunde. Dit zijn abstracte gebieden in de fundamentele wiskunde, die al sinds de oudheid bestudeerd worden: algebraïsche meetkunde is oorspronkelijk ontstaan om (systemen van) veeltermvergelijkingen op te lossen, en de eigenschappen van die oplossingen te bestuderen. In niet-commutatieve algebraïsche meetkunde breiden we de mogelijke soorten van oplossingen uit, om zo een flexibeler framework te hebben waarin we meetkunde kunnen bedrijven. Met die toegenomen flexibiliteit komt echter ook een verhoogde complexiteit, waardoor er de nodige technische bagage vereist is om de resultaten te begrijpen.

De niet-commutatieve algebraïsche meetkunde waar er hier gewag van wordt gemaakt behelst verschillende incarnaties van het onderwerp: niet-commutatieve projectieve meetkunde à la Artin–Zhang, de studie van afgeleide categorieën van gladde projectieve variëteiten à la Bondal–Orlov, en uiteindelijk komt dit allemaal samen in de studie van (gladde en propere) dg categorieën à la Kontsevich. De verbanden die dan gelegd worden in deze thesis helpen ons om moeilijke objecten in niet-commutatieve algebraïsche meetkunde beter te begrijpen, door ze te bestuderen met technieken uit de commutatieve algebraïsche meetkunde die vertrouwder zijn.

In hoofdstuk 1 wordt een uitgebreide inleiding tot de "meetkunde van afgeleide categorieën" gegeven. Op deze manier wordt een van de belangrijkste thema's in deze thesis geïntroduceerd, door een overzicht van de literatuur te geven en telkens de link te leggen met de toepassingen zoals deze in de thesis voorkomen.

In hoofdstuk 2 wordt bestudeerd wanneer de afgeleide categorie van een eindig-dimensionale algebra kan ingebed worden in de afgeleide categorie van een glad en projectief oppervlak. Omdat de afgeleide categorie van een kromme onontbindbaar is, is dit het eerste interessante geval, en er worden belangrijke obstructies gevonden voor het bestaan van zo'n inbedding.

In hoofdstuk 3 wordt een inbedding van de afgeleide categorie van een niet-commutatieve kwadriek in de afgeleide categorie van een deformatie van het hilbertschema van een kwadriek bestudeerd. Dit is een speciaal geval van een vermoeden van Orlov, en we bespreken ook een concrete infinitesimale versie van dit vermoeden. Op basis van dit vermoeden is het vervolgens interessant om de hochschildcohomologie van niet-commutatieve oppervlakken exact te kennen, en in hoofdstuk 4 rekenen we deze uit voor niet-commutatieve vlakken en kwadrieken.

Op basis van de numerieke classificatie van niet-commutatieve oppervlakken van rang 4 construeren we voorbeelden van alle mogelijke gevallen in hoofdstuk 5. Hier gebruiken we een alternatieve constructie van de blowup van een niet-commutatief oppervlak, die voorheen nog niet bestudeerd was. In hoofdstuk 6 vergelijken we vervolgens deze constructie in een speciaal geval met een eerdere constructie via niet-commutatieve $\mathbb{P}^1$-bundels. Hiermee krijgen we een niet-commutatieve versie van het klassieke isomorfisme $\mathrm{Bl}_x \mathbb{P}^2 \cong \mathbb{F}_1$, dat echter geen deformatie is van dit commutatieve fenomeen.

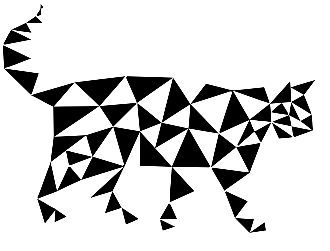
In hoofdstuk 7 wordt de puntvariëteit van scheve veeltermenringen berekend. Dit is een moduliruimte die belangrijke informatie over dit niet-commutatieve object bevat, en we zijn in staat om in arbitraire dimensie te beschrijven hoe deze eruitziet, en in voldoende lage dimensie geven we zelfs een volledige classificatie.

In hoofdstuk 8 wordt een niet-commutatieve versie van chowgroepen bestudeerd. Deze worden ingevoerd via tensorgetrianguleerde meetkunde, en er wordt aangetoond dat deze overeenkomen met invarianten die voorheen ad hoc werden geïntroduceerd.

Tot slot zijn er twee korte appendices. In appendix A wordt de stelling van Borel–Weil–Bott gebruikt om de hochschildcohomologie van $\mathbb{P}^n$ te berekenen in functie van representaties van de liealgebra $\mathfrak{sl}_{n+1}$, waardoor er een onverwacht verband ontstaat met de theorie van orthogonale veeltermen. In appendix B wordt er een voorbeeld gegeven van een delingsalgebra wiens afgeleide categorie kan ingebed worden in de afgeleide categorie van een gladde projectieve variëteit van lagere dimensie dan verwacht.

*Chapter 1*

# A reminder on triangulated and dg categories



(*a triangulated cat*)

In all but one of the chapters of this thesis, triangulated categories and their (dg) enhancements play a prominent role. In this preliminary chapter we discuss various results, examples and techniques which will be important for these chapters.

In section 1.1 we recall the basic definitions, and discuss the main examples and properties of triangulated and dg categories as they appear in this thesis. The important motivating result here is Bondal–Van den Bergh's derived affineness of schemes, as recalled in theorem 1.12.

In section 1.2 we discuss tools to describe the structure of a sufficiently nice triangulated or dg category. These tools are exceptional collections and semiorthogonal decompositions. We discuss many examples of these, all of which are relevant to the various chapters of this thesis, and we will point out where they appear in the thesis.

In section 1.3 we describe invariants associated to triangulated and dg categories, which can be used to understand the structure of such a category. For the purpose of this thesis it suffices to restrict ourselves to algebraic K-theory (or even just Grothendieck groups) and Hochschild (co)homology, but many others exist.

Finally, in section 1.4 we discuss a particularly well-behaved class of triangulated and dg categories, for which the structural results and invariants from the previous sections behave particularly well. These are the smooth and proper dg categories, and we discuss the role of the Serre functor for these categories, Orlov's notion of geometricity, and Bondal's notion of geometric t-structures.

## 1.1 Triangulated and differential graded categories

Not many arguments in this thesis explicitly use the axioms of a triangulated category, but it seems apt to at least formally define them given their prominent role. Likewise for dg categories we will restrict ourselves to giving the definition. The main point of this section, and the entire chapter, is not to introduce the actual formalism, but to set the stage by giving the appropriate examples of the special types of triangulated and dg categories that will be studied in this thesis, and explain how we will study them. Unlike triangulated (and to a lesser extent dg) categories the notion of semiorthogonal decompositions in algebraic geometry has not yet reached the stage of being text book material.

### 1.1.1 Preliminaries

There are different versions of the definition of a triangulated category (mostly with variations in (TR4)), the following definition is the classical one [97, 169].

**Definition 1.1.** A *triangulated category* is an additive category $\mathcal{T}$ together with

1. a *translation functor* (or *shift functor*, or *suspension functor*) $[1]$, which is an automorphism of $\mathcal{T}$;
2. a class of *distinguished triangles* closed under isomorphisms, which are sextuples $(X, Y, Z, u, v, w)$ where $u\colon X \to Y$, $v\colon Y \to Z$, $w\colon Z \to X[1]$ are morphisms in $\mathcal{T}$;

such that

(TR0) For every object $X$ of $\mathcal{T}$ we have that $(X, X, 0, \mathrm{id}, 0, 0)$ is distinguished.

(TR1) For every morphism $u\colon X \to Z$ there exists an object $Z$ called the[1] *mapping cone* of $u$ and morphisms $v\colon Y \to Z$, $w\colon Z \to X[1]$ such that $(X, Y, Z, u, v, w)$ is a distinguished triangle.

(TR2) The triangle $(X, Y, Z, u, v, w)$ is a distinguished triangle if and only if the triangle $(Y, Z, X[1], -v, -w, -u[1])$ is.

(TR3) If $(X, Y, Z, u, v, w)$ and $(X', Y', Z', u', v', w')$ are distinguished triangles, and $f\colon X \to X'$ and $g\colon Y \to Y'$ are morphisms such that the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \exists h} & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array} \tag{1.1}$$

commutes, then there exists a (not necessarily unique) morphism $h\colon Z \to Z'$ making the diagram commute.

(TR4) If $(X, Y, Z', u, j, k)$, $(Y, Z, X', v, l, i)$ and $(X, Z, Y', v \circ u, m, n)$ are distinguished triangles, then there exists a distinguished triangle $(Z', Y', X', f, g, j[1] \circ i)$ such that

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{j} & Z' & \xrightarrow{k} & X[1] \\ \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \exists f} & & \| \\ X & \xrightarrow{v \circ u} & Z & \xrightarrow{m} & Y' & \xrightarrow{n} & X[1] \\ & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \exists g} & & \\ & & X' & = & X' & & \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j[1] \circ i} & & \\ & & Y[1] & \xrightarrow{j[1]} & Z'[1] & & \end{array} \tag{1.2}$$

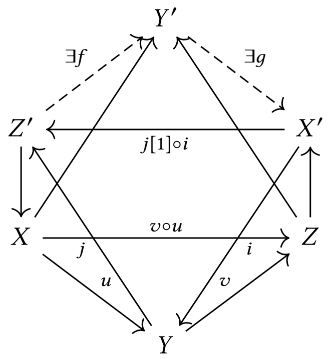
commutes, and moreover $u[1] \circ n = i \circ g$.

One way of interpreting (TR4) is as an analogue of the third isomorphism theorem in an abelian category: given the morphisms $u\colon X \to Y$ and $v\colon Y \to Z$ we can consider their composition and from (TR1) we get three distinguished triangles, where (TR4) asserts that the mapping cones in these distinguished triangles can be made into a distinguished triangle themselves in a way compatible with all other triangles. It is still not clear whether (TR4) follows from the other axioms [169, remark 1.3.15].

[1] This object is only unique up to non-unique isomorphism, the source of some issues with triangulated categories

**Remark 1.2.** Axiom (TR4) is also called the *octahedral axiom.* To understand the origin of this name one has to collapse the identities, remove the shifts and fold the triangles on the left and right so that they meet in a point, resulting in (with some imagination) the octahedron

(1.3)




where the distinguished triangles correspond to four non-adjacent faces, whilst the other four faces correspond to the commutative squares in (1.3) containing an identity morphism.

As already remarked in the definition of a triangulated category, there is a source of potential issues arising with these axioms because of the non-functoriality of the mapping cone. If on the other hand functoriality were to be imposed in the definition, then it can be shown that the triangulated category is necessarily semisimple (as an additive category) [231, proposition II.1.2.14], so this would not be very useful. The non-functoriality of the cone induces problems when trying to perform constructions with triangulated categories, such as tensor products or internal Hom's, and this type of constructions is of course essential when trying to do some form of algebraic geometry with triangulated categories.

Before explaining the solution using dg enhancements in section 1.1.2, we will introduce some notions which already make sense for triangulated categories without the need for an enhancement.

An important finiteness condition for a triangulated category is being compactly generated.

**Definition 1.3.** Let $\mathcal{T}$ be a triangulated category. It is *compactly generated* if it has coproducts, and there exists a set of objects $\mathcal{G}$ (called compact generators) such that

1. $G$ is *compact* for all $G \in \mathcal{G}$, i.e. for every set of objects $\{T_i \in \mathcal{T} \mid i \in I\}$ the natural morphism

   $$\coprod_{i\in I} \mathrm{Hom}_{\mathcal{T}}(G, T_i) \to \mathrm{Hom}_{\mathcal{T}}\left(G, \coprod_{i\in I} T_i\right) \tag{1.4}$$

   is an isomorphism;

2. $\mathcal{G}$ *generates* $\mathcal{T}$, i.e. if $T$ is an object of $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(G, T[n]) = 0$ for all $G \in \mathcal{G}$ and $n \in \mathbb{Z}$, then $T = 0$.

In this case one can go back and forth between the "big" triangulated category $\mathcal{T}$ and its "small" subcategory $\mathcal{T}^c$ of compact objects. This is an important technique used in chapter 8. The main example of compactly generated categories is provided by theorem 1.12, where it turns out $\mathcal{G}$ can even be taken to be a singleton.

Another finiteness condition is Hom-finiteness. For this notion to make sense we will need the triangulated category to not just be additive, but actually $k$-linear where $k$ is a field.

**Definition 1.4.** Let $k$ be a field, and $\mathcal{T}$ a $k$-linear triangulated category. Then $\mathcal{T}$ is Hom-*finite* if $\dim_k \operatorname{Hom}_{\mathcal{T}}(A,B) < +\infty$ for all $A, B \in \mathcal{T}$.

This will be an important notion, required in section 1.4, especially for the definition of the Serre functor to make sense as this involves duals of vector spaces.

### 1.1.2 Enhancements

To solve the issue of non-functoriality of the cone one needs an *enhancement*, for which the triangulated category of interest is a truncated version. Whilst performing constructions one uses this enhanced version, and truncating afterwards yields the desired triangulated category.

There are various ways of enhancing. In this thesis we will be using a (pretriangulated) differential graded category (from now on abbreviated to dg category) as the main tool, but other enhancements include pretriangulated $\mathrm{A}_\infty$-categories, stable $\infty$-categories or stable derivators. These alternative enhancements give a somewhat more flexible and unified theory, at the expense of needing more machinery. For the purposes of this thesis, the "easy" notion of a dg enhancement will suffice, especially as it turns out that in many cases we are just working with dg algebras.

**Definition 1.5.** Let $k$ be a commutative ring. A *dg category* $\mathcal{C}$ (over $k$) is a category enriched over cochain complexes, i.e. $\operatorname{Hom}_{\mathcal{C}}(A,B)^\bullet$ is a cochain complex of $k$-modules.

There is also the corresponding notion of a *dg functor*, which is just a functor compatible with the enhancement. The resulting category of (small) dg categories will be denoted $\mathrm{dgCat}_k$.

To a dg category we can associate several $k$-linear categories.

**Definition 1.6.** Let $\mathcal{C}$ be a dg category. The *homotopy category* of $\mathcal{C}$ is the $k$-linear category $\mathrm{H}^0(\mathcal{C})$ with the same objects, and

$$(1.5)\quad \operatorname{Hom}_{\mathrm{H}^0(\mathcal{C})}(A,B) \coloneqq \mathrm{H}^0(\operatorname{Hom}_{\mathcal{C}}(A,B)^\bullet).$$

It is also possible to associate a new dg category to a given one. For this we will need the dg category $\mathrm{Ch}_{\mathrm{dg}}(k)$, which is the category of cochain complexes of $k$-modules, equipped with the dg structure given by considering the cochain complex of morphisms of all degrees between two cochain complexes, together with the suitable differential.

**Definition 1.7.** Let $\mathcal{C}$ be a dg category. A *right dg module over* $\mathcal{C}$ is a dg functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathrm{Ch}_{\mathrm{dg}}(k)$. These can be assembled in the *dg category of dg modules* $\mathrm{dgMod}\,\mathcal{C}$.

For the homotopy category of the dg category of dg modules there is an obvious triangulated structure, mimicking the construction of the homotopy category of an additive category. We then define a morphism in this category to be a quasi-isomorphism if it is one for the evaluation in each object of $\mathcal{C}$. Then the *derived category* $\mathbf{D}(\mathcal{C})$ is the localisation of the homotopy category of dg modules along these morphisms. We get the Yoneda embedding $\mathrm{H}^0(\mathcal{C}) \hookrightarrow \mathbf{D}(\mathcal{C})$. This allows us to define the notion of a pretriangulated dg category, which is used to define enhancements.

**Definition 1.8.** Let $\mathcal{C}$ be a dg category. Then $\mathcal{C}$ is *pretriangulated* if the image of the Yoneda embedding is stable under the translation functor and its inverse.

We then define the following.

**Definition 1.9.** Let $\mathcal{T}$ be a triangulated category. A *dg enhancement* of $\mathcal{T}$ is a pair $(\mathcal{C}, \varphi)$ of a pretriangulated dg category $\mathcal{C}$ together with an equivalence of triangulated categories $\mathrm{H}^0(\mathcal{C}) \cong \mathcal{T}$.

For more information and results on enhancements one is referred to [158]. For more on the homotopy theory of dg categories and associated constructions there is [217].

Another important class of dg categories are those for which the object set is a singleton. Alternatively we can say the following.

**Definition 1.10.** A *dg algebra* $A^\bullet$ is an algebra object in the category of cochain complexes.

This actually motivated many of the constructions for dg categories: for a usual algebra we can construct its derived category, and we can extend this definition to (dg) categories.

Often the pretriangulated dg categories we are interested in are *Morita equivalent* to a dg algebra, i.e. their derived categories are equivalent, see also section 1.4 and theorem 1.12.

### 1.1.3 Examples

The triangulated categories which appear in this thesis are mainly related to derived categories of some sort. In algebraic geometry we will consider the category of coherent sheaves $\operatorname{coh} X$, and the category of quasicoherent sheaves $\operatorname{Qcoh} X$. The associated derived categories that we will consider are $\mathbf{D}^{\mathrm{b}}(\operatorname{coh} X)$ and $\mathbf{D}(\operatorname{Qcoh} X)$, no other flavours are required. Starting from a sufficiently nice graded (not dg) algebra $A$ we will also consider $\operatorname{qgr} A$ as a replacement for $\operatorname{coh} X$, and its derived category $\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A)$. If $B$ is a $k$-algebra, then we will also consider the derived categories $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} B)$ and $\mathbf{D}(\operatorname{Mod} B)$.

A mixture of these two categories is considered in chapters 5, 6 and 8: if $\mathcal{A}$ is a sheaf of algebras on a scheme $X$ we can consider $\mathbf{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{A})$ and $\mathbf{D}(\operatorname{Qcoh} \mathcal{A})$. For more on the formalism of quasicoherent sheaves over a sheaf of algebras one is referred to section 8.4.

For triangulated categories we have introduced the notion of compact objects. The compact objects of $\mathbf{D}(\operatorname{Qcoh} X)$ can be characterised geometrically as the *perfect complexes*. These are the objects of $\mathbf{D}(\operatorname{Qcoh} X)$ which are Zariski-locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. We will denote $\mathbf{D}(\mathrm{Qcoh}\,X)^{\mathrm{c}}$ by $\mathbf{D}^{\mathrm{perf}}(X)$. We have that $\mathbf{D}^{\mathrm{perf}}(X) \subseteq \mathbf{D}^{\mathrm{b}}(\mathrm{coh}\,X)$ if $X$ is noetherian, and usually we will work with $X$ a smooth projective variety, where this inclusion is even an equivalence.

**Remark 1.11.** We will denote $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}\,X)$ by $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,A)$ by $\mathbf{D}^{\mathrm{b}}(A)$. The only point where unbounded derived categories play a role is in chapter 8, where we will be explicit about which derived categories we are considering.

For each of these categories it is possible to construct dg enhancements. We will not need any specifics about these constructions, only that in the situation that we are working in, these enhancements turn out to be unique in a suitable sense [158]. A priori it might seem like there is a choice of enhancement involved, and the invariants defined in section 1.3 might depend on it, but this will not be the case.

**Derived affineness**

An interesting observation is that if we only care about the homotopy category of a pretriangulated dg category or the derived category of a dg algebra, we can often find a single dg algebra which completely describes the object we are interested in. In this way a triangulated category can be *derived affine*. Morita theory for abelian categories gives us conditions under which a (Grothendieck) abelian category is of the form $\mathrm{Mod}\,B$ for some ring $B$, which is a very strong property in algebraic geometry as this is only satisfied for affine schemes. For non-affine schemes, we can consider $\mathbf{D}(\mathrm{Qcoh}\,X)$, and now something similar happens [51].

**Theorem 1.12** (Bondal–Van den Bergh)**.** Let $X$ be a quasi-compact separated scheme. Then $\mathbf{D}(\mathrm{Qcoh}\,X)$ is compactly generated by a single perfect complex $E^{\bullet}$, and its dg endomorphism algebra is derived Morita equivalent to $\mathbf{D}(\mathrm{Qcoh}\,X)$.

This is just an existential result: there is no control whatsoever over the dg algebra $B$, which depends on the choice of a compact generator. But in favourable cases we have a particularly good choice of generator, allowing us to use tilting theory to describe the derived category. This is explained in section 1.2.2. Then it often becomes much easier to describe the derived invariants introduced in section 1.3.

## 1.2 Decomposing triangulated categories

An important way of studying the structure of triangulated categories is by decomposing them into smaller and more tractable pieces. This process is known as taking semiorthogonal decompositions, and in special cases these are given by exceptional collections. In this section we recall their definitions, and give many examples, indicating where they will be used in this thesis. An overview article for these kinds of decompositions, containing some which are not discussed here, is [145].

### 1.2.1 Exceptional collections

The easiest decomposition of a triangulated category is given by an exceptional collection. For this notion to be well-behaved, we let $\mathcal{T}$ be a Hom-finite triangulated category. We will usually take $k$ to be algebraically closed, there exist more general notions if this is not the case but we will not need these [178].

**Definition 1.13.** We say that $E \in \mathcal{T}$ is an *exceptional object* if

$$\mathrm{Hom}_{\mathcal{T}}(E, E[m]) \cong \begin{cases} k & m = 0 \\ 0 & m \neq 0. \end{cases} \tag{1.6}$$

Exceptional objects are generalisations of rigid objects in an abelian category, i.e. those for which $\mathrm{Ext}^1$ vanishes. Historically this was a very important connection at the beginning of the study of derived categories of smooth projective varieties [90, 99, 133]. As it turns out, in good cases they can be used as the easiest type of building blocks for triangulated categories.

**Definition 1.14.** A sequence $(E_1, \ldots, E_n)$ of objects $E_i \in \mathcal{T}$ is an *exceptional collection* if each $E_i$ is exceptional, and $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[m]) = 0$ for all $i > j$ and $m$.

We will use the terminology of an exceptional *collection*, even though the order is important. There also exist notions where the total order is replaced by a partial order [4], but we will not need these here.

If the triangulated category comes with a dg enhancement, then it makes sense to look at the dg endomorphism algebra of the direct sum of these objects. We can use this algebra to describe the triangulated category as the derived category of dg modules over it. Because arbitrary dg algebras can be somewhat unwieldy to work with, and indeed, by theorem 1.12 their study encompasses the entire study of derived categories of schemes, we will often restrict ourselves to exceptional collections whose derived endomorphism algebra is actually concentrated in a single degree.

**Definition 1.15.** An exceptional collection $(E_1, \ldots, E_n)$ is said to be *strong* if moreover $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[m]) = 0$ for all $m \neq 0$ and $i \neq j$.

In order to use an exceptional collection to completely describe a triangulated category we need to impose another condition.

**Definition 1.16.** An exceptional collection $(E_1, \ldots, E_n)$ is said to be *full* if it generates the category $\mathcal{T}$, or in other words, any full triangulated subcategory of $\mathcal{T}$ containing the objects $E_i$ is necessarily equivalent to $\mathcal{T}$ via the inclusion functor.

In section 1.2.3 we will see that exceptional collections are a very special cases of semiorthogonal decompositions, one in which each subcategory is actually equivalent to $\mathbf{D}^{\mathrm{b}}(k)$, it being generated by the exceptional object $E_i$. Therefore we will also (ab)use the notation which will be introduced for semiorthogonal decompositions, by writing

$$\mathcal{T} = \langle E_1, \ldots, E_n \rangle . \tag{1.7}$$

By taking the Grothendieck group (see also section 1.3.1) of $\mathcal{T}$ one can understand these definitions in terms of linear algebra. Indeed, the Hom of $\mathcal{T}$ induces a bilinear form $(-,-)$ on $\mathrm{K}_0(\mathcal{T})$ using the Euler characteristic of $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(A, B[n])$, where $A, B \in \mathrm{Ob}(\mathcal{T})$. We can consider the classes $[E_i]$ as giving a basis for $\mathrm{K}_0(\mathcal{T}) = \mathbb{Z}^{\oplus n}$ for which the matrix describing the Euler form is upper triangular, and has $(E_i, E_i) = 1$ on the diagonal.

An important feature of exceptional objects is that they lift through deformation (of abelian categories), as in [154]. This motivaties many early results on the use of helices and the use of exceptional collections to study derived categories of noncommutative deformations [50, 99].

There is also the notion of mutation of an exceptional collection (and more generally a semiorthogonal decomposition), which describes an action of the braid group on the set of exceptional collections. For more details one is referred to section 5.2.2.

**Tilting**

The situation of a full and strong exceptional collection gives a special case of a tilting object. These are usually considered for derived categories of smooth projective varieties, or derived categories of finite-dimensional algebras (of finite global dimension). The corresponding definitions are the following.

**Definition 1.17.** Let $X$ be a smooth projective variety. A coherent sheaf $\mathcal{T}$ on $X$ is said to be a *tilting sheaf* if

1. $\mathrm{End}_X(\mathcal{T})$ has finite global dimension;
2. $\mathrm{Ext}^k_X(\mathcal{T},\mathcal{T}) = 0$ for $k \geq 1$;
3. $\mathcal{T}$ (classically) generates $\mathbf{D}^{\mathrm{b}}(X)$, i.e. the smallest strictly full triangulated subcategory closed under direct summands containing $\mathcal{T}$ is equal to $\mathbf{D}^{\mathrm{b}}(X)$.

If the tilting sheaf happens to be a vector bundle, it is called a *tilting bundle*.

The main application of tilting sheaves is given by the following theorem [21, 46]. It gives a way to study the derived category of a variety as the derived category of an algebra. In this way it is a very special case of the "derived affineness" from theorem 1.12, where we actually *do* have control over the algebra that we get because the object we used to construct is not obtained through an existence result[2].

**Theorem 1.18** (Baer, Bondal)**.** Let $\mathcal{T}$ be a tilting sheaf for $X$. Denote $A \coloneqq \mathrm{End}_X(\mathcal{T})$. The functor

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{T},-)\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(A^{\mathrm{op}}) \tag{1.8}$$

is an equivalence of triangulated categories, whose inverse is given by $-\otimes^{\mathbf{L}}_A \mathcal{T}$.

The definition for a finite-dimensional algebra is the following straightforward adaptation.

**Definition 1.19.** Let $A$ be a finite-dimensional algebra of finite global dimension. An $A$-module $T$ is a *generalised tilting module* if

1. $T$ has finite projective dimension;
2. $\mathrm{Ext}^k_A(T,T) = 0$ for $k \geq 1$;

[2]For a smooth projective variety one could use a very ample line bundle $\mathcal{L}$ and use the endomorphism algebra of $\bigoplus_{i=0}^{\dim X} \mathcal{L}^{\otimes i}$. Nevertheless, this somewhat obvious choice does not give an algebra which is easy to work with.

3. there exists an exact sequence

   $$0 \to A \to T_1 \to T_2 \to \ldots \to T_n \to 0 \tag{1.9}$$

   where $T_i \in \operatorname{add} T$.

The third condition is just a rephrasing of the classical generation from the definition of a tilting sheaf, because if we can generate $A$ using $\operatorname{add} T$, then every module can be generated. Recall that $\operatorname{add} T$ are the objects which can be obtained by taking direct sums and summands of $T$.

Another aspect of tilting theory is that it relates t-structures through a derived equivalence. On both sides of the equivalence we have the standard t-structure on the derived category of an abelian category. There is an important distinction between the t-structure coming from a smooth projective variety, and that coming from a (finite-dimensional) algebra, concerning the compatibility with the Serre functor. We come back to this point in section 1.4.4.

Using a full and strong exceptional collection of sheaves (or similarly modules) we can easily obtain tilting objects: it suffices to consider $\bigoplus_{i=1}^{n} \mathcal{E}_i$, where $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ is a full and strong exceptional collection of sheaves. The endomorphism algebra obtained in this way has the extra property that it is a triangular algebra, see also proposition 1.27.

**Rationality?**

For the triangulated category associated to a smooth projective variety $X$ the existence of a full and strong exceptional collection in $\mathbf{D}^{\mathrm{b}}(X)$ imposes very strong conditions on the varieties. For surfaces, a folklore conjecture by Orlov states that existence of such a collection is equivalent to $X$ being rational. The implication that a rational surface indeed has a full and strong exceptional collection is known. In arbitrary dimension, Lunts conjectured that it is equivalent to $X$ being *strongly rational* [79, conjecture 1.2]. All of the smooth projective varieties which make an appearance in chapters 2 and 3 and appendix A admit a full and strong exceptional collection, and so do the noncommutative surfaces considered in chapters 4 to 6.

### 1.2.2 Examples of exceptional collections

We now discuss some exceptional collections which are relevant to this thesis. For more information one is referred to [145].

**Projective space**

This is the exceptional collection that started the whole study of the structure of derived categories [27, 39]. Before, derived categories were used as an appropriate staging ground for functors in Grothendieck duality [97], but not as an object in its own right.

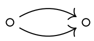
**Theorem 1.20** (Beilinson)**.** The collection

$$\langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle \tag{1.10}$$

is a full and strong exceptional collection (of line bundles) for $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$.

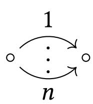
It will be interesting to describe the exceptional collection (1.10) for small values of $n$, as they will be used throughout this thesis.

**Example 1.21.** If we consider $\mathbb{P}^1$, then the structure of the exceptional collection (1.10) is completely described by the Kronecker quiver

(1.11) $\mathsf{K}_2\colon$  .

There are no relations, in this case the algebra is actually (tame) hereditary, it being the extended Dynkin quiver $\widetilde{\mathrm{A}_1}$.

An interesting motivating question to ask is: *what makes the Kronecker quiver special*, when compared to the generalised Kronecker quivers? Recall that these are of the form

(1.12) $\mathsf{K}_n\colon$ 

.

The word "special" does not have an intended specific meaning in this question, but should be understood as a vague notion of "being of geometric nature". A posteriori it is of course possible to point towards the description of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1)$ and the indecomposability of $\mathbf{D}^{\mathrm{b}}(C)$ for $g_C \geq 1$ as in theorem 1.41, but it turns out that there is a more abstract answer to this question. By [50, lemma 3.1] we have that the action of the Serre functor (see section 1.4.2) of the derived category of a smooth projective variety on the Grothendieck group is unipotent. Now the Gram matrix of the (generalised) Kronecker quiver is

$$A = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \tag{1.13}$$

and the action of the Serre functor is given by the Coxeter matrix

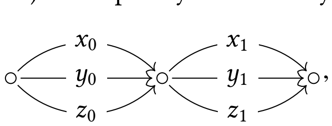
$$C = -A^{-1}A^{\mathrm{t}} = \begin{pmatrix} 1-n^2 & -n \\ -n & 1 \end{pmatrix}. \tag{1.14}$$

Then the characteristic polynomial of $-C$ is $t^2 + (-n^2+2)t + 1$. The matrix $-C$ is unipotent if and only if the characteristic polynomial is a power of $t-1$, which is only the case if $n = 2$.

It is a strengthening of this property in dimension 2 which is used in chapter 2 to provide necessary conditions for the existence of embeddings of derived categories of algebras in derived categories of smooth projective surfaces, and explains the classification from [212] motivating the constructions in chapter 5.

Moreover, as we will see in theorem 1.41 we have that $\mathbb{P}^1$ (and non-split smooth conics) are the only curves for which there exists a semiorthogonal decomposition.

**Example 1.22.** The next case is $\mathbb{P}^2$, for which the structure of the exceptional collection (1.10) is completely described by the quiver

(1.15) 

,

and the relations are

$$\begin{cases} x_0y_1 = y_0x_1 \\ x_0z_1 = z_0x_1 \\ y_0z_1 = z_0y_1. \end{cases} \tag{1.16}$$

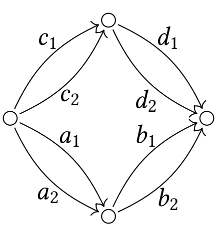
We will use this exceptional collection, and its analogues for noncommutative planes, in chapter 4 and chapter 5. It turns out that on the level of finite-dimensional algebras a noncommutative plane is nothing more than changing the relations accordingly, see e.g. example 5.8.

The exceptional collection from example 1.21 can be used to construct an exceptional collection on $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1 \times \mathbb{P}^1)$. More generally, if we have exceptional collections for two smooth projective varieties $X$ and $Y$ we have one for their product, by using the external tensor product.

**Example 1.23.** For the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ we have a full and strong exceptional collection of the form

$$\langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,0), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\rangle\,. \tag{1.17}$$

Its structure is described by the quiver

(1.18) 


with relations $b_ia_j = d_jc_i$, for $i, j \in \{1, 2\}$. This exceptional collection, and its analogues for noncommutative quadrics, plays an important role in chapter 3 and chapter 4. Its Cartan matrix

$$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.19}$$

moreover appears in the classification of rank 4 lattices from [212] which is the starting point for chapter 5.

There is another important full and strong exceptional collection on $\mathbb{P}^1 \times \mathbb{P}^1$, given by

$$\langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,0), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)\rangle \tag{1.20}$$

which will make an appearance in chapter 3.

### Quadrics

On a smooth quadric hypersurface $Q_n \subseteq \mathbb{P}^{n+1}$ it is possible to define *spinor bundles*. Depending on the parity of $n$, there is either a unique spinor bundle $\mathcal{S}$ of rank $2^{(n-1)/2}$ if $n$ is odd or there are two spinor bundles $\mathcal{S}_+, \mathcal{S}_-$ of rank $2^{n/2}$ if $n$ is even [179].

For low-dimensional quadric hypersurfaces these spinor bundles can be described by using a different interpretation for the quadric [179, examples 1.5], which in particular applies to example 1.23.

**Example 1.24.** The quadric surface $Q_2$ in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and we can completely describe the Picard group in this way. Now the spinor bundles are line bundles, and it can be shown that $\mathcal{S}_+ \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)$ and $\mathcal{S}_- \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)$.

**Example 1.25.** The quadric hypersurface $Q_4$ in $\mathbb{P}^5$ is also the Grassmannian $\mathrm{Gr}(2,4)$. But on the Grassmannian we have the universal subbundle and quotient bundle, see also section 3.2. These are (after dualising the quotient bundle) the spinor bundles.

In [114] the following exceptional collection was constructed.

**Theorem 1.26** (Kapranov)**.** Let $Q_n$ be a smooth $n$-dimensional quadric hypersurface. The following collection

$$\mathbf{D}^{\mathrm{b}}(Q_n) = \begin{cases} \left\langle \mathcal{S}, \mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(1), \ldots, \mathcal{O}_{Q_n}(n-1) \right\rangle & n \text{ odd} \\ \left\langle \mathcal{S}_-, \mathcal{S}_+, \mathcal{O}_{Q_n}, \mathcal{O}_{Q_n}(1), \ldots, \mathcal{O}_{Q_n}(n-1) \right\rangle & n \text{ even} \end{cases} \tag{1.21}$$

is a full and strong exceptional collection of vector bundles for $\mathbf{D}^{\mathrm{b}}(Q_n)$.

The proof uses graded Clifford algebras, which also make an appearance in chapter 6.

### Path algebras of quivers with relations

We will also be concerned with derived categories of finite-dimensional algebras, and for an important subclass of these it is very easy to describe (several) full and strong exceptional collections and their structure. Indeed, every finite-dimensional algebra is Morita equivalent to a basic algebra, and every basic algebra is isomorphic to the path algebra $kQ/I$ of a (finite) quiver with relations.

If this quiver is moreover *acyclic* (or ordered), we can easily describe its derived category [44]. In this case, the algebra is sometimes called a *triangular algebra*. Note that the global dimension of such an algebra is necessarily finite. We will denote by $P_i$ (resp. $I_i$, $S_i$) the projective (resp. injective, simple) module associated to the vertex $i$.

**Proposition 1.27.** Let $Q$ be an acyclic quiver, and $I$ an admissible ideal of relations. Then

$$\begin{aligned} \mathbf{D}^{\mathrm{b}}(kQ/I) &= \langle P_1, \ldots, P_n \rangle \\ &= \langle I_1, \ldots, I_n \rangle \\ &= \langle S_n, \ldots, S_1 \rangle \end{aligned} \tag{1.22}$$

where the first two are full and strong exceptional collections, and the third is full.

One way in which these algebras arise is as the endomorphism algebras of full and strong exceptional collections:

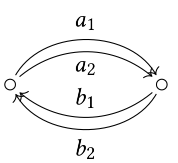
1. vertices correspond to the exceptional objects;
2. arrows correspond to morphisms between these objects;
3. relations correspond to the composition law in the category.

In this way we can reduce the study of the derived category of a smooth projective variety which admits a full and strong exceptional collection to that of an explicit finite-dimensional algebra using theorem 1.18. A well-known example comes from example 1.21, for which one can use [44, example 6.3].

An interesting class of examples to which the previous proposition does not apply are Happel's Fibonacci algebras [91]. We describe this algebra for $n = 2$, but they exist for all $n \in \mathbb{N}$.

**Example 1.28** (Fibonacci algebras)**.** Consider the quiver

(1.23) $Q$:




which is not acyclic, and the ideal $I$ of $kQ$ generated by the following relations:

(1.24) $b_1a_1 = b_2a_1 = b_2a_2 = a_2b_1 = 0.$

The algebra $kQ/I$ is 13-dimensional, and can be shown to be of global dimension 4. Moreover, its derived category *does not* admit a full exceptional collection: one can explicitly compute the Cartan matrix, from which it follows that the orthogonal to an exceptional object cannot be generated by an exceptional object itself.

### 1.2.3 Semiorthogonal decompositions

The existence of a full (and strong) exceptional collection is a very strong condition for a triangulated category. A less restrictive notion is that of a semiorthogonal decomposition, in which the building blocks do not have to be single objects, but they are allowed to be subcategories.

In section 1.3 we will discuss invariants for triangulated and dg categories. Most of these invariants are functorial with respect to arbitrary functors, but for admissible embeddings arising in semiorthogonal decompositions they will have even better behaviour. Indeed, most of them are *additive* invariants, i.e. they send semiorthogonal decompositions to direct sums.

The main non-additive invariant studied in this thesis is Hochschild cohomology, but again there is an important relationship with semiorthogonal decompositions, as explained in example 1.45.

The first notion we will use is that of an admissible subcategory.

**Definition 1.29.** A full subbcategory $i\colon \mathcal{N} \hookrightarrow \mathcal{T}$ is *left* (respectively *right*) *admissible* if there is a left (respectively right) adjoint functor $q\colon \mathcal{T} \to \mathcal{N}$ to $i$. It is *admissible* if $i$ is both left and right admissible.

In the setting of most of the results of this thesis, admissibility follows immediately from the nice properties of the categories which are involved, as we will discuss in section 1.4.

We will denote by $\mathcal{N}^\perp$ the right orthogonal (in $\mathcal{T}$) to $\mathcal{N}$: it is the full subcategory of $\mathcal{T}$ consisting of objects $M$ such that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{N}, M) = 0$. The left orthogonal is defined similarly and is denoted ${}^\perp\mathcal{N}$.

**Definition 1.30.** The triangulated category $\mathcal{T}$ has a *semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle \tag{1.25}$$

for full triangulated subcategories $\mathcal{N}_i$, if $\mathcal{T}$ has an increasing filtration

$$0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_{n-1} \subset \mathcal{T}_n = \mathcal{T} \tag{1.26}$$

by left admissible subcategories $\mathcal{T}_i$ such that in $\mathcal{T}_i$, one has ${}^\perp\mathcal{T}_{i-1} = \mathcal{N}_i \cong \mathcal{T}_i/\mathcal{T}_{i-1}$.

This means that for every object $T \in \mathcal{T}$ there exists a chain of morphisms

$$0 \to T_n \to T_{n-1} \to \ldots \to T_0 = T \tag{1.27}$$

for which $\mathrm{cone}(T_i \to T_{i-1}) \in \mathcal{T}_i$, for $i = 1, \ldots, n$. This in turn can be described as the existence of a diagram

$$\begin{array}{ccccccccc} 0 = T_n & \longrightarrow & T_{n-1} & \longrightarrow & \ldots & \longrightarrow & T_1 & \longrightarrow & T_0 = T \\ & {}_{+1}\nwarrow & \swarrow & & & & & {}_{+1}\nwarrow & \swarrow \\ & C_n & & & & & & C_1 & \end{array} \tag{1.28}$$

where $C_i = \mathrm{cone}(T_i \to T_{i-1})$, and triangles are distinguished triangles. In other words, a semiorthogonal decompositions gives for every object $T \in \mathcal{T}$ a filtration such that the graded piece in degree $i$ belongs to $\mathcal{T}_i$.

**Remark 1.31.** When using dg enhancements, the notion of a semiorthogonal decomposition is closely related to that of a gluing. By using the different functors which are available in a semiorthogonal decompositions, it is possible to define a *gluing functor* and construct a dg category from (enhancements of) the pieces and the gluing functor which gives back (up to Morita equivalence) the original dg category [178]. We will not need the exact machinery of gluings here, but observe that in some way the main point of chapter 4 is the influence the choice of gluing functor has on the Hochschild cohomology of a dg category.

**Remark 1.32.** A related notion of decomposition for a triangulated category is that of a t-structure, which will be introduced in section 1.4.4.

### 1.2.4 Examples of semiorthogonal decompositions

We now come to a list of semiorthogonal decompositions for smooth projective varieties which will occur in this thesis. Again we refer to [145] for more information.

**Remark 1.33.** Many of the semiorthogonal decompositions discussed in this section can actually be generalised to a non-smooth setting, replacing $\mathbf{D}^{\mathrm{b}}(X)$ by $\mathbf{D}^{\mathrm{perf}}(X)$. But all of the applications will be for smooth and projective varieties, hence we will phrase them for regular schemes and explicitly state this condition each time, to make the notation consistent throughout.

**Projective bundle formula**

In [177] it was realised that the resolution of the diagonal used in the proof of theorem 1.20 to study the exceptional collection on $\mathbb{P}^n$ also works in a relative situation, i.e. for a projective bundle over a base scheme $S$. The absolute case is then obtained by taking $S = \operatorname{Spec} k$.

**Theorem 1.34** (Orlov)**.** Let $S$ be a regular locally noetherian scheme. Let $\mathcal{E}$ be a locally free sheaf of rank $n+1$ on $S$. Consider the projective bundle $f\colon \mathbb{P}(\mathcal{E}) \to S$.

Then $\mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathcal{E}))$ has a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathcal{E})) = \left\langle \Phi_0(\mathbf{D}^{\mathrm{b}}(S)), \Phi_1(\mathbf{D}^{\mathrm{b}}(S)), \ldots, \Phi_n(\mathbf{D}^{\mathrm{b}}(S)) \right\rangle \tag{1.29}$$

where the functors $\Phi_i$ are given by

$$\Phi_i\colon \mathbf{D}^{\mathrm{b}}(S) \to \mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathcal{E})) : \mathcal{F}^\bullet \mapsto \mathbf{L}f^*(\mathcal{F}^\bullet) \otimes_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}} \mathcal{O}_{\mathbb{P}(\mathcal{E})/S}(i). \tag{1.30}$$

**Brauer–Severi schemes**

It is moreover possible to consider *twisted* projective bundles, or equivalently Brauer–Severi schemes [36]. Associated to an Azumaya algebra $\mathcal{A}$ there is a class $\alpha \in \mathrm{Br}(S)$ and a Brauer–Severi scheme $\mathrm{BS}(\mathcal{A}) \to S$.

The class $\alpha$ can be interpreted as an element of $\mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_{\mathrm{m}})_{\mathrm{tors}}$, and in this way it defines a twist of the cocycle condition in the definition of coherent sheaves using descent data. This allows us to define the category $\mathrm{coh}(S,\alpha)$ of twisted sheaves on $S$, and its derived category $\mathbf{D}^{\mathrm{b}}(S,\alpha)$. Translating this back to Azumaya algebras, there exists an equivalence

$$\mathbf{D}^{\mathrm{b}}(S,\alpha) \cong \mathbf{D}^{\mathrm{b}}(S,\mathcal{A}), \tag{1.31}$$

where the latter category is the bounded derived category of coherent left $\mathcal{A}$-modules [111, §2.13].

**Theorem 1.35** (Bernardara)**.** Let $S$ be a regular locally noetherian scheme. Let $\mathcal{A}$ be an Azumaya algebra on $S$ of degree $n+1$, with associated class $\alpha$ in the Brauer group. Consider the Brauer–Severi scheme $f\colon \mathrm{BS}(\mathcal{A}) \to S$.

Then $\mathbf{D}^{\mathrm{b}}(\mathrm{BS}(\mathcal{A}))$ has a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\mathrm{BS}(\mathcal{A})) = \left\langle \Phi_0(\mathbf{D}^{\mathrm{b}}(S)), \Phi_1(\mathbf{D}^{\mathrm{b}}(S,\alpha)), \Phi_2(\mathbf{D}^{\mathrm{b}}(S,2\alpha)), \ldots, \Phi_n(\mathbf{D}^{\mathrm{b}}(S,n\alpha)) \right\rangle, \tag{1.32}$$

where the functors $\Phi_i$ are given by

$$\Phi_i\colon \mathbf{D}^{\mathrm{b}}(S,i\alpha) \to \mathbf{D}^{\mathrm{b}}(\mathrm{BS}(\mathcal{A})) : \mathcal{F}^\bullet \mapsto \mathbf{L}f^*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{\mathrm{BS}(\mathcal{A})/S}(i), \tag{1.33}$$

and $\mathcal{O}_{\mathrm{BS}(\mathcal{A})/S}(i)$ is the $f^*(-i\alpha)$-twisted sheaf associated to the twisted projective bundle.

In appendix B we will use this semiorthogonal decomposition in the special case where $S = \operatorname{Spec} k$ to understand the categorical representability of central simple algebras and Brauer–Severi varieties.

### Blowup formula

Because the exceptional divisor in a blowup situation is a projective bundle over the subscheme which is being blown up (where the locally free sheaf to construct the projective bundle is given by the normal bundle), it is possible to use theorem 1.34 to construct a semiorthogonal decomposition of a blowup.

Observe though that this is not done in the most naive way possible, because the closed immersion of the exceptional divisor does *not* give a fully faithful functor on the level of derived categories. This functor is only fully faithful when restricted to the essential image of the pullback along the projection of the projective bundle, or twists of it by the relative $\mathcal{O}(1)$. The discrepancy is explained by realising that it is important to keep track in which ambient category one is computing the category generated by given subcategories. The version quoted here, where $Y$ is only taken to be lci and not smooth, is proved in [140, theorem 3.4].

**Theorem 1.36** (Orlov)**.** Let $i\colon Y \to X$ be a lci subvariety of codimension $c$ inside a smooth and quasiprojective variety $X$. Let $p\colon \operatorname{Bl}_Y X \to X$ be the blowup, and denote the exceptional divisor by $E$. In other words, we are considering the fibre square

$$\begin{array}{ccc} E \coloneqq \mathbb{P}(\mathrm{N}_Y X) & \xrightarrow{j} & \operatorname{Bl}_Y X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & X \end{array} . \tag{1.34}$$

Then $\mathbf{D}^{\mathrm{b}}(\operatorname{Bl}_Y X)$ has a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\operatorname{Bl}_Y X) = \left\langle \mathbf{L}p^*(\mathbf{D}^{\mathrm{b}}(X)), \Phi_0(\mathbf{D}^{\mathrm{b}}(Y)), \Phi_1(\mathbf{D}^{\mathrm{b}}(Y)), \ldots, \Phi_{c-2}(\mathbf{D}^{\mathrm{b}}(Y)) \right\rangle \tag{1.35}$$

where

$$\Phi_i\colon \mathbf{D}^{\mathrm{b}}(Y) \to \mathbf{D}^{\mathrm{b}}(\operatorname{Bl}_Y X) : \mathcal{F}^\bullet \mapsto i_*(\mathbf{L}p^*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_E} \mathcal{O}_{E/Y}(i)). \tag{1.36}$$

In chapter 2 we will use this semiorthogonal decomposition to construct some explicit embeddings of finite-dimensional algebras into the derived categories of surfaces. In chapter 5 we will give a noncommutative generalisation of this result, where we will replace the structure $\mathcal{O}_X$ by a sufficiently nice sheaf of noncommutative algebras $\mathcal{A}$, together with the induced sheaves of algebras on $Y$, $\operatorname{Bl}_Y X$ and $E$.

### Fibrations in quadrics

Just like theorem 1.34 is a relative version of theorem 1.20, it is possible to find a relative version of theorem 1.26. The setting for this is that of (flat) fibration in quadrics, i.e. a morphism $X \to S$ such that $X$ can be realised as a divisor of relative degree 2 in $\mathbb{P}_S(\mathcal{E})$, associated to a line bundle $\mathcal{S} \subseteq \operatorname{Sym}^2 \mathcal{E}^\vee$, where $\mathcal{E}$ a locally free sheaf of rank $n + 2$ on $S$.

From this data it is possible to define the *sheaf of even parts of Cifford algebras*, which as an $\mathcal{O}_S$-module is

$$\mathrm{Cl}_0(X \to S) := \bigoplus_{i\geq 0} \bigwedge^{2i} \mathcal{E} \otimes \mathcal{L}^{\otimes i} \tag{1.37}$$

with the algebra structure arising from a relative version of the Clifford multiplication. Then the associated semiorthogonal decomposition is the following.

**Theorem 1.37** (Kuznetsov)**.** Let $f\colon X \to S$ be a flat quadric fibration of relative dimension $n$. Then $\mathbf{D}^{\mathrm{b}}(X)$ has a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \left\langle \mathbf{D}^{\mathrm{b}}(S, \mathcal{C}l_0), \Phi_0(\mathbf{D}^{\mathrm{b}}(S)), \Phi_1(\mathbf{D}^{\mathrm{b}}(S)), \dots, \Phi_{n-1}(\mathbf{D}^{\mathrm{b}}(S)) \right\rangle \tag{1.38}$$

where

$$\Phi_i\colon \mathbf{D}^{\mathrm{b}}(S) \to \mathbf{D}^{\mathrm{b}}(X) : \mathcal{F}^\bullet \mapsto \mathbf{L}f^*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{X/S}(i) \tag{1.39}$$

and $\mathcal{C}l_0$ is the sheaf of even parts of Clifford algebras associated to the quadric fibration.

The sheaf of algebras $\mathrm{Cl}_0$ has an explicit description depending on the parity of $n$. If $n$ (which is also the dimension of the fibers of $X \to S$) is odd, then $\mathrm{Cl}_0$ is an order whose Azumaya locus is the complement of the discriminant. If $n$ is even, then there exists the central subalgebra $\mathcal{O}_S \oplus \bigwedge^n \mathcal{E} \otimes \mathcal{L}^{n/2}$, and which can be used to construct a degree-two cover of $S$, as in the description of the central Proj of a noncommutative plane which is recalled in chapter 5. These Clifford algebras will appear in chapter 6, where they are used to construct a derived category with certain favorable properties.

Observe that in the situation of example 1.23 we have that $n$ is even, so we construct a degree 2 cover of $\operatorname{Spec} k$, which accounts for two of the exceptional objects in the collection.

### Fully faithful functors versus moduli spaces

By taking the appropriate orthogonal it is of course possible to construct a semiorthogonal decomposition starting from an admissible subcategory. In this case there is not necessarily an explicit description of the orthogonal, but it does allow one to relate the two categories, e.g. by comparing their additive invariants, or their Hochschild cohomology.

One important example of this situation, which motivates the results in chapter 3 is that of a smooth projective surface versus its Hilbert scheme of points. It uses Fourier–Mukai functors, which are a very important tool in the study of derived categories of smooth projective varieties [109].

**Theorem 1.38** (Krug–Sosna)**.** Let $S$ be a smooth projective surface such that the irregularity and geometric genus are zero (i.e. $\mathrm{H}^1(S, \mathcal{O}_S) = \mathrm{H}^2(S, \mathcal{O}_S) = 0$). Then the functor

$$\Phi_{\mathcal{I}_n}\colon \mathbf{D}^{\mathrm{b}}(S) \to \mathbf{D}^{\mathrm{b}}(\operatorname{Hilb}^n S) \tag{1.40}$$

is fully faithful, where $\mathcal{I}_n$ is the universal ideal sheaf on $S \times \operatorname{Hilb}^n S$.

Another interesting situation where there is a relationship between the derived category of a variety and the derived category of an appropriate moduli space is for curves and their moduli of vector bundles [82].

Historically the first example of this phenomenon is that of K3 surfaces, where the functor is not just fully faithful, but actually a derived equivalence with another (twisted) K3 surface [164]. The moduli space in question is that of stable sheaves, and if it is smooth of dimension 2 it is indeed again a K3 surface. It moreover turns out that all derived equivalences of K3 surfaces are of this type [108].

### Other examples

The list of semiorthogonal decompositions is much longer than the ones we have discussed so far. We have limited ourselves to those semiorthogonal decompositions which are relevant for the purpose of this thesis. In particular we have not discussed semiorthogonal decompositions for Fano varieties [145, §5.3, §5.4] and cyclic coverings [146], or the role that homological projective duality plays in constructing new semiorthogonal decompositions [143]. These are not needed as such in this thesis, but it is interesting to remark that in all these cases there appear sheaves of algebras and geometric t-structures, just like we will study.

### Phantom categories

In general it is a difficult question to show that a given collection of semiorthogonal subcategories, or an exceptional collection, generates a triangulated category. It was hoped that, at least in the setting of derived categories in algebraic geometry, it would be enough to show that the subcategory generated by the semiorthogonal subcategories or exceptional collection is indeed the whole category by checking that they generate the entire Hochschild homology or Grothendieck group [142, conjecture 9.1].

It turns out that this conjecture is false [3, 42, 89]. There exist (non-zero) admissible subcategories of the derived category of a smooth projective variety, for which additive invariants (almost) vanish.

**Definition 1.39.** We say that $\mathcal{T}$ is a *quasi-phantom* if $\mathrm{K}_0(\mathcal{T})$ is torsion and $\mathrm{HH}_\bullet(\mathcal{T}) = 0$. It is a *phantom* if moreover $\mathrm{K}_0(\mathcal{T})$ is actually zero.

The existence of these (quasi)phantoms makes it hard to check that a semiorthogonal decomposition describes the entire triangulated category, and no general recipe exists.

### Non-example: Calabi–Yau varieties

Of course, it is also important to know when there *does not exist* a semiorthogonal decomposition at all. The easiest such example is given by Calabi–Yau varieties [137, §5.1].

More generally there is now the result of Kawatani–Okawa [118], which gives a relation between properties of the canonical line bundle $\omega_X$ and the indecomposability of the derived category. This is certainly not the only property of $\mathbf{D}^{\mathrm{b}}(X)$ which can be deduced from properties of $\omega_X$, see section 1.4.2.

**Theorem 1.40** (Kawatani–Okawa)**.** Let $X$ be a smooth projective variety such that the base locus of the linear system $\mathrm{H}^0(X, \omega_X)$ is a finite set. Then $\mathbf{D}^{\mathrm{b}}(X)$ does not have a semiorthogonal decomposition.

In particular, in the case of a globally generated canonical bundle there exist no semiorthogonal decompositions.

**Non-example: curves of non-negative genus**

There is also the earlier result on the indecomposability of the derived category of a smooth projective curve, due to Okawa [173, theorem 1.1]. This is now a corollary of theorem 1.40, because either $\omega_C = \mathcal{O}_C$ if $g = 1$, or $\omega_C$ is ample (hence globally generated) if $g \geq 2$.

**Theorem 1.41** (Okawa)**.** Let $C$ be a smooth projective curve. Then $\mathbf{D}^{\mathrm{b}}(C)$ doesn't have a semiorthogonal decomposition, unless $g = 0$.

If $C \cong \mathbb{P}^1$, then we have discussed its semiorthogonal decomposition in example 1.21. The only other possible case (if we are working over a non-algebraically closed field) is a non-split conic, but in that case we can use theorem 1.35, as such a curve is the Brauer–Severi variety of a non-split central simple algebra of degree 2.

## 1.3 Invariants for triangulated and dg categories

Besides trying to completely describe the structure of a triangulated or dg category by using semiorthogonal decompositions, it is also possible to probe its structure using invariants, just like we do for algebras or schemes. We can use these to understand the structure if a complete description is not feasible, compare two categories using functoriality, see how invariants change if we modify the category, etc.

Many definitions of these invariants for dg categories are immediately extended from those for algebras (using a suitable modification). For the case of schemes there are often (sometimes several) geometric ways of defining similar invariants, and again we have many agreement results which will not be reviewed here.

A natural setting to study (additive) invariants for dg categories is that of noncommutative motives [209]. Just like ordinary motives are a way of decomposing smooth projective varieties into pieces which are relevant to the study of their invariants (in this case, their Weil cohomologies), noncommutative motives are a way of decomposing (sufficiently nice) dg categories into easier pieces, which are responsible for the different pieces of the invariant for the whole category. There is actually an interesting interplay between the Chow motives for smooth projective varieties and the noncommutative motives of their derived categories [38, 175], but we will not need this here.

In this section we will review the 3 invariants which are important for this thesis: algebraic K-theory, Hochschild homology and Hochschild cohomology. They make an appearance in all but one of the chapters. There are many more [209], but we will not need these.

Observe that, for their definition, the structure of a triangulated category is *not* enough (except for the lowest degrees of algebraic K-theory). Hence we will always work with dg categories, and implicitly an enhancement is used whenever this is necessary. Also, there are various ways of defining each invariant, but this section is not an exhaustive comparison of their definitions.

### 1.3.1 Algebraic K-theory

We will not go into details about the actual definition of algebraic K-theory: it is technical, and not relevant to this thesis. Moreover it turns out that we will only need $\mathrm{K}_0$ and $\mathrm{K}_1$. Rather we will use it to explain the notion of an additive invariant, and explain how we can use semiorthogonal decompositions to compute invariants of a dg category.

#### Additive invariants

Many invariants of dg categories are compatible in a strong sense with the semiorthogonal decompositions from section 1.2. In particular this will be the case for algebraic K-theory and Hochschild homology.

**Definition 1.42.** Let $\mathbb{D}$ be an additive category. A functor $E\colon \mathrm{dgCat}_k \to \mathbb{D}$ is an *additive invariant* if

1. it is Morita invariant, i.e. it sends Morita equivalences of dg categories to isomorphisms;
2. if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ are dg categories for which we have a semiorthogonal decomposition $\mathrm{H}^0(\mathcal{C}) = \langle \mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{B})\rangle$, then the inclusion functors induce an isomorphism

   (1.41) $E(\mathcal{C}) \cong E(\mathcal{A}) \oplus E(\mathcal{B})$.

There is also the notion of a localising invariant, but we will not need it for the purpose of this thesis.

In particular, if $\mathcal{T}$ has a full exceptional collection of length $n$, then $E(\mathcal{T}) = E(k)^{\oplus n}$. In the case of algebraic K-theory we have that $\mathrm{K}_0(k) \cong \mathbb{Z}$ using the dimension function, whilst for Hochschild homology we get that $\mathrm{HH}_\bullet(k) = k$ sitting in degree 0.

In general it is *very* hard to compute algebraic K-theory, even for smooth projective varieties. In this thesis we will only need the information in degree 0 and 1, and often we even restrict ourselves to the numerical Grothendieck group.

#### The Grothendieck group

The easiest part of algebraic K-theory is given by the Grothendieck group. Unlike the full algebraic K-theory, this can be defined on the level of triangulated categories without using an enhancement.

**Definition 1.43.** The *Grothendieck group* $\mathrm{K}_0(\mathcal{T})$ of a triangulated category $\mathcal{T}$ is the free abelian group generated by the isomorphism classes of the objects, modulo the relations induced by distinguished triangles, i.e. if $T_1 \to T_2 \to T_3 \to$ is a distinguished triangle, then

(1.42) $[T_2] = [T_1] + [T_3]$.

The construction of the category of noncommutative motives, and that of the universal additive invariant (which is the universal functor towards this category) uses Grothendieck groups for its construction. As a first step it is easy to see that a semiorthogonal decomposition induces a direct sum decomposition, hence $\mathrm{K}_0$ is indeed an additive invariant. For more information one is referred to [209].

For a scheme the Grothendieck group is closely related to the Chow group, by the Grothendieck–Riemann–Roch theorem. This is used in chapter 2 to describe special properties of the action of the Serre functor, when working with a smooth projective surface.

Another way in which this connection with the Chow group plays a prominent role is in chapter 8, where a notion of Chow groups for tensor triangulated categories is studied. These are defined using $\mathrm{K}_0$ and $\mathrm{K}_1$, and turn out to agree with the usual Chow groups in the scheme-theoretical setting.

A more tractable version of the Grothendieck group (but which also contains less information) is given by the following definition. It uses the notion of a Serre functor which will be introduced in section 1.4.2.

**Definition 1.44.** Let $X$ be a smooth projective variety. The *numerical Grothendieck group* $\mathrm{K}_0^{\mathrm{num}}(X)$ is the quotient of $\mathrm{K}_0(X)$ by the left radical of the Euler form (which by Serre duality agrees with the right radical), i.e. one mods out the subgroup of $\mathrm{K}_0(X)$ defined by

$$\chi(-,\mathrm{K}_0(X)) = \chi(\mathrm{K}_0(X),-) = 0. \tag{1.43}$$

By Grothendieck–Riemann–Roch we have that $\mathrm{K}_0^{\mathrm{num}}(X) \cong \mathbb{Z}^{\oplus n}$ for some $n$. It is also possible to define this for any triangulated category with a Serre functor, but observe that for the derived category of a finite-dimensional algebra the Jordan–Hölder decomposition already gives us that $\mathrm{K}_0(A) \cong \mathbb{Z}^{\oplus n}$.

### 1.3.2 Hochschild homology

Another additive invariant which appears in this thesis is Hochschild homology. Originally defined for associative algebras using an explicit chain complex [153] it can be extended to dg categories using the interpretation of Hochschild homology as derived functor of the tensor product in the category of bimodules, or via a generalisation of the Hochschild complex. It can be shown to be additive, and in the case of sufficiently nice schemes one can compute Hochschild homology using sheaf cohomology as in proposition 1.46. Just like for the Grothendieck group, we will usually only need Hochschild homology in settings where it is nearly trivial to describe, e.g. in the presence of a full and strong exceptional collection.

The way in which Hochschild homology plays an important role is through an appropriate notion of the Lefschetz trace formula, as in chapter 4, where the Hochschild cohomology as defined in the next paragraph is related to Hochschild homology with values in an appropriate bimodule given by the inverse of the Serre functor.

### 1.3.3 Hochschild cohomology

An intriguing invariant, which plays an important role in this thesis, is Hochschild cohomology. Unlike the previous two (and many others not mentioned here), it is *not* an additive invariant, let alone that it satisfies functoriality. Classically, its main use was in Gerstenhaber's deformation theory of associative algebras and it was defined using the Hochschild cochain complex, and this has been extended to abelian and dg categories [124, 155]. Another way of defining Hochschild cohomology is as the self-extensions of the diagonal bimodule, i.e. the extensions of the category considered as a bimodule over itself inside the derived category of the enveloping category.

**Limited functoriality**

An important shortcoming of Hochschild cohomology is that it is not functorial: given an arbitrary functor $F\colon \mathcal{C} \to \mathcal{D}$ of dg categories, there is no way of constructing an associated morphism $\mathrm{HH}^\bullet(F)\colon \mathrm{HH}^\bullet(\mathcal{D}) \to \mathrm{HH}^\bullet(\mathcal{C})$ with the desired properties.

But Hochschild cohomology *is* functorial for a restricted class of functors [120, 156]. The version in which we will use it throughout this thesis is the following.

**Example 1.45.** If we have a (dg enhancement of) a semiorthogonal decomposition

$$\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle, \tag{1.44}$$

then we get induced morphisms

$$\begin{aligned} \mathrm{HH}^\bullet(\mathcal{C}) &\to \mathrm{HH}^\bullet(\mathcal{A}), \\ \mathrm{HH}^\bullet(\mathcal{C}) &\to \mathrm{HH}^\bullet(\mathcal{B}) \end{aligned} \tag{1.45}$$

and long exact sequences relating the Hochschild cohomologies to each other. The problem is that the third term in such a sequence is related to the properties of the gluing functor, and it is often not very feasible to do explicit computations with this.

It turns out that in the geometric setting one does not need to use dg enhancements: it suffices to use Fourier–Mukai transforms [141].

Using chapter 3 we get a morphism between the Hochschild cohomology of (a deformation of) the Hilbert scheme of 2 points on a plane (resp. quadric) and the (noncommutative) plane (quadric). As explained in section 3.4 there seems to be an interesting compatibility with the Hochschild–Kostant–Rosenberg decomposition as in proposition 1.46, identifying the Hochschild cohomology of the (noncommutative) plane (resp. quadric) with the commutative deformations of the Hilbert scheme. If this is indeed the case, then chapter 4 precisely computes these Hochschild cohomologies.

**Gerstenhaber structure**

The way in which Hochschild cohomology governs deformation theory is by using the extra structure which exists on the Hochschild cohomology (and the Hochschild complex). This gives rise to a so called *noncommutative differential calculus* [211]. The reason for this name will become clear from proposition 1.46, which (in a suitable version) will identify Hochschild homology (resp. cohomology) of the derived category of a smooth variety $X$ with the differential forms (resp. multivector fields), and their associated pairings.

**Hochschild–Kostant–Rosenberg**
As hinted at before when discussing the noncommutative differential calculus associated to Hochschild (co)homology, if the dg category is of geometric origin then it is possible to *compute* Hochschild (co)homology in a sheaf-theoretical way. There exist various proofs of this result in the literature [86, 103, 208], we will give the following version [239, corollary 0.6].

**Proposition 1.46** (Hochschild–Kostant–Rosenberg)**.** Let $X$ be a smooth variety over a field of characteristic zero or $p > \dim X$. Then there exists an isomorphism of vector spaces

$$\text{(HKR-1)}\quad \mathrm{HH}^i(X) \cong \bigoplus_{p+q=i} \mathrm{H}^p(X, \bigwedge^q \mathrm{T}_X).$$

Likewise for Hochschild homology we have an isomorphism

$$\text{(HKR-2)}\quad \mathrm{HH}_i(X) \cong \bigoplus_{q-p=i} \mathrm{H}^p(X, \Omega_X^q).$$

Especially when $X$ is moreover projective, we obtain that the Hochschild (co)homology is finite-dimensional, and we can explicitly compute its dimensions. This will be used for the easy cases of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ in chapter 4, whilst in appendix A we will use a version of the Hochschild–Kostant–Rosenberg theorem which also incorporates the Gerstenhaber algebra structure on $\mathrm{HH}^\bullet(X)$ to compute this explicitly for $\mathrm{HH}^\bullet(\mathbb{P}^n)$.

## 1.4 Smooth and proper dg categories

A particularly well-behaved class of dg categories is that of smooth and proper dg categories. These notions were originally introduced by Kontsevich, as an analogue of the particularly well-behaved class of schemes given by the smooth projective varieties [130, 220]. They are the type of dg categories which feature prominently in chapters 2 to 6 and appendices A and B, so some discussion is warranted.

### 1.4.1 Definition

We will quickly recall some of the important definitions and properties, but for a more thorough discussion of the subject one is referred to [178].

**Definition 1.47.** A dg category $\mathcal{C}$ is *smooth* if it is a perfect $\mathcal{C}^{\mathrm{op}} \otimes_k \mathcal{C}$-module, i.e. the identity functor is perfect when considered as a bimodule.

That the definition makes sense, i.e. that the notion of smoothness coincides with that in algebraic geometry, follows from [157, proposition 3.13] for perfect fields, and [178, proposition 3.31] for arbitrary fields.

**Proposition 1.48.** Let $X$ be a separated scheme of finite type over $k$. Then $X$ is smooth over $k$ if and only if $\mathbf{D}^{\mathrm{perf}}(X)$ is smooth as a dg category.

**Remark 1.49.** There is also the (possibly counterintuitive) result that $\mathbf{D}^{\mathrm{b}}(X)$ is always smooth as a dg category, regardless of the smoothness of $X$ itself [157]. In this thesis we will only consider smooth projective varieties, where $\mathbf{D}^{\mathrm{b}}(X) \cong \mathbf{D}^{\mathrm{perf}}(X)$, so no confusion is possible.

The second important notion, that of properness, does not depend on the differential graded structure, only on the properties of the associated triangulated category.

**Definition 1.50.** A triangulated category $\mathcal{C}$ is *proper* if for all $X, Y \in \mathcal{C}$ we have

$$\dim_k \left( \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y[n]) \right) < +\infty. \tag{1.46}$$

By the finiteness for coherent cohomology of proper schemes we get the following result.

**Proposition 1.51.** Let $X$ be a separated scheme of finite type over $k$. Then $X$ is proper over $k$ if and only if $\mathbf{D}^{\mathrm{perf}}(X)$ is proper as a triangulated category.

Besides smooth projective varieties, we also have that finite-dimensional algebras of finite global dimension give rise to smooth and proper dg categories. These are the two guiding examples in this thesis.

Another way of characterising smooth and proper dg categories is by using the tensor product of dg categories: it turns out that smooth and proper dg categories are precisely the dualisable objects in the homotopy category for the Morita model structure [64, §5].

One of the important properties that will be used throughout this thesis is that for smooth and proper dg categories many invariants satisfy various finiteness properties [130, §8]. In particular if $\mathcal{C}$ is a smooth and proper dg category, then

$$\dim_k \left( \bigoplus_{n\in\mathbb{Z}} \mathrm{HH}_n(\mathcal{C}) \right) < +\infty \tag{1.47}$$

and

$$\dim_k \left( \bigoplus_{n\in\mathbb{Z}} \mathrm{HH}^n(\mathcal{C}) \right) < +\infty. \tag{1.48}$$

Similar results are true for various other invariants [209]. Observe that for smooth projective varieties, this type of results follows from the Hochschild–Kostant–Rosenberg decomposition, which expresses the invariants in terms of the cohomology of coherent sheaves, which is indeed finite-dimensional.

An invariant which does not necessarily have nice finiteness properties, even for smooth projective surfaces, is the Grothendieck group (and hence more generally algebraic K-theory). There exist smooth projective surfaces for which it cannot be described as an abelian variety [166], and already for curves it is infinitely generated as an abelian group. We will usually work with smooth projective varieties with a full and strong exceptional collection, in which case there is nothing to worry about, as the Grothendieck group is necessarily $\mathbb{Z}^{\oplus n}$ for suitable $n$.

We remark that the results of chapter 2 are obtained by considering the numerical Grothendieck group (which is always finitely generated by the Grothendieck–Riemann–Roch theorem), hence these results also work for surfaces with infinitely generated Grothendieck group.

A final fact to state is that every smooth and proper dg category is actually Morita equivalent to a smooth and proper dg algebra. So we have a special case of the derived affineness for the class of dg categories we are most interested in.

### 1.4.2 Serre functors

In [47, §3] Bondal–Kapranov have introduced the notion of Serre functor for a triangulated category. It is the categorical incarnation for a triangulated category of Serre duality for a smooth projective variety, see example 1.53. One can show that a (or rather: the) Serre functor always exists for a smooth and proper dg category, and it is an important tool in studying the structure of derived categories. It is used in chapters 2 and 4 and its special properties for the derived category of a smooth projective surface motivate the construction and comparison in chapters 5 and 6.

**Definition 1.52.** Let $\mathcal{T}$ be a Hom-finite triangulated category. A *Serre functor* $\mathbb{S}$ for $\mathcal{T}$ is an additive equivalence for which there are given bifunctorial isomorphisms

$$\varphi_{E,F}\colon \operatorname{Hom}_{\mathcal{T}}(E,F) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{T}}(F,\mathbb{S}(E))^{\vee} \tag{1.49}$$

for all $E, F \in \mathcal{T}$.

If a Serre functor exists, it is automatically exact [47, proposition 3.3]. It is moreover unique up to isomorphism [47, proposition 3.4(a)].

**Example 1.53.** The name Serre functor comes from the notion of Serre duality for smooth projective varieties. Indeed, Serre duality (or rather its generalisation to Grothendieck duality, applied to smooth projective varieties) tells us that

$$\mathbb{S} \cong - \otimes \omega_{X/k}[\dim X] \tag{1.50}$$

is the Serre functor for $\mathbf{D}^{\mathrm{b}}(X)$.

**Example 1.54.** In the representation theory of finite-dimensional algebras, the Serre functor is known as the *Nakayama functor*, or rather its derived version. For a finite-dimensional $k$-algebra $A$ of finite global dimension we get that

$$\mathbb{S} \cong \operatorname{Hom}_k(\mathbf{R}\operatorname{Hom}_A(-,A),k), \tag{1.51}$$

is the Serre functor for $\mathbf{D}^{\mathrm{b}}(A)$, i.e. it is the composition of two dualisation functors: one over $k$, and the other over $A$.

There is an important distinction between the two descriptions: for a smooth projective variety and a coherent sheaf, we see that we end up with another coherent sheaf, but shifted by $\dim X$. For an $A$-module there is nothing that guarantees that we end up with a shifted $A$-module. This will be used in section 1.4.4.

### 1.4.3 Geometric dg categories

In section 1.3 we have seen several invariants of dg categories for which there exists both a categorical definition, and a more geometric definition or interpretation in case the category is of geometric origin. Often it is easier to prove a result in this geometric setting, because of the existence of extra tools. As an example we can give Kontsevich and Soibelman's conjecture regarding the degeneration of the Hochschild-to-cyclic spectral sequence for an arbitrary smooth and proper dg algebra over a field of characteristic 0 [130, §9]. In the geometric setting this spectral sequence is the Hodge-to-de Rham spectral sequence, by using a suitable version of the Hochschild–Kostant–Rosenberg decomposition for cyclic homology. An algebro-geometric proof for this degeneration can be found in [72]. In the noncommutative setting on the other hand, a proof for all cases was only found recently [113], extending the earlier result for dg algebras in positive degree [112].

Another example of such a phenomenon is the lattice conjecture for topological K-theory [41], which is an important ingredient in the homological mirror symmetry program.

Now if a statement regarding invariants of dg categories is compatible with semiorthogonal decompositions, then it becomes possible to bootstrap a proof in the geometric setting to the noncommutative setting, provided that the dg category is a semiorthogonal component of the derived category of a scheme.

This brings us to the important question of which (smooth and proper) dg categories can be realised as admissible subcategories of the derived category of smooth projective varieties. This question was raised by Orlov in [178], and answered positively in the special case where the smooth and proper dg category is actually concentrated in degree 0.

**Theorem 1.55** (Orlov)**.** For a finite-dimensional $k$-algebra $A$ of finite global dimension there is an admissible embedding

$$\mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(X) \tag{1.52}$$

into the derived category of a smooth projective variety $X$.

The construction in this theorem is non-canonical: it proceeds by first considering the Auslander algebra of the given algebra $A$, whose derived category contains $\mathbf{D}^{\mathrm{b}}(A)$, and which has an exceptional collection. Then he provides a construction using iterated projective bundles which inductively realises the derived category of the Auslander algebra (or any category admitting an exceptional collection) inside the derived category of a smooth projective variety, by choosing explicit representatives of the gluing functors. There is no interpretation of the constructed variety in terms of the algebra, given how many choices were involved. In chapter 3 we construct a fully faithful embedding of the derived category of a finite-dimensional algebra arising from noncommutative algebraic geometry in a geometrically meaningful way, by deforming a fully faithful functor obtained from algebraic geometry.

Moreover we have that the constructed variety is high-dimensional, even if one tries to optimise the choices in the construction. This was the original motivation for the question which led to chapter 2, where the case of an embedding in the derived category of a smooth projective surface is studied. This is the first non-trivial case, and there are strong obstructions to the existence of such an embedding.

**Remark 1.56.** It is also interesting to note that the $X$ which is constructed admits a full and strong exceptional collection. That such an embedding (into a non-geometric category) exists is also implicitly contained in Iyama's paper [110], which shows that there always exists a quasi-hereditary algebra $\Lambda$ and an idempotent $e$ such that $A = e\Lambda e$, so there is an embedding $\mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\Lambda)$. Such an algebra $\Lambda$ always admits a full exceptional collection, provided by the standard modules $\Delta(\lambda)$ coming from the quasi-hereditary structure.

### 1.4.4 Geometric t-structures

Despite very similar titles, there is an important difference between this section and the previous: we will now consider t-structures on triangulated categories, trying to describe whether a triangulated category is of geometric origin. Whereas the notion of geometric dg category is a property (namely the existence of an admissible embedding), we will now define a structure where different structures correspond to different ways of being geometric.

In [45] Bondal introduced a program to study triangulated categories with properties similar to those of smooth and projective varieties (just like Orlov's), but where the idea is to put sufficiently many restrictions on a triangulated category, so that their classification problem takes on a form similar to that of objects in algebraic geometry. In particular there should only be finite-dimensional families of these objects. For example, all noncommutative planes and quadrics in chapters 3 and 4 come equipped with geometric t-structures, but there is only one truly geometric (i.e. commutative) such surface. And in chapter 5 finite-dimensional families of geometric triangulated categories are constructed for which there are no commutative examples.

The notion of "geometric" triangulated category imposes a compatibility between the Serre functor and some t-structure, suggesting that the triangulated category can be considered as the derived category of an abelian category in which a form of Serre duality holds. This mimicks the situation for $\mathbf{D}^{\mathrm{b}}(X)$ where $X$ is smooth and projective. First we have to define what a t-structure is [28, §1.3].

**Definition 1.57.** A *t-structure* $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$ consists of two strictly full subcategories $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ of $\mathcal{T}$, such that if we denote $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$ we have

1. $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$,
2. $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 0} \supseteq \mathcal{T}^{\geq 1}$,
3. for every object $T \in \mathcal{T}$ there exists a distinguished triangle

   (1.53) $T^{\leq 0} \to T \to T^{\geq 1} \to$

   where $T^{\leq 0} \in \mathcal{T}^{\leq 0}$ and $T^{\geq 1} \in \mathcal{T}^{\geq 1}$.

t-structures are used to generalise the cohomology objects for objects of the derived category of an abelian category. Moreover, the *heart* $\mathcal{T}^{\geq 0} \cap \mathcal{T}^{\leq 0}$ of the t-structure is an abelian category, but this will not play an important role here.

**Example 1.58.** The *standard t-structure* on the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is obtained by setting

$$(1.54)\quad \begin{aligned} \mathbf{D}(\mathcal{A})^{\leq 0} &:= \{A^\bullet \mid \forall i > 0\colon\ \mathrm{H}^i(A^\bullet) = 0\}, \\ \mathbf{D}(\mathcal{A})^{\geq 0} &:= \{A^\bullet \mid \forall i < 0\colon\ \mathrm{H}^i(A^\bullet) = 0\}. \end{aligned}$$

In this case the heart is nothing but $\mathcal{A}$ itself.

Following Bondal's suggestion we can now ask for a compatibility between a t-structure and the Serre functor.

**Definition 1.59.** Let $\mathcal{T}$ be a triangulated category equipped with a Serre functor $\mathbb{S}$. Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t-structure on $\mathcal{T}$. We say that it is *geometric*[3] if there exists an integer $n$ such that $\mathbb{S}[-n]$ preserves the t-structure, i.e.

$$(1.55)\quad \begin{aligned} \mathbb{S}(\mathcal{T}^{\leq -n}) &\subseteq \mathcal{T}^{\leq 0}, \\ \mathbb{S}(\mathcal{T}^{\geq -n}) &\subseteq \mathcal{T}^{\geq 0}. \end{aligned}$$

In other words, $\mathbb{S}$ is compatible with the heart of the t-structure.

From the description of the Serre functor for a smooth projective variety $X$ as given in example 1.53 it is clear that the standard t-structure of $\mathbf{D}^{\mathrm{b}}(X)$ together with its standard t-structure is indeed geometric. This motivates the following definition.

**Definition 1.60.** A *noncommutative algebraic variety* is a triangulated category $\mathcal{T}$ equipped with a Serre functor and a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ such that

1. $\mathcal{T}$ is saturated[4];

2. $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a geometric t-structure.

Because the Serre functor (if it exists) is intrinsic to the category, regardless of the way it has been constructed, the existence of a geometric t-structure is a difficult question. In order to check that a triangulated category has a geometric t-structure, one will usually look for an abelian category "of geometric origin" where one understands Serre duality and whose derived category will be equipped with the standard t-structure. This is the motivation for the construction in chapters 5 and 6.

**Remark 1.61.** There exist non-isomorphic smooth projective varieties with equivalent derived categories. These are so called *Fourier–Mukai partners*. The first example of such behaviour is that of dual abelian varieties [163]. So these give different geometric t-structures on the same triangulated category.

[3]In Koszul duality there exists the notion of a geometric t-structure on the category of graded modules over a graded algebra, but this is a different notion.

[4]Which will be the case whenever it comes from a smooth and proper dg category.

It turns out that the Kodaira dimension is an important invariant in deciding whether there are any non-trivial Fourier–Mukai partners. Recall that abelian varieties have Kodaira dimension 0, and there is a whole industry in understanding Fourier–Mukai partners for other varieties for which $\kappa = 0$, in particular K3 surfaces and Calabi–Yau threefolds.

On the other hand, for varieties at the edges of the Kodaira dimension spectrum it can be shown that there are no non-trivial Fourier–Mukai partners [43]. After all, the Kodaira dimension is defined using the canonical bundle, which on the level of triangulated categories is closely related to the Serre functor, so having strong properties for the canonical bundle allows one to recover strong properties of the triangulated category and its Serre functor.

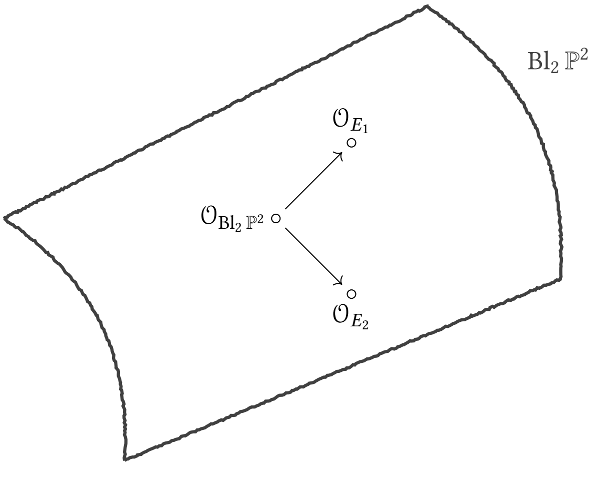
**Example 1.62.** In example 1.21 the derived category of $\mathbb{P}^1$ was described using the path algebra of the Kronecker quiver $\mathsf{K}_2$, where the equivalence $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1) \cong \mathbf{D}^{\mathrm{b}}(k\mathsf{K}_2)$ was induced by the exceptional collection $\langle \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\rangle$.

In this way we get two t-structures: the standard t-structure from $\operatorname{coh}\mathbb{P}^1$ and the standard t-structure from $\operatorname{mod} k\mathsf{K}_2$. The latter is *not* a geometric t-structure. Let us denote $\mathbb{S} \cong - \otimes \mathcal{O}_{\mathbb{P}^1}(-2)[1]$ the Serre functor. Using the interpretation of the equivalence from [119, figure 5.3] we see that $\mathcal{O}_{\mathbb{P}^1}$ corresponds to the indecomposable projective $P_1$. By the Serre functor it gets sent to $\mathcal{O}_{\mathbb{P}^1}(-2)[1]$, which corresponds to $I_1$. But for the Serre functor to be compatible with the t-structure in the appropriate way, we would need an object in $\mathbf{D}^{\mathrm{b}}(k\mathsf{K}_2)^{\geq 1} \cap \mathbf{D}^{\mathrm{b}}(k\mathsf{K}_2)^{\leq 1}$, whereas we actually end up in the heart.

It is also possible to use the formula for the Nakayama functor from example 1.54, relating the computation to Auslander–Reiten triangles.

*Chapter 2*

# Embedding algebras in derived categories of surfaces



## 2.1 Introduction

For an algebraically closed field $k$ of characteristic 0, consider a triangulated category of the form $\mathbf{D}^{\mathrm{b}}(A)$, for some finite-dimensional $k$-algebra $A$ of finite global dimension. In two recent papers, Orlov [176, 178] showed that there always exists an admissible embedding

$$\mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(X), \tag{2.1}$$

for some smooth projective variety $X$.

This construction typically embeds $A$ into a high-dimensional variety, and we consider the existence of an embedding in the derived category of a surface, as the case of curves is completely understood by the indecomposability result of Okawa showing there are no non-trivial embeddings [101]. In particular we give two types of obstructions to the existence of an embedding (2.1) where $X = S$ is a smooth projective surface. These take the form of conditions on the Euler form of the algebra $A$. We summarise these as follows.

**Theorem 2.1.** (see corollaries corollaries 2.10 and 2.11 below) If an embedding

$$\mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(S), \tag{2.2}$$

exists, then $\mathrm{rk}(\chi_A^-) \leq 2$ and $\chi_A^+$ does not admit a 3-dimensional negative-definite subspace.

It is not hard to see that the use of non-commutative motives and additive invariants doesn't yield any strong results. Hence to obtain these results one has to incorporate extra data coming from the Euler form on the triangulated categories, together with an understanding of the structure of the numerical Grothendieck groups of surfaces.

This type of result explains one aspect of the structure of semiorthogonal components of the derived category of a surface. On one hand it is known that there exist (quasi-)phantoms for surfaces, which are components that are very hard to understand [89]. This chapter on the other hand studies certain "easy" components coming from derived categories of finite-dimensional algebras, in particular algebras arising from strong exceptional collections. One interesting result from these obstructions is the non-existence of an exceptional collection of 4 objects whose endomorphism algebra is isomorphic to $k\mathrm{A}_4$.

In section 2.2 we discuss the constraints that are imposed by an embedding (2.1) by using noncommutative motives and incorporating the extra data. In section 2.3 we discuss examples of finite-dimensional algebras that violate the conditions, and we give explicit embeddings for families of algebras that do satisfy the constraints. We also list some open questions on the structure of strong exceptional collections inside derived categories of surfaces raised by the obstructions and examples.

## 2.2 Embedding finite-dimensional algebras into derived categories of surfaces

We now want to discuss properties of $A$ that rule out the existence of an embedding

$$\mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(S), \tag{2.3}$$

for some smooth, projective surface $S$. In section 2.2.1 we consider the (weak) results that one obtains by using noncommutative motives, while we strengthen the results significantly in section 2.2.2 by using extra data.

### 2.2.1 Additive invariants

Looking for restrictions it is natural to start by checking "linear" or additive invariants. An invariant $I(-)$ of a triangulated category is additive if it is additive with respect to semiorthogonal decompositions, i.e.

$$\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle \Rightarrow I(\mathcal{T}) = I(\mathcal{A}) \oplus I(\mathcal{B}). \tag{2.4}$$

Examples of such invariants include algebraic K-theory, nonconnective algebraic K-theory, Hochschild homology, cyclic homology, periodic cyclic homology, negative cyclic homology, topological Hochschild homology and topological cyclic homology, see [210] for some more background.

**Lemma 2.2.** Assuming the existence of an embedding (2.3), for all of the above invariants $I(-)$ one has that $I(\mathbf{D}^{\mathrm{b}}(A))$ is a direct summand of $I(\mathbf{D}^{\mathrm{b}}(S))$.

*Proof.* Since $\mathbf{D}^{\mathrm{b}}(A)$ has a strong generator, it is saturated [51], so (2.3) is admissible and the claim follows from additivity of $I(-)$. □

However, none of these invariants give particularly interesting information with regards to our question, due to the following (corollary of a) result of Keller [122, §2.5] and Tabuada–Van den Bergh [210, corollary 3.20].

**Theorem 2.3.** The additive invariants of a finite-dimensional $k$-algebra of finite global dimension only depend on the number of simple modules.

In other words, additive invariants cannot distinguish between algebras with the same number of simple modules, so they are of limited use for our question. Assuming that a specific variety has an additive invariant that is actually computable, this can give a little information though.

**Example 2.4.** Using the Hochschild–Kostant–Rosenberg decomposition of Hochschild homology, it is easy to see that an embedding (2.3) gives rise to the inequality

$$|Q_0| = \dim_k \mathrm{HH}_0(A) \leq \dim_k \mathrm{HH}_0(S) = 2 + \mathrm{h}^{1,1}, \tag{2.5}$$

where $|Q_0|$ denotes the number of vertices in the quiver of $A$, and $\mathrm{h}^{1,1}$ is the relevant Hodge number.

**Remark 2.5.** Of course, all of the above does not depend on $S$ being a surface and there are obvious generalisations to higher-dimensional varieties.

### 2.2.2 Quadratic invariants

We now give two restrictions on the embeddings of derived categories of finite-dimensional algebras in the derived category of a smooth projective surface $S$. Both results concern the Euler form on the Grothendieck group of such a surface, and are valid for arbitrary surfaces.

Let $X$ again denote a smooth projective variety of dimension $n$. Recall that the bilinear Euler form is defined as

$$\chi\colon\ \mathrm{K}_0(X)\times\mathrm{K}_0(X)\to\mathbb{Z}:(A,B)\mapsto\sum_i(-1)^i\dim_k\mathrm{Ext}^i_X(A,B) \tag{2.6}$$

Moreover one has the natural topological filtration $\mathrm{F}^\bullet$ on $\mathbf{D}^{\mathrm{b}}(X)$ [SGA6, exposé X], where $\mathrm{F}^i\,\mathbf{D}^{\mathrm{b}}(X)$ consists of the complexes of coherent sheaves on $X$ whose cohomology sheaves have support of codimension at least $i$.

Denote by $\mathbb{S}$ the Serre functor on $\mathbf{D}^{\mathrm{b}}(X)$. Recall that this functor is defined as

$$\mathbb{S}(\mathcal{E}^\bullet)=\mathcal{E}^\bullet\otimes\omega_X[n], \tag{2.7}$$

where $\omega_X$ denotes the canonical line bundle on $X$. The Serre functor induces an automorphism on $\mathrm{K}_0(X)$, which we'll denote by the same symbol, and moreover one has

$$\chi(X,Y)=\chi(Y,\mathbb{S}\,X). \tag{2.8}$$

By definition, the filtered piece $\mathrm{F}^i\,\mathrm{K}_0(X)$ is the image of the morphism induced by the inclusion $\mathrm{F}^i\,\mathbf{D}^{\mathrm{b}}(X)\hookrightarrow\mathbf{D}^{\mathrm{b}}(X)$. Using this filtration one proves the following result of Suslin [50, lemma 3.1].

**Lemma 2.6.** The operator $(-1)^n\,\mathbb{S}$ is unipotent on $\mathrm{K}_0(X)$.

To study $\chi$ using linear algebra, we pass to the numerical Grothendieck group, which is better behaved than the usual Grothendieck group in some respects [166].

**Definition 2.7.** The *numerical Grothendieck group* $\mathrm{K}_0^{\mathrm{num}}(X)$ is the quotient of $\mathrm{K}_0(X)$ by left radical of the Euler form (which by Serre duality agrees with the right radical), i.e. one mods out the subgroup of $\mathrm{K}_0(X)$ defined by

$$\chi(-,\mathrm{K}_0(X))=\chi(\mathrm{K}_0(X),-)=0. \tag{2.9}$$

For smooth projective varieties this group is always free of finite rank by the Grothendieck–Riemann–Roch theorem, so from now on we will restrict $\chi$ to $\mathrm{K}_0^{\mathrm{num}}(X)$. A closer inspection of the *antisymmetrisation* of the Euler form

$$\chi^-(A,B)\coloneqq\chi(A,B)-\chi(B,A)=\chi(A,(1-\mathbb{S})B), \tag{2.10}$$

leads to the first observation. The following result was also proved by de Thanhoffer de Völcsey with a different method [71], and later published as [212, proposition B].

**Theorem 2.8.** For a smooth projective surface $S$ one has $\mathrm{rk}\,\chi^-\leq 2$.

*Proof.* By dévissage, we can generate $\mathrm{K}_0(S)$ by $[\mathcal{O}_S]$ and classes $[k_s]$ of skyscrapers for $s \in S$, and structure sheaves of curves $[\mathcal{O}_C]$ for all curves $C$ on $S$, see [SGA6, proposition 0.2.6].

By Grothendieck–Riemann–Roch the Chern character gives an isomorphism

$$\mathrm{K}_0^{\mathrm{num}}(S) \otimes \mathbb{Q} \cong \mathrm{CH}^{\bullet,\mathrm{num}}(S) \otimes \mathbb{Q} \tag{2.11}$$

between the numerical Grothendieck group and algebraic cycles modulo numerical equivalence [138, appendix]. Hence it suffices to consider a single class $[k_s]$ as all points are numerically equivalent [83, §19.3.5], and $\mathrm{CH}^{1,\mathrm{num}}(S)$ is a finitely generated free abelian group of rank $\rho$ [83, example 19.3.1]. Observe that under the isomorphism obtained via the Chern character map, $[k_s]$ is pure of degree 2, $[\mathcal{O}_S]$ is pure of degree 0 and $[\mathcal{O}_C]$ sits in degrees 1 and 2, and the rank of $\mathrm{K}_0^{\mathrm{num}}(S)$ is $\rho + 2$.

We wish to compute the rank of the matrix $\chi^-$ using this choice of basis. For this we need to know the values of $\chi^-(\alpha, \beta)$ for $\alpha \in \mathrm{CH}^i(S)$ and $\beta \in \mathrm{CH}^j(S)$, with $i, j \in \{0, 1, 2\}$. We immediately get that

$$\chi^-([k_s], [k_s]) = 0 \tag{2.12}$$

and using the presentation

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \tag{2.13}$$

we get that

$$\begin{aligned} &\chi^-([\mathcal{O}_C], [k_s]) \\ &\quad = \chi([\mathcal{O}_S], [k_s]) - \chi([k_s], [\mathcal{O}_S]) - \chi([\mathcal{O}_S(-C)], [k_s]) + \chi([k_s], [\mathcal{O}_S(-C)]) \\ &\quad = \chi([\mathcal{O}_S], [k_s]) - \chi([k_s], [\mathcal{O}_S]) - \chi([\mathcal{O}_S], [k_s]) + \chi([k_s], [\mathcal{O}_S]). \\ &\quad = 0 \end{aligned} \tag{2.14}$$

For $i = j = 1$ we have that $C \cdot D = -\chi([\mathcal{O}_C], [\mathcal{O}_D])$, hence $\chi$ is symmetric on this part by the commutativity of the intersection product, therefore it vanishes in the antisymmetric Euler form.

The (skew-symmetric) matrix one obtains is of the form

$$\left(\begin{array}{c|c|c} 0 & \underline{0} & \clubsuit \\ \hline \underline{0} & 0 \cdot \mathrm{id}_\rho & \underline{\spadesuit} \\ \hline \clubsuit & \underline{\spadesuit} & 0 \end{array}\right) \tag{2.15}$$

where we order our generators as $[k_s]$, $[\mathcal{O}_C]$, $[\mathcal{O}_S]$, so there is a block decomposition of a $(\rho + 2) \times (\rho + 2)$-square matrix with some unknown values, but it is of rank $\leq 2$ regardless of the unknowns. ◻

One can also consider the *symmetrised* Euler form

$$\chi^+(A, B) = \chi(A, B) + \chi(B, A). \tag{2.16}$$

This defines a quadratic form on $\mathrm{K}_0^{\mathrm{num}}(X)$, and we can consider its signature, i.e. the tuple $(n_0, n_+, n_-)$ describing the degenerate, positive-definite and negative-definite part of the form. The forms that we consider are non-degenerate over $\mathbb{Q}$ by our restriction to $\mathrm{K}_0^{\mathrm{num}}(X)$, hence it suffices to specify $(n_+, n_-)$.

**Theorem 2.9.** Let $S$ be a smooth projective surface. Then the signature of $\chi^+$ is $(\rho, 2)$.

*Proof.* Similar to the proof of theorem 2.8, we consider the (ordered) basis $[k_s]$, $[\mathcal{O}_S]$, $[\mathcal{O}_C]_C$. The Hodge index theorem [83, example 19.3.1] says that the signature for the intersection product on the curves is $(1, \rho - 1)$. Via the equality $C \cdot D = -\chi(\mathcal{O}_C, \mathcal{O}_D)$, the subspace spanned by the $[\mathcal{O}_C]$ has signature $(\rho - 1, 1)$ for $\chi^+$.

Because

$$\begin{aligned} \chi^+([k_s], [k_s]) &= 0 \\ \chi^+([\mathcal{O}_S], [k_s]) &= 2 \end{aligned} \tag{2.17}$$

using Serre duality and

$$\chi^+([\mathcal{O}_S], [\mathcal{O}_S]) = 2\chi(\mathcal{O}_S) \tag{2.18}$$

is twice the Euler characteristic of $S$, the subspace spanned by $[k_s]$ and $[\mathcal{O}_S]$ is a hyperbolic plane, so it has signature $(1, 1)$.

For the other terms we get that

$$\chi^+([\mathcal{O}_C], [k_s]) = 0 \tag{2.19}$$

just as in (2.14), whilst $\chi^+([\mathcal{O}_C], [\mathcal{O}_S])$ can be arbitrary. Therefore, after a base change, the matrix of $\chi^+$ has the form

$$\left(\begin{array}{c|c|c} 0 & 2 & \underline{0} \\ \hline 2 & 2\chi(\mathcal{O}_S) & \underline{*}^{t} \\ \hline \underline{0} & \underline{*} & D \end{array}\right) \tag{2.20}$$

where $D = \operatorname{diag}(-1, 1, \ldots, 1)$ is a diagonal matrix of size $\rho$. In particular, the decomposition is not orthogonal.

If $\rho = 1$ then the sign of the determinant of the matrix is positive, but there is at least one negative eigenvalue coming from $D$. So the signature must be $(1, 2)$.

If $\rho \geq 2$, denote by $W$ the subspace spanned by the last $\rho - 2$ basis vectors for the choice of basis as in (2.20). The subspace spanned by the vectors $(1, 0, \ldots, 0)$ and $(0, 0, 1, 1, 0, \ldots, 0)$ is a totally isotropic subspace in $W^\perp$.

Since $\chi^+$ is non-degenerate $W^\perp$ contains an orthogonal sum of two hyperbolic planes. Now $\dim \mathrm{K}_0^{\mathrm{num}}(S) \otimes \mathbb{Q} = \dim W + \dim W^\perp$, so $W^\perp \sim 2\mathbb{H}$. We conclude that the signature of $\chi^+$ can be computed as

$$(\rho - 2, 0) + 2(1, 1) = (\rho, 2). \tag{2.21}$$

□

We now apply these observations to give restrictions on embeddings of (bounded derived categories of) finite-dimensional algebras into (bounded derived categories of) smooth projective surfaces. Let $A$ denote a basic finite-dimensional $k$-algebra of finite global dimension with $n$ simple modules. In that case, there is a well-defined Euler form given by

$$\chi\colon \mathrm{K}_0(A) \times \mathrm{K}_0(A) \to \mathbb{Z} : (X, Y) \mapsto \sum_i (-1)^i \dim_k \operatorname{Ext}^i_A(X, Y). \tag{2.22}$$

Since the indecomposable projective modules and the simple modules form dual bases, this bilinear form is non-degenerate. Also submatrices cannot increase in rank and signatures behave well under restriction, so the following corollaries are clear.

**Corollary 2.10.** Given a smooth projective surface $S$ and an embedding

$$\mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(X), \tag{2.23}$$

the rank of $\chi_A^-$ is $\leq 2$.

**Corollary 2.11.** Given a smooth projective surface $S$ and an embedding

$$\mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(X), \tag{2.24}$$

then $\chi_A^+$ does not admit a 3-dimensional negative-definite subspace.

## 2.3 Embedding hereditary algebras

In this section we show how the established criteria can be applied to restrict embeddings as in theorem 1.55 for hereditary algebras, as these have a particularly nice description for their derived categories. Observe that by using tilting theory it is possible to find finite-dimensional algebras of global dimension $\geq 2$ which are derived equivalent to path algebras. Hence all results in this section are also valid for iterated tilted algebras.

The extreme case of the embedding being an equivalence has a particularly easy answer. It is elementary that $A$ being semisimple implies $X$ is a union of points. In global dimension 1 there is the following easy folklore result.

**Proposition 2.12.** If $A = kQ$ is hereditary (and not semisimple), and

$$\mathbf{D}^{\mathrm{b}}(X) \cong \mathbf{D}^{\mathrm{b}}(A) \tag{2.25}$$

is an equivalence, then $X \cong \mathbb{P}^1$ and $A \cong k\mathrm{K}_2$, the path algebra of the Kronecker quiver.

*Proof.* The description of $\mathbb{P}^1$ is standard [27]. To see that this is the only variety with this property consider the skyscraper sheaves $k_x$, which are indecomposable objects (or more precisely, they are point objects).

A triangle equivalence sends these to indecomposable objects of $\mathbf{D}^{\mathrm{b}}(kQ)$, which correspond to the indecomposable modules up to a shift since every object therein is formal. Now by Serre duality

$$\begin{aligned} \mathrm{Ext}_X^d(k_x, k_x) &\cong \mathrm{Hom}_X(k_x[d], k_x \otimes \omega_X[\dim X])^\vee \\ &\cong \mathrm{Hom}_X(k_x, k_x[\dim X - d])^\vee, \end{aligned} \tag{2.26}$$

and since a hereditary algebra is of global dimension 1, $X$ has to be a curve.

By Serre duality we have that for every object $E \in \mathbf{D}^{\mathrm{b}}(X)$

$$\mathrm{Hom}_X(E, E[1]) \cong \mathrm{Hom}_X(E, E \otimes \omega_X)^\vee \neq 0. \tag{2.27}$$

In particular, $\mathbf{D}^{\mathrm{b}}(X)$ only contains exceptional objects if $X \cong \mathbb{P}^1$, otherwise a nonzero section of $\omega_X$ gives a nonzero morphism.

Any hereditary algebra derived equivalent to $X$ is thus derived equivalent to $k\mathrm{K}_2$. It is known, see [93, §4.8], that any derived equivalence between basic hereditary algebras is given by a sequence of sink or source reflections, so there is no possibility other than $\mathrm{K}_2$. □

Having settled the case of an equivalence assume there is an embedding

$$\mathbf{D}^{\mathrm{b}}(kQ) \hookrightarrow \mathbf{D}^{\mathrm{b}}(S), \tag{2.28}$$

for an acyclic quiver $Q$ and a smooth projective surface $S$, i.e. we are in the situation of a strong (but not full) exceptional collection without relations in the composition law. The following questions regarding the structure of possible quivers $Q$ come to mind:

(Q1) Is there a bound on the number of vertices of $Q$?

(Q2) Is there a bound on the number of arrows of $Q$?

(Q3) Is there a bound on the number of paths in $Q$?

(Q4) Is there a bound on the length of paths in $Q$?

(Q5) Is it possible to embed any quiver on 3 vertices?

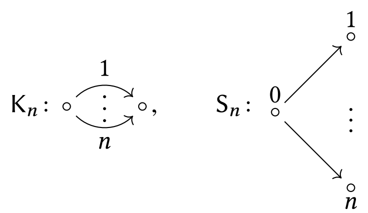
The remainder of this section is dedicated to answering these questions. In the following proposition we provide some explicit embeddings for well known families of quivers.

**Proposition 2.13.** Let $A = kQ$ be the path algebra of an acyclic quiver $Q$.

1. If $A$ is of finite type or tame, i.e. $Q$ is a Dynkin or Euclidean quiver, then an embedding (2.28) exists if and only if

$$Q = \mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{D}_4, \tilde{\mathrm{A}}_1 \text{ or } \tilde{\mathrm{A}}_2.$$

2. If $Q$ occurs in the following families of quivers:

(2.29) 

then an embedding (2.28) exists.

*Proof.* For part (1), it suffices to compute the matrices for the antisymmetric Euler forms which can be obtained from the Cartan matrices, and observe that starting from $\mathsf{A}_4$, $\mathsf{D}_5$ and $\tilde{\mathsf{A}}_3$ the rank is bounded below by 4 as there will be a submatrix of rank 4 in each of these coming from the smallest cases $\mathsf{A}_4$, $\mathsf{D}_5$ and $\tilde{\mathsf{A}}_3$. The exceptional types $\mathsf{E}_{6,7,8}$ or $\tilde{\mathsf{E}}_{6,7,8}$ have corresponding ranks 6, 6, 8, 6, 6 and 8.

For the 5 cases that are not ruled out by this restriction on the antisymmetric Euler form, and the infinite families in part (2), there are explicit embeddings.

Let $n = 2m$, then one can embed $\mathsf{K}_{2m}$ by considering $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, m-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $n = 2m-1$, then one can embed $\mathsf{K}_{2m-1}$ by considering $\mathcal{O}_{\mathrm{Bl}_1\,\mathbb{P}^2}$ and $\mathcal{O}_{\mathrm{Bl}_1\,\mathbb{P}^2}(E+mF)$, on the blow-up of $\mathbb{P}^2$ in a point $p$. Here, as usual, $E$ denotes the divisor associated to the $-1$-curve and $F$ the one associated to the strict transform of any line in $\mathbb{P}^2$ through $p$.

For the family $\mathsf{S}_n$, by Orlov's blow-up formula in theorem 1.36 we obtain a semiorthogonal decomposition

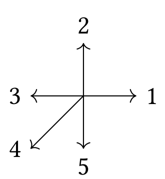
$$(2.30)\quad \mathbf{D}^{\mathrm{b}}(\mathrm{Bl}_1\,\mathbb{P}^2) = \langle \pi^*(\mathbf{D}^{\mathrm{b}}(\mathbb{P}^2)), \mathcal{O}_E\rangle$$

where $\pi\colon \mathrm{Bl}_1\,\mathbb{P}^2 \to \mathbb{P}^2$ is the blow-up morphism, and $\mathcal{O}_E$ is the structure sheaf of the exceptional divisor. The blow-up locus is denoted $p$. Consider the exceptional line bundle $\mathcal{O}_{\mathbb{P}^2}$ on $\mathbb{P}^2$, then one checks by adjunction

$$(2.31)\quad \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{Bl}_1\,\mathbb{P}^2)}(\pi^*(\mathcal{O}_{\mathbb{P}^2}), \mathcal{O}_E) \cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathbb{P}^2)}(\mathcal{O}_{\mathbb{P}^2}, k_p)$$

that the exceptional pair $(\pi^*(\mathcal{O}_{\mathbb{P}^2}), \mathcal{O}_E)$ has endomorphism ring $k\mathsf{S}_1$. Using the blow-up formula inductively, this gives a realisation of $k\mathsf{S}_n$ using $\mathrm{Bl}_n\,\mathbb{P}^2$.

By the identifications $\mathsf{A}_2 = \mathsf{S}_1$, $\mathsf{A}_3 = \mathsf{S}_2$ (using reflection), $\mathsf{A}_4 = \mathsf{S}_3$ (using reflection) and $\mathsf{K}_2 = \tilde{\mathsf{A}}_1$ the only remaining quiver is $\tilde{\mathsf{A}}_2$, and for this one we use some elementary toric geometry [65]. The variety $\mathrm{Bl}_2\,\mathbb{P}^2$ can be represented by the fan

(2.32)



As basis for $\mathrm{Pic}(\mathrm{Bl}_2\,\mathbb{P}^2)$, we choose the first three torus-invariant divisors $D_1, D_2$ and $D_3$. It can be computed that

$$(2.33)\quad (\mathcal{O}_{\mathrm{Bl}_2\,\mathbb{P}^2}, \mathcal{O}_{\mathrm{Bl}_2\,\mathbb{P}^2}(D_1 - D_3), \mathcal{O}_{\mathrm{Bl}_2\,\mathbb{P}^2}(D_1))$$

has the desired structure. □

**Remark 2.14.** Of course, there are many alternatives to the above embeddings. The Kronecker quivers $\mathsf{K}_n$ for example can also be embedded using $\mathcal{O}(E)$ and $\mathcal{O}(E + nF)$ on $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, the $n$th Hirzebruch surface.

Using this proposition, (Q1) and (Q2) have a negative answer. For (Q3) the answer is also no, since one can always reflect the $S_n$-quiver in some non-zero vertex.

The questions (Q4) and (Q5) are more subtle. Let us say a quiver $Q'$ is *forbidden* if the rank of $\chi^-$ is strictly greater than 2.

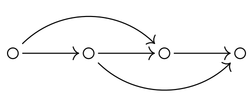
**Lemma 2.15.** If a quiver $Q$ contains a forbidden quiver $Q'$ as a full subquiver, then it cannot be embedded into a smooth projective surface.

*Proof.* The fullness ensures that the $\chi^-$ matrix of $Q'$ occurs as a block in that of $Q$ (for the basis of simples for example), so $\mathrm{rk}(\chi_Q^-) > 2$ and the quiver cannot be embedded. □

Then question (Q4) about path length can be partially answered by plugging $A_4$ into this lemma, as we know from corollary 2.10 that $A_4$ cannot be embedded into the derived category of a surface. Observe that $A_4$ does satisfy the condition on the negative-definite subspaces for $\chi^+$, as in corollary 2.11.

However, if $A_4$ occurs as a non-full subquiver of $Q$, it is not clear what happens. The following example shows that in some cases one *can* have an embedding.

**Example 2.16.** Consider the following quiver:

(2.34) Q: 

It has $\mathrm{rk}(\chi^-) = 2$ and contains $A_4$ but only as a non-full subquiver, so it does not satisfy the condition for the previous proposition. In fact, it can be embedded into $\mathrm{Bl}_2\,\mathbb{P}^2$ by extending the strong exceptional collection we already had for $\tilde{A}_2$. In terms of the fan mentioned in (2.32), the collection

$$(\mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2}, \mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2}(D_1 - D_3), \mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2}(D_1), \mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2}(D_2)) \tag{2.35}$$

can be checked to be a strong exceptional collection with the desired endomorphism ring, after reflecting in the vertex corresponding to $\mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2}$.

It is also possible to find an example of a quiver that satisfies corollary 2.10 but violates corollary 2.11.

**Example 2.17.** Consider an acyclic quiver on 5 vertices $v_1, \ldots, v_n$, whose $\chi$ is given by the matrix

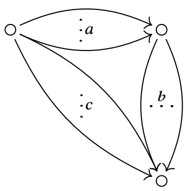
$$\begin{pmatrix} 1 & 2 & 4 & 3 & 0 \\ 0 & 1 & 4 & 5 & 2 \\ 0 & 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.36}$$

It is straightforward to check that $\mathrm{rk}\,\chi^- = 2$, but $\chi^+$ has a negative-definite subspace of dimension 3.

The reason for posing question (Q5) is that any skew-symmetric $3 \times 3$-matrix has rank $\leq 2$, and moreover it cannot have a 3-dimensional negative-definite subspace, since one can always look at the projective indecomposables which yield a nonzero positive-definite subspace. Also, since every quiver on 2 vertices can be embedded, (Q5) naturally arises as the next case.

Such a quiver can be presented as

(2.37) $Q_{a,b,c}$:




To make computations feasible we will only consider line bundles on rational surfaces, i.e. iterated blow-ups of the minimal rational surfaces $\mathbb{P}^2$ and $\mathbb{F}_n$ for $n \neq 1$. In this context we can apply techniques based on Riemann–Roch arithmetic which are for instance also used in [101].

**Theorem 2.18.** The values $(a, b, c)$ as in (2.37) for which there exists a rational surface $S$ and an embedding

$$\mathbf{D}^{\mathrm{b}}(kQ_{a,b,c}) \hookrightarrow \mathbf{D}^{\mathrm{b}}(S) \tag{2.38}$$

given by a strong exceptional collection of line bundles are

$$\begin{gathered} \{(0,n,n) \mid n \in \mathbb{N}\} \cup \{(n,0,n) \mid n \in \mathbb{N}\}, \\ \{(1,n,1) \mid n \in \mathbb{N}\} \cup \{(n,1,1) \mid n \in \mathbb{N}\}, \\ \{(2,2,0)\}. \end{gathered} \tag{2.39}$$

*Proof.* By twisting we have that the exceptional collection is of the form

$$\langle \mathcal{O}_S, \mathcal{O}_S(D), \mathcal{O}_S(E) \rangle, \tag{2.40}$$

for divisors $D$ and $E$.

We first claim that if an embedding (2.38) exists, then

$$a + b = ab + c. \tag{2.41}$$

To see this, first note that on a rational surface $\chi(\mathcal{O}_S) = 1$. From exceptionality we get $0 = (\mathcal{O}_S(D), \mathcal{O}) = \chi(-D)$, and similarly $\chi(-E) = \chi(D-E) = 0$. By Riemann–Roch we obtain

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - K_S \cdot D), \tag{2.42}$$

for any divisor $D$. By antisymmetrising this equation for our divisors $D, E$ and $E - D$ respectively and plugging in the zeroes we found above, we get

$$\chi(D) + \chi(E - D) = \chi(E). \tag{2.43}$$

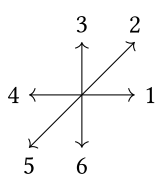
| $(a, b, c)$ | | $D$ | $E$ |
|---|---|---|---|
| $(0, n, n)$ | $n = 2m$ | $D_2 - D_4$ | $(m-1)D_1 + mD_2 + D_3$ |
| | $n = 2m+1$ | $D_1 - D_4$ | $D_1 + D_2 + mD_3 + (m-1)D_4$ |
| $(n, 0, n)$ | $n = 2m$ | $D_2 + mD_3 + (m-1)D_4$ | $D_1 + D_2 + (m-1)D_3 + (m-1)D_4$ |
| | $n = 2m+1$ | $D_2 + mD_3 + mD_4$ | $D_1 + D_2 + mD_3 + (m-1)D_4$ |
| $(1, n, 1)$ | $n = 2m$ | $D_1 + D_2 - D_4$ | $mD_1 + mD_2 + D_3$ |
| | $n = 2m+1$ | $D_4$ | $mD_1 + mD_2 + D_3 + D_4$ |
| $(n, 1, 1)$ | $n = 2m$ | $(m-1)D_1 + mD_2 + D_3$ | $(m-1)D_1 + mD_2 + D_3 + D_4$ |
| | $n = 2m+1$ | $D_2 + mD_3 + mD_4$ | $D_1 + D_2 + mD_3 + mD_4$ |

Table 2.1: Divisors for embeddings of 3-vertex quivers in $\mathbf{D}^{\mathrm{b}}(\mathrm{Bl}_3\,\mathbb{P}^2)$

By strong exceptionality, $\mathrm{h}^i(\mathcal{O}_S(D)) = \mathrm{h}^i(\mathcal{O}_S(E)) = \mathrm{h}^i(\mathcal{O}_S(E-D)) = 0$ for all $i > 0$, so $\chi(D) = \mathrm{h}^0(\mathcal{O}_S(D))$ and similarly for $E$ and $E - D$, so we finally find

(2.44) $$\mathrm{h}^0(\mathcal{O}_S(D)) + \mathrm{h}^0(\mathcal{O}_S(E-D)) = \mathrm{h}^0(\mathcal{O}_S(E)) \Rightarrow a + b = ab + c.$$

Solving this equation yields the solutions listed in the statement. We now give a construction for each of these cases. All of these can be realised on the del Pezzo surface $\mathrm{Bl}_3\,\mathbb{P}^2$ of degree 6, which will be represented by the fan

(2.45) 


and as a basis for $\mathrm{Pic}(X)$ we choose the torus-invariant divisors $D_1, \ldots, D_4$.

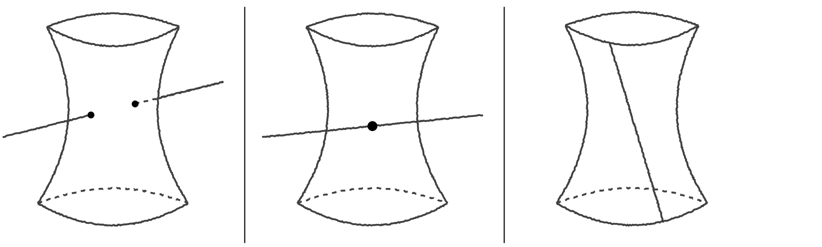
For the families of solutions we give a possible choice of divisors in table 2.1.

For the isolated case $(2, 2, 0)$ it suffices to take $D = L_1$ and $E = L_1 + L_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$, where the $L_i$ form the basis of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ given by the two rulings. ☐

**Remark 2.19.** On a rational surface any exceptional sequence can be mutated into one consisting of rank one objects, and the previous result also holds in this greater generality, since we only used numerical computations to obtain (2.41). We leave open the question whether considering more general exceptional vector bundles (or coherent sheaves) on more general surfaces yields more examples.

*Chapter 3*

# Derived categories of noncommutative quadrics and Hilbert schemes of 2 points



## 3.1 Introduction

In this chapter we study the derived category of a quadric (and its noncommutative analogues) in relationship with the derived category of the Hilbert scheme of two points on a quadric (and commutative deformations thereof). The motivation comes from several seemingly disparate observations.

First of all let $S$ be a smooth projective surface (over an algebraically closed field $k$ throughout). Then it is a classical result of Fogarty that the Hilbert scheme of $n$ points $\mathrm{Hilb}^n S$ is again a smooth projective variety, of dimension $2n$ [80]. If we moreover assume that $\mathrm{H}^1(S, \mathcal{O}_S) = \mathrm{H}^2(S, \mathcal{O}_S) = 0$ (e.g. $S$ is a rational surface, such as a quadric) then Krug–Sosna [132] have proven that the Fourier–Mukai functor

$$(3.1)\quad \Phi_{\mathcal{I}_{\mathcal{U}_n}} : \mathbf{D}^{\mathrm{b}}(S) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S)$$

is a fully faithful functor, where $\mathcal{I}_{\mathcal{U}_n}$ is the ideal sheaf on $S \times \mathrm{Hilb}^n S$ for the universal family $\mathcal{U}_n \subset S \times \mathrm{Hilb}^n S$.

Another piece of motivation stems from the notion of geometric dg categories as introduced by Orlov [178]. He shows that any dg category whose homotopy category has a full exceptional collection can be embedded in (an enhancement of) the derived category of a smooth projective variety. This construction can be applied to the full exceptional collection describing the derived category of a quadric surface, but the resulting variety is constructed using iterated projective bundles and does not seem to have a geometric interpretation in terms of a moduli problem, unlike the embedding (3.1).

It is interesting to try and apply Orlov's algorithm to a dg category with a full exceptional collection which is not of geometric origin. This brings us to the final piece of motivation: deformations of abelian categories. In noncommutative algebraic geometry a central role is played by abelian categories and their derived categories, and there is a framework for describing the deformations of abelian categories [155, 156], so in particular it can be applied to $\mathrm{coh}\,\mathbb{P}^1 \times \mathbb{P}^1$. The quadric is easily seen to be rigid (i.e. $\mathrm{H}^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathrm{T}_{\mathbb{P}^1 \times \mathbb{P}^1}) = 0$), but its category of coherent sheaves has nontrivial deformations ($\mathrm{HH}^2(\mathrm{coh}\,\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1, \bigwedge^2 \mathrm{T}_{\mathbb{P}^1 \times \mathbb{P}^1})$ is 9-dimensional), which can be seen as deformations of $\mathbb{P}^1 \times \mathbb{P}^1$ in the noncommutative direction.

Because the quadric has a strongly ample sequence it is moreover possible to pass from infinitesimal deformations to formal deformations [227], and the theory has been worked out in detail in [199, 226]. A noncommutative quadric is an abelian category $\mathrm{qgr}\,A$, which is a certain quotient category of the category of graded modules for a (generalised) graded algebra $A$ satisfying some natural conditions. On the derived level it is possible to view a family of noncommutative quadrics by varying the relations in the quiver coming from the full and strong exceptional collection [174]. For these new exceptional collections it makes sense to apply Orlov's embedding result, but again the result is an iterated projective bundle construction where arbitrary choices have been made and there is no moduli interpretation.

Yet for $\mathbb{P}^2$ (and its noncommutative deformations) Orlov shows that there exists an embedding in (a commutative deformation of) $\mathrm{Hilb}^2 \mathbb{P}^2$ [176], hence there *is* a moduli interpretation for the derived category of the finite-dimensional algebras whose structure resembles that of the Beĭlinson quiver for $\mathbb{P}^2$.

In this chapter we obtain a result analogous to Orlov's for noncommutative quadrics. The following is a compressed version of theorem 3.26 and is our main result.

**Theorem 3.1.** For a generic noncommutative quadric $A$ there exists a deformation $H$ of $\mathrm{Hilb}^2 \mathbb{P}^1 \times \mathbb{P}^1$ and a fully faithful embedding

$$\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}\, A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{coh}\, H). \tag{3.2}$$

To prove this result we need an explicit geometric model for $\mathrm{Hilb}^2 \mathbb{P}^1 \times \mathbb{P}^1$, which we give in proposition 3.3. In section 3.3.1 we explain how this geometric model depends on so called geometric squares: linear algebra data that describes the composition law in the derived category. In section 3.3.2 it is shown how a sufficiently generic noncommutative quadric gives rise to such a geometric square, and indeed to an embedding as in theorem 3.1.

Note that there exists a notion of Hilbert scheme of points for a general cubic Artin–Schelter regular graded algebra [69], which is a subset of all noncommutative quadrics. We do not address the comparison between these moduli spaces and the deformations constructed in this chapter.

We also formulate a general question regarding limited functoriality of Hochschild cohomology and the Hochschild–Kostant–Rosenberg decomposition, motivated by a conjecture of Orlov. This is done in section 3.4. We discuss some evidence suggesting an interesting relationship between the Hochschild cohomology of a surface and the Hochschild cohomology of the Hilbert scheme of points, showing that the results in this chapter hint towards a much more general picture.

## 3.2 The geometry of $\mathrm{Gr}(1,3)$ and $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$

Throughout the chapter, we will assume $k$ is an algebraically closed field of characteristic 0.

### 3.2.1 Grassmannians

For a vector space $V$ of dimension $n+1$, denote by $\mathbb{P}(V) = \mathbb{P}^n$ the projective space of all hyperplanes of $V$. Denote by $G := \mathrm{Gr}(k,n)$ the Grassmannian of $k$-dimensional linear subspaces in $\mathbb{P}^n$. This is naturally identified with the set of $(k+1)$-dimensional linear subspaces of $V^\vee$, or $(n-k)$-dimensional quotients of $V^\vee$. The Grassmannian of $k$-planes in $\mathbb{P}^n$ is naturally identified with the Grassmannian of $(n-k-1)$-planes in $(\mathbb{P}^n)^\vee$.

There is a tautological exact sequence of vector bundles

$$0 \to \mathcal{R} \to V \otimes \mathcal{O}_G \to \mathcal{Q} \to 0, \tag{3.3}$$

where $\mathrm{rk}(\mathcal{R}) = n-k$, $\mathrm{rk}(\mathcal{Q}) = k+1$. Also, there are identifications $\mathrm{H}^0(G,\mathcal{Q}) \cong V$ and (by considering the dual Grassmannian), $\mathrm{H}^0(G,\mathcal{R}^\vee) \cong V^\vee$. The Grassmannian is a fine moduli space (with universal object $\mathcal{Q}$) for the functor $F_G\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ sending a scheme $X$ to the set of epimorphisms $V \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F}$, where $\mathcal{F}$ is a rank $k+1$ vector bundle on $X$. In particular there is a bijection

$$\mathrm{Hom}_{\mathrm{Sch}}(X, G) \to F_G(X), \tag{3.4}$$

given by pulling back the universal epimorphism from (3.3) along a morphism in $\mathrm{Hom}_{\mathrm{Sch}}(X, G)$. In the other direction, given an epimorphism $V \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F}$, we define an element of $\mathrm{Hom}_{\mathrm{Sch}}(X, G)$ by sending an $x \in X$ to the element of $G$ defined by the induced map on the fibres $V \twoheadrightarrow \mathcal{F}_x \otimes k(x)$.

From now on we will focus on a specific case. Let $V$ denote a 4-dimensional vector space, and take $k = 1$. Then $\mathbb{G} := \mathrm{Gr}(1, 3)$ is the Grassmannian of lines in $\mathbb{P}^3$.

The following lemma will be used in section 3.3.1 to construct strong exceptional collections.

**Lemma 3.2.** Let $V_0, V_1$ denote two-dimensional vector spaces. Then any isomorphism $\phi\colon V \to V_0 \otimes V_1$ induces an isomorphism

$$F_\phi\colon \mathrm{Hom}(\mathcal{R}, \mathcal{K}) \otimes \mathrm{Hom}(\mathcal{K}, \mathcal{O}_{\mathbb{G}}) \to \mathrm{Hom}(\mathcal{R}, \mathcal{O}_{\mathbb{G}}), \tag{3.5}$$

where

$$\mathcal{K} := \ker\left(\mathrm{H}^0(\mathbb{G}, \mathcal{O}_L(1)) \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_L(1)\right), \tag{3.6}$$

and the embedding $L = \mathbb{P}(V_0) \hookrightarrow \mathbb{G}$ is induced by the epimorphism

$$f_\phi\colon \mathrm{H}^0(L, V_1 \otimes \mathcal{O}_L(1)) \otimes \mathcal{O}_L \twoheadrightarrow V_1 \otimes \mathcal{O}_L(1). \tag{3.7}$$

*Proof.* First notice that $V \cong \mathrm{H}^0(L, V_1 \otimes \mathcal{O}_L(1))$, so by (3.4) the epimorphism $f_\phi$ does indeed induce an embedding $L = \mathbb{P}(V_0) \hookrightarrow \mathbb{G}$. This embedding can be explicitly described as follows: a point $p \in L$ (i.e. a linear functional $p\colon V_0 \to k$) determines a linear map $f_p\colon V \to V_1$, which is just contraction by $p$, defining a closed point of $\mathbb{G}$.

Now we consider the evaluation morphism inside

$$\mathrm{Hom}(\mathcal{R} \otimes \mathrm{Hom}(\mathcal{R}, \mathcal{O}_{\mathbb{G}}), \mathcal{O}_{\mathbb{G}}) \cong \mathrm{Hom}(\mathcal{R} \otimes V^\vee, \mathcal{O}_{\mathbb{G}}) \cong V \otimes V^\vee. \tag{3.8}$$

Applying the inverse of $\phi^\vee$, and by duality, we can interpret the evaluation morphism inside

$$\mathrm{Hom}(\mathcal{R} \otimes V_1^\vee, \mathcal{O}_{\mathbb{G}} \otimes V_0). \tag{3.9}$$

Now we show that we actually land in the subsheaf $\mathcal{K}$. Assume that this is not true, then there is some point $p \in L$ where the composition

$$\mathcal{R}_p \otimes k(p) \to V_1 \xrightarrow{p} k \tag{3.10}$$

in the fibers is non-zero. However, by definition of the embedding of $L$ into $\mathbb{G}$, we see that $\mathcal{R}_p \otimes k(p) = \ker(f_p)$, so the composition (3.10) is zero and we obtain a contradiction. Finally, as the pairing remains perfect, we obtain an isomorphism $F_\phi$. ◻

### 3.2.2 Hilbert schemes of points

The Hilbert scheme is a classical object in algebraic geometry, parametrising closed subschemes of a projective scheme. One can associate a Hilbert polynomial to a closed subscheme, and this gives rise to a disjoint union decomposition of the Hilbert scheme. In particular, for the constant Hilbert polynomial we get the *Hilbert scheme of points*.

For a smooth projective curve $C$ one has that $\mathrm{Hilb}^n C = \mathrm{Sym}^n C$. In particular it is again smooth projective and of dimension $n$. For a smooth projective surface $S$ it can be shown that $\mathrm{Hilb}^n S$ is again smooth projective and of dimension $2n$. For higher-dimensional varieties and $n \gg 2$ the Hilbert scheme becomes (very) singular.

We will identify $\mathbb{P}^1 \times \mathbb{P}^1$ with its image under the Segre embedding

$$(3.11)\quad \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 : ([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1],$$

which we denote by $Q$, a smooth quadric surface. This surface has two rulings, and every line on $Q$ defines a point of $\mathbb{G}$. We denote

$$(3.12)\quad L := L_0 \sqcup L_1 = \{l \in \mathbb{G} \mid l \subset Q\} \subset \mathbb{G},$$

where $L_0$ (respectively $L_1$) corresponds to the lines in the first (respectively second) ruling. Note that $L_0 \cap L_1 = \emptyset$, and each of these two lines determines a factorization of $V$ as in lemma 3.2.

The following proposition provides our main model for working with the Hilbert scheme $\mathbb{H} := \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ of two points on $\mathbb{P}^1 \times \mathbb{P}^1$. A reference for this description is [181, theorem 1.1].

**Proposition 3.3.** There is an isomorphism $\mathbb{H} \cong \mathrm{Bl}_L \mathbb{G}$.

*Proof.* Using the Segre embedding (3.11) there exists a surjective morphism

$$(3.13)\quad f\colon \mathbb{H} \to \mathbb{G}\colon [Z] \mapsto l_{[Z]}$$

where the line $l_{[Z]}$ for a point $[Z] \in \mathbb{H}$ is defined to be the line through the two points if $[Z]$ corresponds to two distinct points, otherwise we use the tangent vector to define the line.

This morphism is defined by using the (1-)very ample line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$, which by [58] gives the morphism $f$. In the functorial language used in section 3.2.1 one uses the tautological bundle on $\mathbb{H}$ given by the Fourier–Mukai transform of the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$, which is of rank 2 and which comes equipped with a surjective morphism from $\mathcal{O}_{\mathbb{H}} \otimes_k V$.

On the open set $\mathbb{G} \setminus L$ this is a bijection, the inverse being given by the morphism mapping a line in $\mathbb{P}^3$ to its intersection with the quadric.

On the closed set $L \subseteq \mathbb{G}$ the fiber over $l \in L$ can be identified with $\mathbb{P}^2$: it is formed by the set of pairs of points on the line $l$, hence $\mathbb{H}_l \cong \mathrm{Sym}^2 \mathbb{P}^1 \cong \mathbb{P}^2$.

By the universal property property of blow-ups we get a morphism $\mathbb{H} \to \mathrm{Bl}_L \mathbb{G}$. We can then apply the factorisation result from [67], and as the Picard groups are both of rank 3 (which for $\mathbb{H}$ follows from [81]) we get that the morphism is necessarily an isomorphism. □

**Remark 3.4.** In [176] the embedding of a noncommutative $\mathbb{P}^2$ into the derived category of a deformation of $\mathrm{Hilb}^2 \mathbb{P}^2$ is based on the description of the Hilbert scheme as $\mathrm{Hilb}^2 \mathbb{P}^2 \cong \mathbb{P}(\mathrm{Sym}^2 \mathrm{T}_{\mathbb{P}^2}(-1)^\vee)$.

To find an exceptional collection on $\mathbb{H}$ that is compatible with deformations we need to describe some bundles on $\mathbb{G}$ and on $\mathbb{H}$ more explicitly. Based on proposition 3.3 we will use the following notation for the rest of the chapter.

$$\begin{array}{ccc} E = E_0 \sqcup E_1 & \xrightarrow{j=j_0\sqcup j_1} & \mathbb{H} := \mathrm{Bl}_{L_0\sqcup L_1} \mathbb{G} \\ \Big\downarrow {\scriptstyle q=q_0\sqcup q_1} & & \Big\downarrow {\scriptstyle p} \\ L = L_0 \sqcup L_1 & \xrightarrow{i=i_0\sqcup i_1} & \mathbb{G}. \end{array} \tag{3.14}$$

**Lemma 3.5.** There are isomorphisms

$$\begin{aligned} \mathcal{Q}|_{L_i} &\cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}, \\ \mathcal{R}|_{L_i} &\cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, \\ \mathcal{N}_{L_i}\mathbb{G} &\cong \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}. \end{aligned} \tag{3.15}$$

*Proof.* The first two isomorphisms follows from (3.4) since the $L_i$ are embedded using the exact sequence

$$0 \to \mathcal{O}_{L_i}(-1)^{\oplus 2} \to V \otimes \mathcal{O}_{L_i} \to \mathcal{O}_{L_i}(1)^{\oplus 2} \to 0 \tag{3.16}$$

as in lemma 3.2.

For the third isomorphism we get for the normal bundle

$$\begin{aligned} \mathcal{N}_{L_i}\mathbb{G} &= \mathrm{coker}(\mathrm{T}_{\mathbb{P}^1} \to \mathcal{H}om(\mathcal{R}, \mathcal{Q})|_{\mathbb{P}^1}) \\ &= \mathrm{coker}(\mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 4}) \\ &= \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}, \end{aligned} \tag{3.17}$$

since the tangent bundle $\mathrm{T}_{\mathbb{G}}$ can be expressed as $\mathcal{H}om(\mathcal{R}, \mathcal{Q})$, ◻

From the description of the normal bundle $\mathcal{N}_{L_i}\mathbb{G}$ in lemma 3.5 we find that

$$E_i \cong \mathbf{Proj}_{\mathbb{P}^1}\left(\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3})^\vee\right) \cong \mathbb{P}^1 \times \mathbb{P}^2 . \tag{3.18}$$

We will use the following notation

$$\mathcal{O}_{E_i}(m,n) := \mathcal{O}_{\mathbb{P}^1}(m) \boxtimes \mathcal{O}_{\mathbb{P}^2}(n). \tag{3.19}$$

Whenever we write $\mathcal{O}_E(m,n)$ this means that we use this construction for both connected components.

For the final lemma, recall that $\mathcal{O}_E(E)$ is shorthand for $\mathcal{O}_{\mathbb{H}}(E)|_E = j^*(\mathcal{O}_{\mathbb{H}}(E))$, which can also be written as $\mathcal{N}_E\mathbb{H}$. Using this notation we can describe $\omega_{\mathbb{H}}$ and two bundles on the exceptional locus $E$ as follows.

**Lemma 3.6.** There are isomorphisms

$$\omega_{\mathbb{H}} \cong p^* \left( \bigwedge^2 \mathcal{Q} \right)^{\otimes -4} (2E), \tag{3.20}$$

and

$$\begin{aligned} \mathcal{O}_E(E) &\cong \mathcal{O}_E(2, -1), \\ \omega_{\mathbb{H}}|_E &\cong \mathcal{O}_E(-4, -2). \end{aligned} \tag{3.21}$$

*Proof.* Applying the adjunction formula and the isomorphism $\mathcal{N}_{L_i}\mathbb{G} \cong \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}$ from lemma 3.5, we find

$$\begin{aligned} \omega_{\mathbb{P}^1} &\cong i^*(\omega_{\mathbb{G}}) \otimes \det(\mathcal{N}_{L_i}\mathbb{G}) \\ \Leftrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) &\cong \omega_{\mathbb{G}}|_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(6) \\ \Leftrightarrow \mathcal{O}_{\mathbb{P}^1}(-8) &\cong \omega_{\mathbb{G}}|_{\mathbb{P}^1}. \end{aligned} \tag{3.22}$$

For the canonical bundles, we get

$$\omega_{\mathbb{H}} \cong p^*(\omega_{\mathbb{G}}) \otimes \mathcal{O}_{\mathbb{H}}(2E), \tag{3.23}$$

and

$$\omega_E \cong (\omega_{\mathbb{H}} \otimes \mathcal{O}_{\mathbb{H}}(E))|_E. \tag{3.24}$$

Now plug (3.23) into (3.24) and use $\omega_{E_i} \cong \mathcal{O}_{E_i}(-2, -3)$ to get

$$\begin{aligned} \mathcal{O}_E(-2, -3) &\cong \mathcal{O}_E(-8, 0) \otimes \mathcal{O}_E(3E) \\ \Leftrightarrow \mathcal{O}_E(6, -3) &\cong \mathcal{O}_E(3E) \\ \Leftrightarrow \mathcal{O}_E(E) &\cong \mathcal{O}_E(2, -1). \end{aligned} \tag{3.25}$$

Finally (3.23) provides

$$\omega_{\mathbb{H}}|_E \cong \mathcal{O}_E(-8, 0) \otimes \mathcal{O}_E(4, -2) \cong \mathcal{O}_E(-4, -2), \tag{3.26}$$

completing the proof. □

### 3.2.3 The derived category of Gr(1, 3)

The following theorem is a particular case of a more general result obtained in [53, 116].

**Theorem 3.7.** The derived category of $\mathbb{G}$ has a full and strong exceptional collection

$$\mathbf{D}^{\mathrm{b}}(\mathbb{G}) = \left\langle \bigwedge^2 \mathcal{R} \otimes \bigwedge^2 \mathcal{R}, \bigwedge^2 \mathcal{R} \otimes \mathcal{R}, \bigwedge^2 \mathcal{R}, \mathrm{Sym}^2 \mathcal{R}, \mathcal{R}, \mathcal{O}_{\mathbb{G}} \right\rangle. \tag{3.27}$$

**Remark 3.8.** In fact, we will only need the exceptional pair $\langle \mathcal{R}, \mathcal{O}_{\mathbb{G}} \rangle$, which can also be established by elementary means.

We know from proposition 3.3 that $\mathbb{H} \cong \mathrm{Bl}_L(\mathbb{G})$, so Orlov's blowup formula (see theorem 1.36) describes the derived category of $\mathbb{H}$.

**Corollary 3.9.** There is a semiorthogonal decomposition

$$\begin{aligned} \mathbf{D}^{\mathrm{b}}(\mathbb{H}) &= \left\langle \mathbf{D}^{\mathrm{b}}(\mathbb{G}), \mathbf{D}^{\mathrm{b}}(L)_0, \mathbf{D}^{\mathrm{b}}(L)_1 \right\rangle \\ &= \left\langle \mathbf{D}^{\mathrm{b}}(\mathbb{G}), \mathbf{D}^{\mathrm{b}}(L_0)_0, \mathbf{D}^{\mathrm{b}}(L_0)_1, \mathbf{D}^{\mathrm{b}}(L_1)_0, \mathbf{D}^{\mathrm{b}}(L_1)_1 \right\rangle \end{aligned} \tag{3.28}$$

In particular there exists a full exceptional collection of length 14 in $\mathbf{D}^{\mathrm{b}}(\mathbb{H})$.

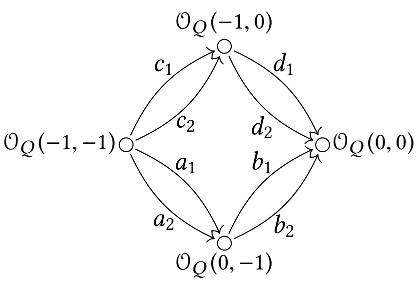
**Remark 3.10.** This is not the only way of obtaining a semiorthogonal decomposition of the Hilbert scheme in this situation. For an arbitrary surface $S$ one obtains using equivariant derived categories [78] and the description of the Hilbert scheme of points as a quotient that there exists a full (and strong) exceptional collection in $\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S)$ provided there exists a full (and strong) exceptional collection in $\mathbf{D}^{\mathrm{b}}(S)$ [132, proposition 1.3 and remark 4.6].

## 3.3 Embedding derived categories of noncommutative quadrics

### 3.3.1 Geometric squares and deformations of $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$

Recall [116] that for the derived category of the quadric $Q$ there is a full and strong exceptional collection

(3.29)




with relations $b_i a_j = d_j c_i$, for $i, j \in \{1, 2\}$. We isolate some of the properties of this exceptional collection in the following definition.

**Definition 3.11.** A *geometric square* is a septuple $\square = (V, U_0^0, U_1^0, U_0^1, U_1^1, \phi_0, \phi_1)$, where $V$ is a 4-dimensional vector space, the $U_j^i$ are 2-dimensional vector spaces, and the $\phi_i$ are isomorphisms

$$\phi_i \colon V \to U_0^i \otimes U_1^i. \tag{3.30}$$

Using lemma 3.2, the two isomorphisms $\phi_i$ in a geometric square give rise to two embeddings $L_i \coloneqq \mathbb{P}(U_0^i) \hookrightarrow \mathbb{G}$ and sheaves

$$\mathcal{K}_i \coloneqq \ker\left(\mathrm{H}^0(\mathbb{G}, \mathcal{O}_{L_i}(1)) \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{L_i}(1)\right). \tag{3.31}$$

**Proposition 3.12.** For a sufficiently generic geometric square $\square$, the Ext-quiver of the endomorphism algebra

$$Q_\square \coloneqq \mathrm{End}_{\mathbb{G}}(\mathcal{R} \oplus \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{O}_{\mathbb{G}}) \tag{3.32}$$

is of the form (3.29), and moreover

$$\dim \mathrm{Hom}(\mathcal{R}, \mathcal{O}_{\mathbb{G}}) = 4. \tag{3.33}$$

*Proof.* We first check that there are no Hom's going backwards. Applying $\mathrm{Hom}(-, \mathcal{R})$ to (3.31) we see that by exceptionality of the pair $\langle \mathcal{R}, \mathcal{O}_{\mathbb{G}} \rangle$ we need to prove that

$$\mathrm{Ext}^1(\mathcal{O}_{L_i}(1), \mathcal{R}) = 0. \tag{3.34}$$

This is the case by Serre duality:

$$\begin{aligned} \mathrm{Ext}^1_{\mathbb{G}}(\mathcal{O}_{L_i}(1), \mathcal{R}) &\cong \mathrm{Ext}^3_{\mathbb{G}}(\mathcal{R} \otimes \omega_{\mathbb{G}}^\vee, \mathcal{O}_{L_i}(1))^\vee \\ &\cong \mathrm{Ext}^3_{L_i}((\mathcal{R} \otimes \omega_{\mathbb{G}}^\vee)|_{L_i}, \mathcal{O}_{L_i}(1))^\vee \\ &= 0. \end{aligned} \tag{3.35}$$

Applying $\mathrm{Hom}(\mathcal{O}_{\mathbb{G}}, -)$ to (3.31) we get that $\mathcal{K}_i$ indeed does not have global sections because we get the identity morphism between $\mathrm{Hom}(\mathcal{O}_{\mathbb{G}}, \mathrm{H}^0(\mathbb{G}, \mathcal{O}_{L_i}(1)) \otimes \mathcal{O}_{\mathbb{G}})$ and $\mathrm{Hom}(\mathcal{O}_{\mathbb{G}}, \mathcal{O}_{L_i}(1))$.

Now to each of the isomorphisms $\phi_i$ we can apply lemma 3.2, and for a generic geometric square, the $\mathbb{P}(U_0^i)$ don't intersect in $\mathbb{G}$, hence $\mathrm{Hom}(\mathcal{K}_i, \mathcal{K}_{1-i}) = 0$, and the algebra $Q_\square$ does indeed have the form (3.29). $\square$

These four coherent sheaves cannot be used to realise an admissible embedding $\mathbf{D}^{\mathrm{b}}(Q_\square) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{G})$ since they do not form an exceptional collection. To ensure that they do, we need to blow up $\mathbb{G}$ in the two $L_i$, mimicking the description in proposition 3.3. Let us denote by $E_i$ the corresponding exceptional divisors on $\mathbb{H}_\square \coloneqq \mathrm{Bl}_{L_0 \sqcup L_1} \mathbb{G}$, so we have a cartesian square

$$\begin{array}{ccc} E_\square = E_0 \sqcup E_1 & \xhookrightarrow{j = j_0 \sqcup j_1} & \mathbb{H}_\square = \mathrm{Bl}_{L_0 \sqcup L_1} \mathbb{G} \\ \Big\downarrow{\scriptstyle q = q_0 \sqcup q_1} & & \Big\downarrow{\scriptstyle p} \\ L_\square = L_0 \sqcup L_1 & \xhookrightarrow{i = i_0 \sqcup i_1} & \mathbb{G} \end{array} \tag{3.36}$$

similar to (3.14).

We are now ready to show how a generic geometric square gives rise to a strong exceptional collection of vector bundles. In theorem 3.14 we will describe the structure of this strong exceptional collection.

In the proof we will compute mutations of exceptional collections. If $\langle E, F \rangle$ is an exceptional collection we will denote the left mutated collection as $\langle \mathrm{L}_E F, E \rangle$. A special property of the exceptional collection in (3.29) is that it is a 3-block collection, and one can also mutate blocks, for which similar notation will be used.

**Theorem 3.13.** For a generic geometric square, there is a strong exceptional collection of vector bundles

$$\langle p^*\mathcal{R}, \mathcal{C}_0, \mathcal{C}_1, \mathcal{O}_{\mathbb{H}_\square}\rangle \tag{3.37}$$

of ranks $2, 2, 2, 1$ on $\mathbb{H}_\square$, where

$$\mathcal{C}_i := \ker(\mathcal{O}_{\mathbb{H}_\square}^{\oplus 2} \twoheadrightarrow \mathcal{O}_{E_i}(1,0)). \tag{3.38}$$

*Proof.* The first and last object are clearly vector bundles. For the middle objects, this can be checked in the fibers by tensoring the defining short exact sequence of $\mathcal{C}_i$ with the residue field in a point, and using that $\mathcal{O}_{E_i}(1,0)$ is the pushforward of a line bundle on $E_i$, hence locally has the divisor short exact sequence as a flat resolution.

The derived pullback $\mathbf{L}p^*$ is fully faithful, and $\mathbf{L}p^* = p^*$ when applied to vector bundles. Since $\langle \mathcal{R}, \mathcal{O}_{\mathbb{G}}\rangle$ is a strong exceptional pair by theorem 3.7, so is $\langle p^*\mathcal{R}, \mathcal{O}_{\mathbb{H}_\square}\rangle$.

We first check that $\langle E, [F, G]\rangle = \langle \mathcal{O}_{\mathbb{H}_\square}, [\mathcal{O}_{E_0}(1,0), \mathcal{O}_{E_1}(1,0)]\rangle$ is a strong (block) exceptional collection. The sheaves $\mathcal{O}_{E_i}(1,0)$ are exceptional by the fully faithfulness of (1.36); moreover

$$\begin{aligned} \operatorname{Hom}(\mathcal{O}_{E_i}(1,0), \mathcal{O}_{\mathbb{H}_\square}[k]) &= \operatorname{Hom}(\mathcal{O}_{\mathbb{H}_\square}, \mathcal{O}_{E_i}(1,0) \otimes \omega_{\mathbb{H}_\square}[4-k])^\vee \\ &= \mathrm{H}^{4-k}(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(-3,-2))^\vee \\ &= 0, \end{aligned} \tag{3.39}$$

where we used (3.21) in the second equality. Also,

$$\operatorname{Hom}(\mathcal{O}_{\mathbb{H}_\square}, \mathcal{O}_{E_i}(1,0)[k]) = \mathrm{H}^k(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(1,0)), \tag{3.40}$$

and $\mathcal{O}_{E_0}(1,0), \mathcal{O}_{E_1}(1,0)$ are orthogonal because they have disjoint support, so the collection $\langle E, [F, G]\rangle$ is indeed a strong (block) exceptional collection. Hence the mutated collection

$$\langle [\mathrm{L}_E(F), \mathrm{L}_E(G)], E\rangle = \langle [\mathcal{C}_0, \mathcal{C}_1], \mathcal{O}_{\mathbb{H}_\square}\rangle \tag{3.41}$$

is also exceptional. By applying $\operatorname{Hom}(-, \mathcal{O}_{\mathbb{H}_\square})$ to the defining short exact sequence for $\mathcal{C}_i$

$$0 \to \mathcal{C}_i \to \mathcal{O}_{\mathbb{H}_\square}^{\oplus 2} \to \mathcal{O}_{E_i}(1,0) \to 0, \tag{3.42}$$

obtained from the mutation and using that $\langle \mathcal{O}_{\mathbb{H}_\square}, \mathcal{O}_{E_i}(1,0)\rangle$ is strong exceptional, we see that $\langle [\mathcal{C}_0, \mathcal{C}_1], \mathcal{O}_{\mathbb{H}_\square}\rangle$ is a strong exceptional collection.

It remains to check that $\langle p^*\mathcal{R}, [\mathcal{C}_0, \mathcal{C}_1]\rangle$ is a strong exceptional collection. We first check strongness: applying $\operatorname{Hom}(p^*\mathcal{R}, -)$ to (3.42) and using that $\langle p^*(\mathcal{R}), \mathcal{O}_{\mathbb{H}_\square}\rangle$ is exceptional, we find an exact sequence

$$0 \to \operatorname{Hom}(p^*\mathcal{R}, \mathcal{C}_i) \to \operatorname{Hom}(p^*\mathcal{R}, \mathcal{O}_{\mathbb{H}_\square}^{\oplus 2}) \to \operatorname{Hom}(p^*\mathcal{R}, \mathcal{O}_{E_i}(1,0)) \to \operatorname{Ext}^1(p^*\mathcal{R}, \mathcal{C}_i) \to 0, \tag{3.43}$$

and

$$\operatorname{Ext}^{m+1}(p^*\mathcal{R}, \mathcal{C}_i) \cong \operatorname{Ext}^m(p^*\mathcal{R}, \mathcal{O}_{E_i}(1,0)), \tag{3.44}$$

for all $i \geq 1$. Now

$$\mathrm{Ext}^m(p^*\mathcal{R}, \mathcal{O}_{E_i}(1,0)) \cong \mathrm{H}^m(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(2,0)^{\oplus 2}), \tag{3.45}$$

which is zero for $m \geq 1$. Also

$$\begin{aligned} \dim \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{\mathbb{H}_\square}^{\oplus 2}) &= 8, \\ \dim \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{E_i}(1,0)) &= 6, \end{aligned} \tag{3.46}$$

so it suffices to note that

$$\begin{aligned} \mathrm{Hom}(p^*\mathcal{R}, \mathcal{C}_i) &\cong \mathrm{Hom}_{\mathbb{G}}(\mathcal{R}, \mathbf{R}p_*\mathcal{C}_i) \\ &\cong \mathrm{Hom}_{\mathbb{G}}(\mathcal{R}, \mathcal{K}_i), \end{aligned} \tag{3.47}$$

which is 2-dimensional by proposition 3.12. Finally we check exceptionality: again one can apply $\mathrm{Hom}(-, p^*\mathcal{R})$ to (3.42) to see that

$$\mathrm{Ext}^m(\mathcal{C}_i, p^*\mathcal{R}) \cong \mathrm{Ext}^{m+1}(\mathcal{O}_{E_i}(1,0), p^*\mathcal{R}), \tag{3.48}$$

and this last group can be calculated using Serre duality, lemma 3.6 and lemma 3.5 as follows:

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_{E_i}(1,0), p^*\mathcal{R}[k+1]) &\cong \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{E_i}(1,0) \otimes \omega_{\mathbb{H}_\square}[4-k-1])^\vee \\ &\cong \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{E_i}(-3,-2)[4-k-1])^\vee \\ &\cong \mathrm{H}^{4-k-1}(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(-2,-2)^{\oplus 2})^\vee, \end{aligned} \tag{3.49}$$

which is easily seen to be zero for all $k$. □

**Theorem 3.14.** For a generic geometric square $\square$, there is an admissible embedding

$$\mathbf{D}^{\mathrm{b}}(Q_\square) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{H}_\square), \tag{3.50}$$

where $Q_\square$ is the endomorphism algebra as in (3.32), and $\mathbb{H}_\square$ is a deformation of $\mathbb{H}$.

*Proof.* By theorem 3.13, there is an admissible embedding

$$\mathbf{D}^{\mathrm{b}}(\mathrm{End}(p^*\mathcal{R} \oplus \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{O}_{\mathbb{H}_\square})) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{H}_\square). \tag{3.51}$$

Because $i$ is a closed immersion we get the exact sequence

$$0 \to \mathrm{L}^1 p^*(\mathcal{O}_{L_i}(1)) \to p^*(\mathcal{K}_i) \to \mathcal{O}_{\mathbb{H}}^2 \to \mathcal{O}_{E_i}(1,0) \to 0 \tag{3.52}$$

after applying $\mathbf{L}p^*$ to (3.31) and hence by quotienting out the torsion in $p^*(\mathcal{K}_i)$ we obtain an isomorphism

$$\mathcal{C}_i \cong p^*(\mathcal{K}_i)/\mathrm{L}^1 p^*(\mathcal{O}_{L_i}(1)). \tag{3.53}$$

The last thing to observe is that the action of $Q_\square$ remains faithful, which gives us the isomorphism

$$\mathrm{End}_{\mathbb{H}}(p^*\mathcal{R} \oplus \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{O}_{\mathbb{H}_\square}) \cong Q_\square \tag{3.54}$$

and the admissible embedding (3.50).

To see this, it suffices to realise that the action of $Q_\square$ generically does not change under taking the quotient with the torsion subsheaf. So if an element of $Q_\square$ were to act as zero on the exceptional collection on $\mathbb{H}$, it would also act as zero on the original collection of sheaves on $\mathbb{G}$ because all sheaves are torsionfree, arriving at a contradiction. □

### 3.3.2 Noncommutative quadrics

We will now recall the necessary definitions and some properties of noncommutative quadrics, all of which are proven in [226]. Then we will explain how a generic noncommutative quadric gives rise to a geometric square, such that we can prove the embedding result in theorem 3.26.

A $\mathbb{Z}$-algebra is a pre-additive category with objects indexed by $\mathbb{Z}$, generalising the theory of graded algebras and modules. All usual notions like right and left modules, bimodules, ideals, etc. make sense in this context and we will freely make use of them. For more details, consult [226, §2].

Let $\operatorname{Gr} A$ denote the category of right $A$-modules, and (if $A$ is noetherian) $\operatorname{gr} A$ the full subcategory of noetherian objects. Also $\operatorname{QGr} A$ (respectively $\operatorname{qgr} A$) is the quotient of $\operatorname{Gr} A$ (respectively $\operatorname{gr} A$) by the torsion modules. The quotient functor is denoted $\pi\colon \operatorname{gr} A \to \operatorname{qgr} A$.

We write $A_{i,j} = \operatorname{Hom}_A(j,i)$, and $e_i = i \xrightarrow{\mathrm{id}} i$, for $i,j \in \mathbb{Z}$. Then $P_i = e_i A$ are projective generators for $\operatorname{Gr} A$ and if $A$ is connected, $S_i$ will be the unique simple quotient of $P_i$.

**Definition 3.15.** A $\mathbb{Z}$-algebra $A$ is a *three-dimensional cubic Artin–Schelter-regular algebra* if

1. $A$ is connected,
2. $\dim A_{ij}$ is bounded by a polynomial in $j-i$,
3. $\sum_{j,k} \dim \operatorname{Ext}^j_{\operatorname{Gr} A}(S_k, P_i) = 1$, for every $i$,
4. the minimal resolution of $S_i$ has the form

$$(3.55)\quad 0 \to P_{i+4} \to P_{i+3}^{\oplus 2} \to P_{i+1}^{\oplus 2} \to P_i \to S_i \to 0.$$

Using this definition, we can now define noncommutative quadrics.

**Definition 3.16.** A *noncommutative quadric* is a category of the form $\operatorname{QGr} A$, where $A$ is a three dimensional cubic Artin–Schelter regular algebra.

An important subclass of the cubic Artin–Schelter regular $\mathbb{Z}$-algebras is given by the $\mathbb{Z}$-algebra associated to a cubic Artin–Schelter regular graded algebra [11]. In general one gets a $\mathbb{Z}$-algebra $\check{B}$ from a $\mathbb{Z}$-graded algebra by setting

$$(3.56)\quad \check{B}_{i,j} := B_{j-i}.$$

The $\mathbb{Z}$-algebras obtained in this way are called 1-periodic.

The motivation for this definition comes from the following theorem. For details and unexplained terminology we refer to [226, 227].

**Theorem 3.17.** [226, theorem 1.5] Let $(R, \mathfrak{m})$ be a complete commutative Noetherian local ring with $k = R/\mathfrak{m}$. Any $R$-deformation of the abelian category $\operatorname{coh}\mathbb{P}^1 \times \mathbb{P}^1$ is of the form $\operatorname{qgr}\mathcal{A}$, where $\mathcal{A}$ is an $R$-family of three-dimensional cubic Artin-Schelter regular $\mathbb{Z}$-algebras.

One of the main results of [226] is the classification of cubic Artin–Schelter regular $\mathbb{Z}$-algebras in terms of linear algebra data. We will now recall this description for use in proposition 3.22.

A three-dimensional cubic AS-regular algebra satisfies $A_{i,i+n} = 0$ for $n < 0$. It is generated by the $V_i = A_{i,i+1}$ and the relations are generated by the

$$R_i = \ker(V_i \otimes V_{i+1} \otimes V_{i+2} \to A_{i,i+3}), \tag{3.57}$$

which are of dimension two. Denote by

$$W_i = V_i \otimes R_{i+1} \cap R_i \otimes V_{i+3} \subset V_i \otimes V_{i+1} \otimes V_{i+2} \otimes V_{i+3}, \tag{3.58}$$

which are of dimension one. Any non-zero element of $W_i$ is a rank two tensor, both as an element of $V_i \otimes R_{i+1}$ and as an element of $R_i \otimes V_{i+3}$. Finally, $A$ is determined up to isomorphism by its truncation $\bigoplus_{i,j=0}^{3} A_{ij}$, which motivates the following definition.

**Definition 3.18.** A quintuple $(V_0, V_1, V_2, V_3, W)$, where the $V_i$ are two-dimensional vector spaces and $0 \neq W = kw \subset V_0 \otimes V_1 \otimes V_2 \otimes V_3$ is called *geometric* if for all $j \in \{0, 1, 2, 3\}$, and for all $0 \neq \phi_j \in V_j^\vee$, the tensor

$$\langle \phi_j \otimes \phi_{j+1}, w \rangle \tag{3.59}$$

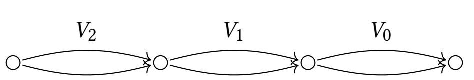
is non-zero, where indices are taken modulo four.

In the sequel we will sometimes identify a quintuple by a non-zero element of $W$, and we will omit the tensor product.

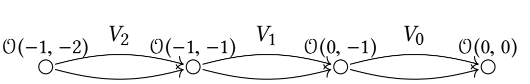
From the previous discussion, it is clear how to associate a quintuple to a noncommutative quadric. In fact, this quintuple is geometric and there is the following classification theorem that tells us that it suffices to consider geometric quintuples.

**Theorem 3.19.** [226, theorem 4.31] There is an isomorphism preserving bijection between noncommutative quadrics and geometric quintuples.

By construction a noncommutative quadric has a full and strong exceptional collection

(3.60) 


with relations $R = W \otimes V_3^\vee$. We will use the (purely formal) notation

(3.61) 

**Example 3.20** (Linear quadric)**.** We can now explain how the (commutative) quadric surface gives rise to a cubic Artin–Schelter regular $\mathbb{Z}$-algebra. On $\mathbb{P}^1 \times \mathbb{P}^1$ there are the line bundles

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(m,n) = \mathcal{O}_{\mathbb{P}^1}(m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(n). \tag{3.62}$$

The following defines an ample sequence:

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(n) = \begin{cases} \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,k) & \text{if } n = 2k, \\ \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k+1,k) & \text{if } n = 2k+1. \end{cases} \tag{3.63}$$

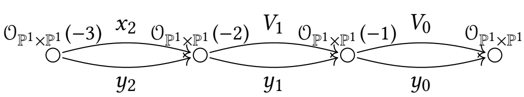
Put $A = \bigoplus_{i,j} \operatorname{Hom}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-j), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-i))$. Then $\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1 \cong \operatorname{qgr} A$, and $A$ is a 3-dimensional cubic AS-regular algebra. One may choose bases $x_i, y_i$ for $V_i$ such that the relations in $A$ are given by

$$\begin{aligned} x_i x_{i+1} y_{i+2} - y_i x_{i+1} x_{i+2} &= 0 \\ x_i y_{i+1} y_{i+2} - y_i y_{i+1} x_{i+2} &= 0. \end{aligned} \tag{3.64}$$

The tensor $w \in W_0$ corresponding to these relations is given by

$$w = x_0x_1y_2y_3 - y_0x_1x_2y_3 - x_0y_1y_2x_3 + y_0y_1x_2x_3. \tag{3.65}$$

The corresponding exceptional collection has quiver

(3.66) 


with relations (3.64), corresponding to (3.61).

The relationship between the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding and the $\mathbb{Z}$-algebra $A$ is obtained by taking the 2-Veronese of $A$, giving an isomorphism

$$\left( k\langle x,y\rangle / \begin{pmatrix} x^2y - yx^2 \\ xy^2 - y^2x \end{pmatrix} \right)_2 \cong k[a,b,c,d]/(ad-bc), \tag{3.67}$$

where we descibed the $\mathbb{Z}$-algebra as a graded algebra, because in this case $A$ is 1-periodic.

Another important class of noncommutative quadrics is given by the so called type-A cubic algebras.

**Example 3.21** (Type-A cubic algebras)**.** We will consider the generic class of cubic algebras from [11]. In this case the (graded) algebra $A$ has two generators $x$ and $y$ and relations

$$\begin{aligned} ay^2x + byxy + axy^2 + cx^3 &= 0 \\ ax^2y + bxyx + ayx^2 + cy^3 &= 0 \end{aligned}. \tag{3.68}$$

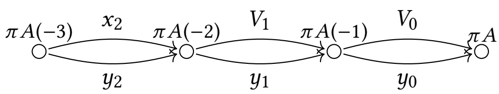
These algebras are Artin–Schelter regular for $(a : b : c) \in \mathbb{P}^2 - S$, where

(3.69) $S = \{(a : b : c) \in \mathbb{P}^2 \mid a^2 = b^2 = c^2\} \cup \{(0 : 0 : 1), (0 : 1 : 0)\}$,

The tensor $w \in W_0$ corresponding to these relations in the $\mathbb{Z}$-algebra setting is given by

(3.70)
$$\begin{aligned} w = {} & a y_0 y_1 x_2 x_3 + b y_0 x_1 y_2 x_3 + a x_0 y_1 y_2 x_3 + c x_0 x_1 x_2 x_3 \\ & + a x_0 x_1 y_2 y_3 + b x_0 y_1 x_2 y_3 + a y_0 x_1 x_2 y_3 + c y_0 y_1 y_2 y_3. \end{aligned}$$

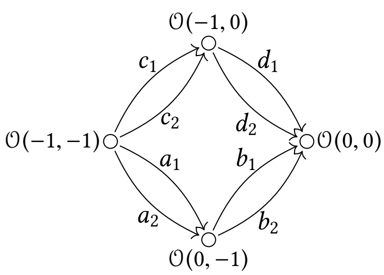
The corresponding full and strong exceptional collection is given by

(3.71)



with relations coming from (3.68).

Since our model for theorem 3.14 was the 3-block exceptional collection (3.29) and not the linear collection (3.66), we first have to mutate a linear exceptional collection as in (3.61) to a square one as in (3.29).

**Proposition 3.22.** The exceptional collection obtained from (3.61) by right mutating the first two objects is strong and has the structure

(3.72)



where we used the notation $\mathcal{O}(-1, 0) = \mathrm{R}_{\mathcal{O}(-1,-1)}\mathcal{O}(-1, -2)$.

*Proof.* By construction the right mutation $\mathcal{O}(-1, 0)$ fits in a short exact sequence

(3.73) $0 \to \mathcal{O}(-1, -2) \to V_2^\vee \otimes_k \mathcal{O}(-1, -1) \to \mathcal{O}(-1, 0) \to 0$

because we can compute the mutation entirely in $\operatorname{qgr} A$ as the morphism on the left is indeed a monomorphism by definition.

To see that $\operatorname{Hom}(\mathcal{O}(-1, 0), \mathcal{O}(0, 0)) = R$ one can use the proof of [226, lemma 4.3]. By applying $\operatorname{Hom}(-, \mathcal{O}(0, 0))$ to (3.73) we get a long exact sequence, which by the canonical isomorphism $A_{0,2} = V_0 \otimes V_1 = \operatorname{Hom}(\mathcal{O}(-1, -1), \mathcal{O}(0, 0))$ corresponds to

(3.74) $0 \to \operatorname{Hom}(\mathcal{O}(-1, 0), \mathcal{O}(0, 0)) \to V_0 \otimes V_1 \otimes V_2 \to A_{0,3} \to 0$,

hence $R = \operatorname{Hom}(\mathcal{O}(-1, 0), \mathcal{O}(0, 0)$. This also shows that the higher Ext's vanish.

Finally, to see that $\mathcal{O}(-1, 0)$ and $\mathcal{O}(0, -1)$ are completely orthogonal, we can apply $\operatorname{Hom}(-, \mathcal{O}(0, -1))$ to (3.73). By the resulting long exact sequence where the isomorphism $V_1 \otimes V_2 \cong A_{1,3}$ is the only non-zero map we get the desired orthogonality. ∎

We are almost in a situation where we can apply theorem 3.14. However, an arbitrary geometric quintuple does not give rise to a geometric square since the induced map $R \otimes V_2^\vee \to V_0 \otimes V_1$ in (3.72) is not necessarily an isomorphism. The next proposition describes a dense subset for which this is the case. Recall that $w \in V_0 \otimes V_1 \otimes V_2 \otimes V_3$, and we have an action of $\mathbb{G}_{\mathrm{m}}$ on this space, so $w$ can be interpreted as a point in $\mathbb{P}^{15}$.

**Proposition 3.23.** A generic geometric quintuple $(V_0, V_1, V_2, V_3, w)$ gives rise to a geometric square. More precisely, for $w$ in a Zariski open subset $\mathcal{U}'$ of $\mathbb{P}^{15}$,

(3.75) $\square_w = (V_0 \otimes V_1, V_0, V_1, V_2^\vee, V_3^\vee, \mathrm{id}, \phi_w)$

is a geometric square, where $\phi_w = \langle -, w \rangle^{-1}$.

*Proof.* The condition that the morphism

(3.76) $\langle -, w \rangle \colon V_2^\vee \otimes V_3^\vee \to V_0 \otimes V_1$

induced by an element $w \in V_0 \otimes V_1 \otimes V_2 \otimes V_3$ is an isomorphism is given by the nonvanishing of the determinant. The open subset $\mathcal{U}'$ is defined as the intersection of the locus of geometric quintuples with the complement of this vanishing locus in $\mathbb{P}^{15}$. So starting from a geometric quintuple with $w \in \mathcal{U}'$ we can define the associated square (3.75). □

**Remark 3.24.** We remark that the condition required for proposition 3.23 is indeed stronger than the geometricity condition for a quintuple. This geometricity condition ensures that the morphism $\langle -, w \rangle$ sends the pure tensors to nonzero elements. This does not imply that the morphism is an isomorphism, only that the kernel has to intersect the quadric cone corresponding to the pure tensors trivially in the origin. This implies that the kernel is necessarily of dimension 1.

**Example 3.25** (Linear quadric)**.** For the geometric quintuple (3.65) it is easy to see that $w \in \mathcal{U}$, so we get an associated geometric square and an exceptional collection (3.72), which is exactly the 3-block collection (3.29). Another small calculation shows that the two $\mathbb{P}^1$'s don't intersect so $w \in \mathcal{U}'$. As expected, the two $\mathbb{P}^1$'s correspond to the two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$, which we used in proposition 3.3.

Let us denote by $\mathbb{H}_w := \mathbb{H}_{\square_w}$, for $w \in \mathcal{U}'$, and by $\operatorname{qgr} A_w$ the associated noncommutative quadric. The following is then our main result.

**Theorem 3.26.** The varieties $\mathbb{H}_w$ form a smooth projective family $\mathcal{H}$ over a Zariski open $\mathcal{U} \subset \mathcal{U}'$ containing $\mathbb{H}$, and for each $w \in \mathcal{U}$ there is an admissible embedding

(3.77) $\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A_w) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{H}_w)$.

by vector bundles of ranks 2, 2, 2 and 1.

*Proof.* This is now immediate from the combination of theorem 3.14, proposition 3.23 and proposition 3.22. Note that we have to restrict to a Zariski open $\mathcal{U} \subset \mathcal{U}'$ since theorem 3.14 only works for a generic geometric square for which the corresponding $\mathbb{P}^1$'s do not intersect. Also, $\mathbb{H}$ is a member of the family by example 3.25. □

**Example 3.27.** Consider the type-A cubic algebra from example 3.21 for the parameters $(0 : 1 : 1)$. In this case the matrix describing $\phi_w$ is the identity matrix, hence the two $\mathbb{P}^1$'s coincide and theorem 3.14 does not apply.

## 3.4 Further remarks

Based on the result for $\mathbb{P}^2$ from [176] Orlov conjectured informally that every noncommutative deformation can be embedded in some commutative deformation, i.e. for every smooth projective variety $X$ there exists a smooth projective variety $Y$ and a fully faithful functor $\mathbf{D}^{\mathrm{b}}(X) \hookrightarrow \mathbf{D}^{\mathrm{b}}(Y)$ such that for every noncommutative deformation of $X$ there is a commutative deformation of $Y$ such that there is again a fully faithful functor between the bounded derived categories.

The result in this chapter adds some further evidence to this, by proving the result for $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $Y = \mathrm{Hilb}^2 \mathbb{P}^1 \times \mathbb{P}^1$. The general construction from [176] seems to prove this conjecture in case $\mathbf{D}^{\mathrm{b}}(X)$ has a full and strong exceptional collection: noncommutative deformations of $X$ correspond to changing the relations in the quiver, and these changes are reflected by changing the vector bundles in the iterated projective bundle construction.

However, it would be interesting to know whether one can always choose for $Y$ a natural moduli space associated to $X$, as is the case for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ where one can take the Hilbert scheme of two points. To investigate this in a more general setting we formulate an infinitesimal version of this conjecture in terms of limited functoriality for Hochschild cohomology, and explain how results on Poisson structures on surfaces give some substance to this conjecture in special cases.

The infinitesimal deformation theory of abelian categories is governed by their Hochschild cohomology [156], and one has the Hochschild–Kostant–Rosenberg decomposition for Hochschild cohomology of smooth varieties. In particular there is the decomposition

$$\mathrm{HH}^2(X) = \mathrm{H}^0(X, \bigwedge^2 \mathrm{T}_X) \oplus \mathrm{H}^1(X, \mathrm{T}_X) \oplus \mathrm{H}^2(X, \mathcal{O}_X) \tag{3.78}$$

where the first term can be understood as the noncommutative deformations, the second as the commutative (or geometric) deformations and the third one corresponding to gerby deformations [74, 216].

The natural categorical framework for Hochschild cohomology is that of dg categories. It is easily checked that Hochschild cohomology is not functorial for arbitrary functors: it only satisfies a limited functoriality. Indeed, in the case of a dg functor inducing a fully faithful embedding on the level of derived categories there is an induced morphism on the Hochschild cohomologies [120], which in the case of Fourier–Mukai transforms is treated in [141].

Combining limited functoriality with the Hochschild–Kostant–Rosenberg decomposition one could formulate an infinitesimal version of Orlov's conjecture as follows.

**Question 3.28.** Let $X$ be a smooth projective variety. Does there exist a smooth projective variety $Y$ and a fully faithful embedding $\mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$, such that the induced morphism on Hochschild cohomologies induces a surjective morphism

$$\mathrm{H}^1(Y, \mathrm{T}_Y) \twoheadrightarrow \mathrm{H}^0(X, \bigwedge^2 \mathrm{T}_X). \tag{3.79}$$

Sadly, we do not even know the answer for the embeddings obtained for $X = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ and $Y = \operatorname{Hilb}^2 X$.

Some positive evidence comes from a result by Hitchin who shows in [102] the existence of the split exact sequence

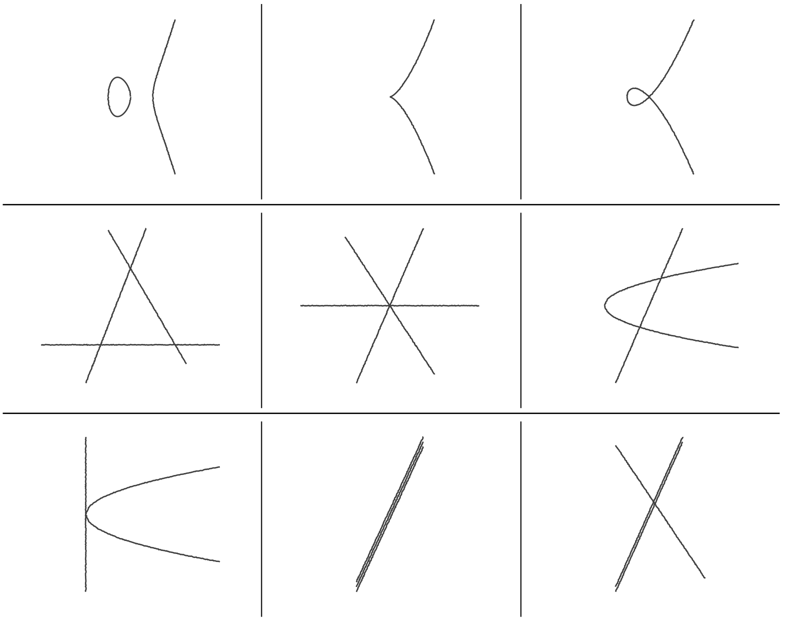
$$0 \to \mathrm{H}^1(S, \mathrm{T}_S) \to \mathrm{H}^1(\operatorname{Hilb}^n S, \mathrm{T}_{\operatorname{Hilb}^n S}) \to \mathrm{H}^0(S, \omega_S^\vee) \to 0 \tag{3.80}$$

where $S$ is a smooth projective surface over the complex numbers.

Again one does not know that the morphism on the right is related to (3.79), but it does show that a possible approach might be to choose for $Y$ a smooth projective variety representing a moduli problem associated to $X$. The choice of the Hilbert scheme of $n$ points seems to be a natural choice in the case of a surface, but for higher-dimensional varieties the Hilbert scheme fails to be smooth in general.

*Chapter 4*

# Hochschild cohomology of noncommutative planes and quadrics



## 4.1 Introduction

Noncommutative algebraic geometry in the sense of Artin–Zhang concerns the abelian categories associated to (not necessarily commutative) graded algebras, giving rise to the category of coherent sheaves on (nonexistent) noncommutative projective varieties. An important class of such algebras is given by the analogues of the (commutative) polynomial ring, and their properties have been axiomatised by Artin–Schelter [11]. A full classification of 3-dimensional Artin–Schelter regular algebras has been obtained by Artin–Tate–Van den Bergh [12, 13], and it can be seen as the classification of *noncommutative planes* and *noncommutative quadrics*. Indeed, there are precisely two types of graded algebras in this case, distinguished by their Hilbert series, and which correspond to the noncommutative analogues of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

More recently, Lowen–Van den Bergh developed a deformation theory for abelian categories, generalising that of associative algebras [155, 156, 227]. They introduced a notion of Hochschild cohomology for abelian categories, which in turn describes the infinitesimal deformations as an abelian category. Based on the computation of the Hochschild cohomology of $\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1$ it becomes clear that the classification of noncommutative quadrics is incomplete if one restricts to graded algebras as there should be more degrees of freedom [226], and in op. cit. Van den Bergh generalised the construction of noncommutative quadrics to use $\mathbb{Z}$-algebras. Previously the classification of noncommutative planes (where graded algebras do suffice) was approached differently by Bondal–Polishchuk using $\mathbb{Z}$-algebras to give a more streamlined approach [50].

In this chapter we describe the Hochschild cohomology of *all* noncommutative planes and quadrics using geometric techniques. For the commutative case this computation is an easy application of the Hochschild–Kostant–Rosenberg theorem as in examples 4.4 and 4.5, but in the noncommutative case no such techniques are available. The main observation from tables 4.1 and 4.2 is that noncommutative planes and quadrics have *less (infinitesimal) symmetries* or equivalently that they are *more (infinitesimally) rigid*, which explains the *dimension drop* we see happening in tables 4.1 and 4.2, and we give an explicit description of how this dimension drop behaves for all cases in the classification.

In section 4.2.2 we describe the technicalities regarding $\mathbb{Z}$-algebras which are necessary to define noncommutative planes and quadrics and describe their classification. In section 4.2.3 we give an overview of a completely classical but not so well-known classification of base loci of pencils of quadrics, which arise as the point schemes of noncommutative quadrics.

The method of computing the Hochschild cohomology is given in section 4.3: it is based on the Lefschetz trace formula for Hochschild cohomology as recalled in section 4.3.1, and the description of the first Hochschild cohomology of a finite-dimensional algebra as the Lie algebra of its outer automorphisms in section 4.3.2. In section 4.3.3 it is explained how this group of outer automorphisms of the finite-dimensional algebra can be related to the automorphism groups of the corresponding $\mathbb{Z}$-algebra, and hence by the geometric classification of noncommutative planes (resp. quadrics) by Bondal–Polishchuk (resp. Van den Bergh) we can reduce the problem to a purely geometric question regarding automorphism groups of certain (possibly singular, possibly reducible, possibly nonreduced) curves.

In section 4.4 we then give a description of the automorphism groups necessary to apply corollary 4.33. For noncommutative planes this description is given in [50, table 1], and recalled in table 4.4. For noncommutative quadrics this description is new, and the details for the computations are given in section 4.4.2.

Another class of finite-dimensional algebras arising from algebraic geometry for which the Hochschild cohomology is completely known is that of weighted projective lines [92]. Remark that in that case it turns out that there is *no* dependence on the parameters (i.e. the location of the stacky points).

**Conventions** Throughout we will assume $k$ to be an algebraically closed field of characteristic not equal to $2, 3$. When comparing Hochschild cohomology to Poisson cohomology we will only consider fields of characteristic $0$.

## 4.2 Preliminaries

In this section we recall the definition of Hochschild cohomology for different types of objects, and its properties that we will need later.

### 4.2.1 Hochschild cohomology of abelian and differential graded categories

It is well-known that Hochschild cohomology for associative algebras governs their deformation theory. Its definition has been generalised to arbitrary dg categories [120], see also section 1.3.3.

**Definition 4.1.** Let $\mathcal{C}$ be a $k$-linear dg category. Its *Hochschild complex* $\mathrm{CC}^\bullet_{\mathrm{dg}}(\mathcal{C})$ is the $\mathrm{Tot}^\Pi$ of the complex whose term in degree $p \geq 0$ is given by

(4.1)
$$\prod_{C_0,\ldots,C_p} \mathrm{Hom}_k\left(\mathcal{C}(C_{p-1},C_p)^\bullet \otimes_k \mathcal{C}(C_{p-2},C_{p-1})^\bullet \otimes_k \ldots \otimes_k \mathcal{C}(C_0,C_1)^\bullet, \mathcal{C}(C_0,C_p)^\bullet\right)$$

with differentials as for the Hochschild complex of an associative algebra, and which is zero in degree $p \leq -1$. The cohomology of this complex is the *Hochschild cohomology of* $\mathcal{C}$, and will be denoted $\mathrm{HH}^\bullet_{\mathrm{dg}}(\mathcal{C})$.

We will also need the Hochschild cohomology of an abelian category $\mathcal{A}$. This is *not* the Hochschild cohomology of this abelian category considered as a $k$-linear category giving rise to a dg category in degree 0. Rather, the definition is as follows.

**Definition 4.2.** Let $\mathcal{A}$ be a $k$-linear abelian category. The *Hochschild cohomology* $\mathrm{HH}^\bullet_{\mathrm{ab}}(\mathcal{A})$ is defined as the Hochschild cohomology of the dg category associated to the $k$-linear subcategory of injective objects of the Ind-completion of $\mathcal{A}$, i.e.

(4.2) $$\mathrm{HH}^\bullet_{\mathrm{ab}}(\mathcal{A}) \coloneqq \mathrm{HH}^\bullet_{\mathrm{dg}}(\mathrm{Inj}\,\mathrm{Ind}\,\mathcal{A}).$$

This definition makes sense because for any abelian category we have that $\operatorname{Ind}\mathcal{A}$ is a Grothendieck abelian category, hence has enough injective objects.

In [156, theorem 6.1] it is shown that the Hochschild cohomology of abelian categories agrees with the Hochschild cohomology (as a dg category) of a dg enhancement of the (bounded) derived category of an abelian category in the following way.

**Proposition 4.3** (Lowen–Van den Bergh)**.** Let $\mathcal{A}$ be an abelian category. Let $\mathbf{D}^{\mathrm{b}}_{\mathrm{dg}}(\mathcal{A})$ be the dg enhancement of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ given by the full subcategory (of the dg category of complexes in $\operatorname{Ind}\mathcal{A}$ with injective components) on the left bounded complexes of injectives with bounded cohomology inside $\mathcal{A}$. Then there exists an isomorphism

$$\mathrm{HH}^{\bullet}_{\mathrm{ab}}(\mathcal{A}) \cong \mathrm{HH}^{\bullet}_{\mathrm{dg}}(\mathbf{D}^{\mathrm{b}}_{\mathrm{dg}}(\mathcal{A})). \tag{4.3}$$

This allows us to prove properties of the Hochschild cohomology using purely algebraic techniques, in particular proposition 4.25, by taking the derived equivalence of our category of interest $\operatorname{qgr} A$ with $\operatorname{mod} kQ/I$ given by proposition 4.15.

The definition of Hochschild cohomology for a differential graded category as given in section 4.2.1 agrees with the classical definition if one considers an associative algebra as a dg category with a single object, and morphisms concentrated in a single degree, Moreover the Hochschild cohomology of a finite-dimensional algebra agrees with the Hochschild cohomology of its derived category. All these things are recalled in section 1.3.3. We now explicitly describe these results for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

**Example 4.4.** For the projective plane we have

$$\begin{aligned} \mathrm{HH}^0(\mathbb{P}^2) &\cong \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}), \\ \mathrm{HH}^1(\mathbb{P}^2) &\cong \mathrm{H}^0(\mathbb{P}^2, \mathrm{T}_{\mathbb{P}^2}), \\ \mathrm{HH}^2(\mathbb{P}^2) &\cong \mathrm{H}^0(\mathbb{P}^2, \textstyle\bigwedge^2 \mathrm{T}_{\mathbb{P}^2}) \cong \mathrm{H}^0(\mathcal{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)). \end{aligned} \tag{4.4}$$

In particular, the dimensions are 1, 8 and 10 respectively.

Another way of computing these dimensions, closer to the approach taken for noncommutative planes, is by using that

$$\begin{aligned} \mathrm{HH}^1(\mathbb{P}^2) &\cong \mathrm{H}^0(\mathbb{P}^2, \mathrm{T}_{\mathbb{P}^2}) \\ &\cong \operatorname{Lie}\operatorname{Aut}(\mathbb{P}^2) \\ &\cong \operatorname{Lie}\mathrm{PGL}_3 \\ &\cong \mathfrak{sl}_3 \end{aligned} \tag{4.5}$$

is 8-dimensional, combined with corollary 4.28.

**Example 4.5.** For the quadric surface we have

$$\begin{aligned} \mathrm{HH}^0(\mathbb{P}^1 \times \mathbb{P}^1) &\cong \mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}), \\ \mathrm{HH}^1(\mathbb{P}^1 \times \mathbb{P}^1) &\cong \mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathrm{T}_{\mathbb{P}^1\times\mathbb{P}^1}), \\ \mathrm{HH}^2(\mathbb{P}^1 \times \mathbb{P}^1) &\cong \mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1, \textstyle\bigwedge^2 \mathrm{T}_{\mathbb{P}^1\times\mathbb{P}^1}) \cong \mathrm{H}^0(\mathcal{P}^2, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)). \end{aligned} \tag{4.6}$$

In particular, the dimensions are 1, 6 and 9 respectively.

Another way of computing these dimensions, closer to the approach taken for noncommutative quadrics, is by using that

$$\begin{aligned} \mathrm{HH}^1(\mathbb{P}^1 \times \mathbb{P}^2) &\cong \mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathrm{T}_{\mathbb{P}^1\times\mathbb{P}^1}) \\ &\cong \mathrm{Lie}\,\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \\ &\cong \mathrm{Lie}\,((\mathrm{PGL}_2 \times \mathrm{PGL}_2) \rtimes \mathrm{Sym}_2) \\ &\cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \end{aligned} \tag{4.7}$$

is 6-dimensional, combined with corollary 4.28.

### 4.2.2 Artin–Schelter regular $\mathbb{Z}$-algebras

In noncommutative algebraic geometry à la Artin–Zhang [14] one studies the analogues of projective varieties by considering noncommutative graded algebras and their "categories of (quasi)coherent sheaves" on them, by mimicking the construction of $\operatorname{coh} X$ provided by Serre's theorem [190, §59]. An important class of such noncommutative graded algebras is given by the appropriate analogues of the (commutative) polynomial ring (which defines $\mathbb{P}^n$), and a suitable definition for these was found by Artin–Schelter [11].

For the purposes of this chapter it will be important to extend the class of algebras from which we will construct noncommutative projective varieties to include $\mathbb{Z}$-algebras [50, §4]. Recall that a $\mathbb{Z}$-algebra is a non-unital associative algebra $A = \bigoplus_{i,j\in\mathbb{Z}} A_{i,j}$ for which $A_{i,j}$ is a finite-dimensional vector space, the subalgebra $A_{i,i}$ is isomorphic to $k$ for all $i \in \mathbb{Z}$, and $A_{i,j} = 0$ if $j > i$. The multiplication law takes the grading into account, in the sense that it is of the form

$$A_{l,k} \otimes A_{i,j} \to \delta_l^j A_{i,k}. \tag{4.8}$$

The constructions from [14] can be generalised to this setting [226, §2], in particular we have an abelian category $\operatorname{qgr} A$ for every $\mathbb{Z}$-algebra. As in op. cit. we denote $P_{i,A} = e_i A$, where $e_i$ is the local unit associated to $i \in \mathbb{Z}$, and we denote $S_{i,A}$ the unique simple quotient of $P_i$.

**Definition 4.6.** Let $A$ be a $\mathbb{Z}$-algebra. Then $A$ is *Artin–Schelter regular* if

1. $A$ is connected, i.e. $\dim_k A_{i,j} < +\infty$ for all $i \leq j$, $A_{i,j} = 0$ for all $i > j$, and $A_{i,i} \cong k$ for all $i$;
2. $\dim_k A_{i,j}$ has growth bounded by a polynomial in $j - i$;
3. $\operatorname{pdim} S_{i,A} < +\infty$, and is moreover bounded independently of $i$;
4. for all $i \in \mathbb{Z}$

$$\sum_{j,k\in\mathbb{Z}} \dim_k \operatorname{Ext}^j_{\operatorname{Gr} A}(S_{k,A}, P_{i,A}) = 1. \tag{4.9}$$

There is no classification of 3-dimensional Artin–Schelter regular $\mathbb{Z}$-algebras as such, but based on the classification of 3-dimensional Artin–Schelter regular graded algebras into two classes distinguished by their Hilbert series (or similarly by the shape of the minimal projective resolution of the simple module $k$) [11] the following definition is taken from [226, definition 3.1].

**Definition 4.7.** Let $A$ be an Artin–Schelter regular $\mathbb{Z}$-algebra. Then

1. $A$ is *3-dimensional quadratic* if the minimal resolution of $S_{i,A}$ is of the form

$$0 \to P_{i+3,A} \to P_{i+2,A}^{\oplus 3} \to P_{i+1,A}^{\oplus 3} \to P_{i,A} \to S_{i,A} \to 0; \tag{4.10}$$

2. $A$ is *3-dimensional cubic* if the minimal resolution of $S_{i,A}$ is of the form

$$0 \to P_{i+4,A} \to P_{i+3,A}^{\oplus 2} \to P_{i+1,A}^{\oplus 2} \to P_{i,A} \to S_{i,A} \to 0. \tag{4.11}$$

Based on various properties of the graded algebras and their associated abelian categories, the category $\operatorname{qgr} A$ associated to a 3-dimensional quadratic (resp. cubic) algebra $A$ is a *noncommutative plane* (resp. *noncommutative quadric*).

**Example 4.8.** Let $B = k[x, y, z]$ be the commutative polynomial ring. One can associate a $\mathbb{Z}$-algebra $\check{B}$ to it by setting $\check{B}_{i,j} = B_{j-i}$. Then $\operatorname{qgr} B \cong \operatorname{qgr} \check{B} \cong \operatorname{coh} \mathbb{P}^2$.

**Example 4.9.** The $\mathbb{Z}$-algebra for the category $\operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1$ is *not* given by mimicking example 4.8 for the algebra $k[x, y, z, w]/(xy - zw)$. The trivial module $k$ has infinite global dimension in this case, so its minimal projective resolution cannot be of the prescribed form. This reasoning actually tells us that the only commutative Artin–Schelter regular graded algebra is the commutative polynomial ring.

Rather we need to look for a noncommutative graded algebra (or $\mathbb{Z}$-algebra) $B$ such that $\operatorname{qgr} B \cong \operatorname{coh} \mathbb{P}^1 \times \mathbb{P}^1$. This graded algebra is given by [226, example 5.1.1]

$$B := k\langle x, y\rangle/(x^2y - yx^2, xy^2 - y^2x). \tag{4.12}$$

We will now summarise the main result of [50, 226], which is the classification of the Artin–Schelter regular $\mathbb{Z}$-algebras corresponding to noncommutative planes (resp. quadrics). It gives an equivalence of categories between

1. the category of $\mathbb{Z}$-algebras with morphisms being the isomorphisms,
2. the category of elliptic triples (resp. quadruples) with morphisms being the isomorphisms (whose precise definition is given in section 4.4),
3. the category of linear algebra data that describes the composition law in the quiver associated to the canonical full and strong exceptional collection from proposition 4.15.

**Remark 4.10.** Observe that isomorphism of $\mathbb{Z}$-algebras corresponds to equivalence of the graded module categories [192]. We will come back to this in the context of noncommutative planes (resp. quadrics) in remark 4.37 (resp. remark 4.42).

In corollary 4.33 we will relate the outer automorphisms of the path algebra described in the third part of the comparison to certain automorphisms of the geometric data describing the noncommutative plane (resp. cubic), which will allow us to give a description of the Hochschild cohomology.

We will change the terminology and notation from [50, §5, §6] to be consistent with that of [226]. In that case the linear algebra data describing the $\mathbb{Z}$-algebra of a noncommutative plane is given by the following definition.

**Definition 4.11.** A *geometric quadruple* is a quadruple of vector spaces $(V_0, V_1, V_2, W)$ such that $\dim_k V_i = 3$, and $W \subseteq V_0 \otimes V_1 \otimes V_2$ has $\dim_k W = 1$, hence we may assume that $W = kw$ for some tensor $w$. We moreover ask that for all $j = 0, 1, 2$ and for all restrictions $w|_{v_j}$ for $v_j \in V_j$ the associated bilinear form obtained $V_{j-1} \otimes V_{j+1} \to k$ is of rank at least two.

We will say that it is *elliptic* if the determinant associated to the restriction $w_{\phi_j}$ is a cubic in $\mathbb{P}(V_j)$.

For the $\mathbb{Z}$-algebra describing a noncommutative quadric the linear algebra data is described as follows.

**Definition 4.12.** A quintuple of vector spaces $(V_0, V_1, V_2, V_3, W)$ is called a *geometric quintuple* if $\dim_k V_i = 2$, and $W \subseteq V_0 \otimes V_1 \otimes V_2 \otimes V_3$ has $\dim_k W = 1$, hence we may assume that $W = kw$ for some tensor $w$. We moreover ask that for all $j = 0, 1, 2, 3$ and for all $\phi_j \in V_j^\vee \setminus \{0\}$, $\phi_{j=1} \in V_{j+1}^\vee \setminus \{0\}$

(4.13) $\langle \phi_j \otimes \phi_{j+1}, w \rangle \neq 0.$

Using [226, theorem 4.4] we will say that it is *elliptic* if it is not of the form $F(A)$ where $A$ is a linear quadric, in the notation of loc. cit.

An isomorphism of quintuples is an isomorphism of the vector spaces $V_i$ that preserves the tensor $w$.

The classification result for noncommutative planes (resp. quadrics) is now given by a triangle of categorical equivalences. For noncommutative planes it is a combination of [50, theorem 6.1 and theorem 6.2], together with the results of §4 of op.cit.

**Proposition 4.13** (Bondal–Polishchuk)**.** There exist equivalences of categories between

1. elliptic quadratic Artin–Schelter regular $\mathbb{Z}$-algebras;
2. elliptic triples;
3. elliptic geometric quadruples;

where the morphisms in each category are the isomorphisms of these objects.

For noncommutative quadrics it is given in [226, §4].

**Proposition 4.14** (Van den Bergh)**.** There exist equivalences of categories between

1. elliptic cubic Artin–Schelter regular $\mathbb{Z}$-algebras;
2. elliptic quadruples;

3. elliptic geometric quintuples;

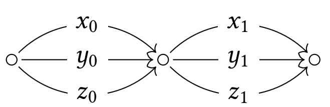
where the morphisms in each category are the isomorphisms of these objects.

One way of interpreting the linear algebra data in the classification is by the following description of the derived category of a noncommutative plane (resp. quadric).

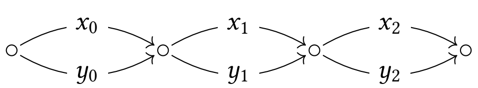
**Proposition 4.15.** Let $A$ be a quadratic (resp. cubic) three-dimensional Artin–Schelter regular $\mathbb{Z}$-algebra. Then $\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A)$ admits a full and strong exceptional collection

$$\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A) = \begin{cases} \left\langle \widetilde{A}, \widetilde{A}(1), \widetilde{A}(2) \right\rangle & A \text{ quadratic} \\ \left\langle \widetilde{A}, \widetilde{A}(1), \widetilde{A}(2), \widetilde{A}(3) \right\rangle & A \text{ cubic} \end{cases} \tag{4.14}$$

whose structure is described by the quiver

(4.15) 


resp.

(4.16) 


whose relations in the elliptic case can be obtained from the linear algebra data in propositions 4.13 and 4.14 by considering the tensor $w$ as a superpotential.

In particular we can use the agreement of various notions of Hochschild cohomology to get the following description.

**Corollary 4.16.** Let $A$ be a quadratic (resp. cubic) three-dimensional Artin–Schelter regular $\mathbb{Z}$-algebra. Let $kQ/I$ be the endomorphism algebra of the full and strong exceptional collection from proposition 4.15. Then we have an isomorphism of Gerstenhaber algebras

$$\mathrm{HH}^{\bullet}_{\mathrm{ab}}(\operatorname{qgr} A) \cong \mathrm{HH}^{\bullet}(kQ/I). \tag{4.17}$$

### 4.2.3 Segre symbols

The classification of noncommutative planes using geometric data depends on the classification of plane cubic curves [50, §6]. This classification is a classical and well-known result, and will be used without further comment.

The classification of noncommutative quadrics using geometric data depends on the classification of $(2,2)$-divisors on $\mathbb{P}^1 \times \mathbb{P}^1$ [226, §4.4]. This is also classical, but not so well-known. We will give some background to this classification, and in table 4.3 we will introduce the labeling used in this chapter for these divisors.

Such a $(2,2)$-divisor is the base locus of a pencil of quadrics in $\mathbb{P}^3$, so we wish to classify these base loci. We can represent elements of the pencil as a symmetric $4 \times 4$-matrix $\alpha M + \beta N$, where $M$ is the matrix describing $\mathbb{P}^1 \times \mathbb{P}^1$ in its Veronese embedding (in particular, it is of maximum rank) whilst $N$ is any other (non-zero) element of the pencil.

**Definition 4.17.** Let $M$ be a square matrix of size $n+1$. Let $\lambda_1,\ldots,\lambda_e$ be the eigenvalues of $M$, in decreasing order of algebraic multiplicity, and in case of ambiguity increasing order of number of Jordan blocks with said eigenvalue.

Let $(m_{j,1},\ldots,m_{j,f_j})$ be the sizes in decreasing order of the Jordan blocks with eigenvalue $\lambda_j$ in the Jordan normal form of $M$. Then the *Segre symbol* of $M$ is

$$\left[(m_{1,1},\ldots,m_{1,f_1}),\ldots,(m_{e,1},\ldots,m_{e,f_e})\right]. \tag{4.18}$$

If $f_j=1$, then we will write $m_{j,1}$ instead of $(m_{j,1})$.

The following result is the main reason why we are interested in Segre symbols [75, theorem 8.6.3].

**Proposition 4.18** (Segre)**.** The base loci of two pencils of quadrics in $\mathbb{P}^n$ are projectively equivalent if and only if the Segre symbols coincide and there exists a projective isomorphism of the pencils that preserves the singular quadrics in the pencil.

The main example we are interested in is for $n=3$, in which case there are precisely 14 possible Segre symbols, because there are 14 two-dimensional partitions of 4. The case $[(1,1,1,1)]$ corresponds to the situation where the two quadrics coincide, so we ignore this. Then the possible Segre symbols and a description of the base locus are given in table 4.3. They are not ordered based on the degeneration of the coincidence of the eigenvalues, rather they are grouped based on their geometric properties, see also table 4.5. This mimicks the ordering of the plane cubics used in [50, §6].

## 4.3 A description of Hochschild cohomology

In this section we give a general procedure to determine the dimensions of the Hochschild cohomology groups, the actual computations are performed in section 4.4. The idea is to use the Euler characteristic of the Hochschild cohomology as in section 4.3.1, which in our case will only involve three terms by proposition 4.27. As $\mathrm{HH}^0$ will always be one-dimensional in this setting, it suffices to compute $\mathrm{HH}^1$ to determine $\mathrm{HH}^2$, as in corollary 4.28. In section 4.3.3 we explain how the Lie algebra of outer automorphisms of the finite-dimensional algebra obtained by tilting theory is related to automorphisms of the geometric data used in the classification of noncommutative planes and quadrics.

### 4.3.1 Euler characteristic of Hochschild cohomology

A first ingredient in the computation of the Hochschild cohomology of noncommutative planes and quadrics is a Lefschetz type formula describing the Euler characteristic of Hochschild cohomology of a smooth and proper dg category. Recall that any such category admits a Serre functor on its derived category, inducing by functoriality an automorphism $\mathbb{S}$ of the Hochschild homology.

For any smooth and proper dg category $\mathcal{C}$ we will denote

$$\chi(\mathrm{HH}^\bullet(\mathcal{C})) = \dim_k \mathrm{HH}^{\mathrm{even}}(\mathcal{C}) - \dim_k \mathrm{HH}^{\mathrm{odd}}(\mathcal{C}) = \sum_{i\in\mathbb{Z}}(-1)^i \dim_k \mathrm{HH}^i(\mathcal{C}) \tag{4.19}$$

the Euler characteristic of Hochschild cohomology, and likewise for the Hochschild homology of $\mathcal{C}$. We then have the following result [180, corollary 3.11].

**Proposition 4.19.** Let $\mathcal{C}$ be a smooth and proper dg category. Then

$$\chi(\mathrm{HH}^\bullet(\mathcal{C})) = \mathrm{tr}\left(\mathbb{S}^{-1}|_{\mathrm{HH}_{\mathrm{even}}(\mathcal{C})}\right) - \mathrm{tr}\left(\mathbb{S}^{-1}|_{\mathrm{HH}_{\mathrm{odd}}(\mathcal{C})}\right). \tag{4.20}$$

Because $(-1)^{\dim X}\,\mathbb{S}^{-1}$ is upper triangular we get the following corollary [180, example 3.12].

**Corollary 4.20.** Let $X$ be a smooth, projective variety. Then

$$\chi(\mathrm{HH}^\bullet(X)) = (-1)^{\dim X}\chi(\mathrm{HH}_\bullet(X)). \tag{4.21}$$

Observe that in general the sum on the left-hand side of (4.21) runs from 0 to $2\dim X$, whilst the sum on the right-hand side runs from $-\dim X$ to $\dim X$, using the Hochschild–Kostant–Rosenberg theorem.

The main examples we are interested in are $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$, which admit a full and strong exceptional collection. In general, whenever we have a tilting object (e.g. the direct sum of a full and strong exceptional collection) we can simplify the right-hand side of (4.21).

**Corollary 4.21.** Let $X$ be a smooth and projective variety with a tilting object. Then

$$\chi(\mathrm{HH}^\bullet(X)) = (-1)^{\dim X}\dim_k \mathrm{HH}_0(X). \tag{4.22}$$

*Proof.* By [52, theorem 4.1] we have that $\mathrm{HH}_i(X)$ is concentrated in degree 0. □

Observe that for smooth projective varieties admitting a full and strong exceptional collection we have that $\dim_k \mathrm{HH}_0(X) = \mathrm{rk}\,\mathrm{K}_0(X)$ is the number of exceptional objects.

**Remark 4.22.** In the situation of corollary 4.21 it is possible to give a purely algebraic proof of (4.22) using [94, theorem 2.2]. In this case one uses that the Serre functor acts unipotently on K-theory [50, lemma 3.1], which is an invariant of the derived category hence applies to the finite-dimensional algebra $kQ/I$.

It is important to note that the Serre functor acting unipotently *only depends* on the structure of the quiver with relations, hence for any noncommutative plane or quadric we will obtain the same result.

This already allows for a heuristic interpretation of the moduli spaces in the case of noncommutative surfaces having a full and strong exceptional collection. Observe that we always have $\mathrm{HH}^0(X) \cong k$, so if $\mathrm{HH}^i(X) = 0$ for $i \geq 3$ (i.e. that all deformations are unobstructed), we have that

$$\dim \mathrm{HH}^2(X) - \dim \mathrm{HH}^1(X) = \mathrm{rk}\,\mathrm{K}_0(X) - 1 \tag{4.23}$$

describes the number of moduli for (noncommutative) deformations of $X$. This is indeed confirmed in the case of noncommutative planes (resp. quadrics) by the classification of [13, 50] (resp. [226]). Observe that in both these cases the global dimension of the algebra associated to the full and strong exceptional collection is 2, see proposition 4.27.

**Example 4.23.** The classification of noncommutative projective planes is performed in [50] in terms of quadratic Artin–Schelter regular $\mathbb{Z}$-algebras. As discussed in section 4.4.1 the geometric data classifying these algebras is an elliptic triple. In the generic case we are considering elliptic curves, and we see that there are $1+2-1=2$ moduli: one from the $j$-line, two from $\operatorname{Pic}^3 C$ and we subtract one from the one-dimensional automorphism group of $C$.

**Example 4.24.** The classification of noncommutative quadrics is performed in [226] in terms of cubic Artin–Schelter regular $\mathbb{Z}$-algebras. As discussed in section 4.4.2 the geometric data classifying these algebras is an elliptic quadruple. In the generic case we are considering elliptic curves, and we see that there are $1+3-1=3$ moduli: one from the $j$-line, three from $\operatorname{Pic}^2 C$ and we subtract one from the one-dimensional automorphism group of $C$.

Similar observations apply to the other del Pezzo surfaces by [222].

### 4.3.2 First Hochschild cohomology as Lie algebra

As explained in [121], we have that Hochschild cohomology and its structure as a super-Lie algebra is an invariant of the derived equivalence class. In particular, when computing $\mathrm{HH}^\bullet_{\mathrm{ab}}(\operatorname{qgr} A)$ we can compute this by using tools from the representation theory of finite-dimensional algebras to compute $\mathrm{HH}^\bullet(kQ/I)$.

We are interested in the first Hochschild cohomology group, for which we have the following description [85, 121].

**Proposition 4.25.** Let $B := kQ/I$ be a finite-dimensional algebra. There exists an isomorphism of Lie algebras

$$(\mathrm{HH}^1(B), [-,-]) \cong \operatorname{Lie}\operatorname{Out}(B) = \operatorname{Lie}\operatorname{Out}^0(B). \tag{4.24}$$

Observe that this therefore describes the Lie algebra structure of $\mathrm{HH}^1_{\mathrm{ab}}(\operatorname{qgr} A)$. It would be interesting to describe both the algebra structure of $\mathrm{HH}^\bullet_{\mathrm{ab}}(\operatorname{qgr} A)$ and the Lie module structure of $\mathrm{HH}^{\geq 2}_{\mathrm{ab}}(\operatorname{qgr} A)$. For the algebra structure it is known (using a purely algebraic proof) for the commutative plane and quadric that the cup products of elements in degree 1 generate the degree 2 part. On the other hand, computations suggest that for noncommutative planes and quadrics the algebra structure is usually trivial, i.e. all cup products are zero, except for a few cases in the classification.

In appendix A we will describe the whole Gerstenhaber algebra structure on the Hochschild cohomology of $\mathbb{P}^n$, because in the commutative case we can use the Hochschild–Kostant–Rosenberg decomposition to describe the higher components too.

**Remark 4.26.** It is important to observe that $\operatorname{Out}(A)$ is not necessarily an invariant of the derived category, but $\operatorname{Out}^0(A)$ is. Also, this description is valid independent of the characteristic of $k$. Because we avoid $\operatorname{char} k = 2, 3$ we will have that the $\operatorname{Out}(A)$ in question are smooth algebraic groups.

To compute all dimensions of the Hochschild cohomology of a noncommutative plane (resp. quadric) we will use that the finite-dimensional algebra obtained by tilting is of global dimension 2, which limits the possibly nonzero Hochschild cohomology spaces. Observe that in the commutative case we could have used the Hochschild–Kostant–Rosenberg decomposition from (HKR-1).

**Proposition 4.27.** Let $A$ be a quadratic (resp. cubic) Artin–Schelter regular $\mathbb{Z}$-algebra. Then

$$\mathrm{HH}^i_{\mathrm{ab}}(\operatorname{qgr} A) = 0 \tag{4.25}$$

for $i \geq 3$.

*Proof.* Let $B$ be a finite-dimensional algebra. We will denote the *Hochschild cohomology dimension* by

$$\mathrm{HH}^\bullet\text{-}\dim B := \sup\{n \in \mathbb{N} \mid \mathrm{HH}^n(B) \neq 0\}. \tag{4.26}$$

Because $\operatorname{pdim}_{B^{\mathrm{e}}} B = \operatorname{gldim} B$ we have that

$$\mathrm{HH}^\bullet\text{-}\dim B \leq \operatorname{gldim} B. \tag{4.27}$$

We have that $\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A)$ admits the tilting object constructed as the direct sum of the $A(i)$'s for $i = 0, \ldots, 2$ (resp. $i = 0, \ldots, 3$).

In the *quadratic* case it is immediate that $\operatorname{gldim} B = 2$ as $\operatorname{gldim} B \leq \#Q_0 - 1$, and $B$ is not hereditary.

In the *cubic* case use that there are precisely two relations of length 3 to conclude by a computation that the projective dimension of the simple object associated to $A$ is 2. □

Hence under the assumption that $\operatorname{gldim} kQ/I = 2$ and that the Serre functor acts unipotently on Hochschild homology, we can compute the dimension as follows:

**Corollary 4.28.**

$$\dim_k \mathrm{HH}^i_{\mathrm{ab}}(\operatorname{qgr} A) = \begin{cases} 1 & i = 0 \\ \dim_k \operatorname{Lie} \operatorname{Out}^0(kQ/I) & i = 1 \\ \#Q_0 + \dim \mathrm{HH}^1(kQ/I) - 1 & i = 2 \\ 0 & i \geq 3 \end{cases}. \tag{4.28}$$

*Proof.* We have that

$$\mathrm{HH}^0_{\mathrm{ab}}(\operatorname{qgr} A) \cong \mathrm{HH}^0(kQ/I) \cong \mathrm{Z}(kQ/I) \cong k \tag{4.29}$$

because the quiver is acyclic and connected.

The result now follows from proposition 4.25, proposition 4.27 and the proof of corollary 4.21 because the unipotency of the Serre functor only depends on the structure of the Cartan matrix, which is invariant under modifying the relations in the quiver associated to a noncommutative plane (resp. quadric). □

**Remark 4.29.** Observe that one expects a strong correspondence between the Hochschild cohomology of a noncommutative plane (resp. quadric) and the Poisson cohomology of its semiclassical limit. In particular, working over the complex numbers the dimension formula in corollary 4.28 is closely related to the results of [106].

### 4.3.3 Identifying automorphism groups

The following theorem is the main tool in computing the Hochschild cohomology of noncommutative planes and quadrics, and depends crucially on the classification result for these objects. The lack of a classification is the main obstruction in systematically generalising the description of Hochschild cohomology for more general noncommutative objects, such as the other noncommutative del Pezzo surfaces, or noncommutative $\mathbb{P}^3$'s.

To a $\mathbb{Z}$-algebra we can associate its *truncation*. This is a finite-dimensional algebra, which is used in the classification of noncommutative planes and quadrics. If $A$ is an elliptic quadratic (resp. cubic) Artin–Schelter regular $\mathbb{Z}$-algebra we define the *truncation* $A_\sigma$ as

$$(4.30)\quad A_\sigma := \bigoplus_{i,j=0}^{n} A_{i,j}$$

where $n = 3$ (resp. $n = 4$). It has the following easy but important interpretation in terms of the exceptional collection of proposition 4.15.

**Lemma 4.30.** There exists an isomorphism

$$(4.31)\quad A_\sigma \cong kQ/I,$$

where $kQ/I$ is the algebra associated to the full and strong exceptional collection from proposition 4.15.

*Proof.* Because $A$ is generated in degree 1 we can associate the arrows in the quiver with a basis for the components $A_{i,i+1}$, and the obvious morphism induced from this is an isomorphism by a dimension count. □

We will denote $\mathcal{T}$ (resp. $\mathcal{Q}$) for the category of elliptic triples (resp. quadruples) as introduced in section 4.2.2. Then for an object $(C, \mathcal{L}_0, \mathcal{L}_1)$ we will consider the automorphism group $\mathrm{Aut}_{\mathcal{T}}(C, \mathcal{L}_0, \mathcal{L}_1)$ (and likewise for elliptic quadruples).

**Theorem 4.31.** Let $A$ be an elliptic 3-dimensional quadratic (resp. cubic) $\mathbb{Z}$-algebra. Let $A_\sigma$ be the truncation of $A$ as in lemma 4.30. Let $(C, \mathcal{L}_0, \mathcal{L}_1)$ (resp. $(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$) be the elliptic triple (resp. quadruple) associated to $A$. Then

$$(4.32)\quad \mathrm{Aut}_k(A_\sigma) \cong \mathrm{Aut}_{\mathcal{T}}(C, \mathcal{L}_0, \mathcal{L}_1)$$

resp.

$$(4.33)\quad \mathrm{Aut}_k(A_\sigma) \cong \mathrm{Aut}_{\mathcal{Q}}(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2).$$

*Proof.* For noncommutative planes it is shown in [50, proposition 5.8] that the category $\widetilde{\mathrm{Ord}}_3$ is equivalent to the category of non-degenerate quantum determinants $\widetilde{\mathrm{Qd}}_3$. This first category is precisely the category of path algebras on 3 vertices with relations, with morphisms being the isomorphisms. We can moreover restrict ourselves now to the subcategory $\widetilde{\mathrm{Qd}}(3,3,3)$ of $\widetilde{\mathrm{Qd}}_3$, consisting of non-degenerate quantum determinants of the form $\varphi\colon U \otimes_k V \otimes_k W \to k$ where $\dim_k U = \dim_k V = \dim_k W = 3$.

In [50, theorem 6.2] it is shown that the subcategory Ed of $\widetilde{\mathrm{Qd}}(3,3,3)$ of tensors for which the determinants of the restriction of $\varphi$ to $U$, $V$ or $W$ are cubics in $\mathbb{P}^2$ (i.e. we are in the elliptic case) is equivalent to the category of elliptic triples $\mathcal{T}$. In particular, the automorphism groups for the $\mathbb{Z}$-algebras for noncommutative planes are identified with the automorphism groups of the associated finite-dimensional algebra, as in (4.32).

Similarly for noncommutative quadrics, we can use [226, corollaries 4.3 and 4.5] for the relation between finite-dimensional algebras and noncommutative quadrics, and [226, corollary 4.32] for the correspondence between automorphisms of the finite-dimensional algebras with the automorphisms of the elliptic quadruples. □

**Remark 4.32.** The structure of the objects in the subcategory $\widetilde{\mathrm{Qd}}(3,3,3)$ in the proof of theorem 4.31 rather corresponds to the exceptional collection $\mathcal{O}_{\mathbb{P}^2}, \mathrm{T}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$. For the actual computation of the Hochschild cohomology this does not matter.

To apply proposition 4.25 we need to understand the outer automorphisms of the endomorphism algebra of the full and strong exceptional collection. This is achieved by the following corollary, where we denote $\mathrm{Aut}_{\mathcal{L}_0,\mathcal{L}_1}(C)$ for the subgroup of $\mathrm{Aut}(C)$ preserving $\mathcal{L}_0$ and $\mathcal{L}_1$ (and likewise for an elliptic quadruple).

**Corollary 4.33.** In the notation of theorem 4.31 we have that

(4.34) $\mathrm{Out}_k(kQ/I) \cong \mathrm{Aut}_{\mathcal{L}_0,\mathcal{L}_1}(C)$

resp.

(4.35) $\mathrm{Out}_k(kQ/I) \cong \mathrm{Aut}_{\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2}(C)$.

*Proof.* By lemma 4.30 we can replace the truncated algebra $A_\sigma$ by the endomorphism algebra of the full and strong exceptional collection.

Now we need to determine the role of the inner automorphisms in the description (4.32) (resp. (4.33)). The inner automorphisms for $kQ/I$ are given by conjugation with an element of the form $\sum_{i=1}^n \alpha_i e_i$, with $\alpha_i \in k^\times$ and $e_i$ the idempotents corresponding to the vertices, for $n = 3$ (resp. $n = 4$). These give an inner automorphism group of the form $k^\times \times k^\times$ (resp. $k^\times \times k^\times \times k^\times$).

In the description of the automorphism of elliptic triples (resp. quadruples), these are precisely the automorphisms of the pairs (resp. triples) of line bundles: we have that $\mathrm{Aut}(\mathcal{L}_i) = k^\times$. So we obtain a short exact sequence

(4.36) $0 \to k^\times \times k^\times \to \mathrm{Aut}_{\mathcal{T}}(C, \mathcal{L}_0, \mathcal{L}_1) \to \mathrm{Aut}_{\mathcal{L}_0,\mathcal{L}_1}(C) \to 0$

resp.

(4.37) $0 \to k^\times \times k^\times \to \mathrm{Aut}_{\mathcal{Q}}(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) \to \mathrm{Aut}_{\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2}(C) \to 0$

Hence we have the description of the outer automorphisms of $kQ/I$ as in (4.34) (resp. (4.35)). □

In section 4.4 we describe the automorphism groups in (4.34) and (4.35). For noncommutative planes this description is already available in [50, table 6.1], whilst for noncommutative quadrics it is new.

### 4.3.4 Hochschild cohomology of Artin–Schelter regular algebras

In the graded case one can also compute the Hochschild cohomology of *A as an algebra* [153, §1.5]. In the quadratic case the Hochschild (co)homology is computed for skew polynomial algebras in [236], and for the generic Sklyanin algebra in [224]. In the generic Sklyanin-like cubic case there is [160].

Indeed, using Poincaré–Van den Bergh duality the Hochschild cohomology is just dual to the Hochschild homology [221]. One could ask whether there is a connection between the Hochschild cohomology as an algebra, and the Hochschild cohomology of its associated abelian category. The following examples shows that there is no straightforward connection, by exhibiting two quadratic Artin–Schelter regular algebras, for which the zeroth Hochschild cohomology is different, but the abelian categories are equivalent.

**Example 4.34.** Consider

$$\begin{aligned} A &:= k[x,y,z] \\ B &:= k_{\mathbf{q}}[x,y,z] \end{aligned} \tag{4.38}$$

where $k_{\mathbf{q}}[x,y,z]$ is the skew polynomial ring associated to the coefficients in the matrix $\mathbf{q} = (q_{i,j})$ for which $q_{i,j} = q_{j,i}^{-1}$ and $q_{i,i} = 1$. If $q_{0,1}q_{1,2}q_{0,2}^{-1} = 1$, then the point variety of $B$ is isomorphic to $\mathbb{P}^2$ as discussed in chapter 7, and in particular $\operatorname{qgr} A \cong \operatorname{qgr} B$.

But if the parameters $q_{i,j}$ are not roots of unity, then the ring is not finite over its center [162]. In particular we have that

$$\mathrm{HH}^0(A) = \mathrm{Z}(A) = A \ncong \mathrm{HH}^0(B) \tag{4.39}$$

This subtle dependence on the choice of the presentation disappears when considering the abelian category $\operatorname{qgr} A$.

Also, the Hochschild homology of the algebra $B$ lives in degrees $0,\ldots,3$ and again depends on the choice of parameters, whereas the Hochschild homology of $\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} B)$ is computed from the exceptional collection, and hence isomorphic to 3 (resp. 4) copies of $k$ [209]. Likewise there is no immediate connection for the usual commutative polynomial ring $A$, or for other examples from the classification of Artin–Schelter regular algebras.

## 4.4 Automorphisms of 3-dimensional Artin–Schelter regular ℤ-algebras

We now describe the automorphism groups which are necessary to use corollary 4.33.

### 4.4.1 Noncommutative planes

The classification of three-dimensional quadratic Artin–Schelter regular algebras from proposition 4.13 by the classification of elliptic (and linear) triples. We are mostly interested in the elliptic case here, as the linear case corresponds to the commutative projective plane, for which one can appeal to Hochschild–Kostant–Rosenberg to compute the Hochschild cohomology as in example 4.4.

The notion of elliptic triple used in proposition 4.13 is defined as follows.

**Definition 4.35.** An *elliptic triple* is a triple $(C, \mathcal{L}_0, \mathcal{L}_1)$ where

1. $C$ is a curve;
2. $\mathcal{L}_0$ and $\mathcal{L}_1$ are non-isomorphic very ample line bundles of degree 3;

such that $\deg(\mathcal{L}_0|_{C_i}) = \deg(\mathcal{L}_1|_{C_i})$ for each irreducible component $C_i$ of $C$.

The curve $C$ can be interpreted as the *point scheme*, which is a fine moduli space for point modules. We will not need this interpretation, but it is interesting to observe that the infinitesimal automorphisms of the noncommutative object are intimately related to the automorphisms of an associated commutative object.

To apply corollary 4.33 for the computation of $\mathrm{HH}^\bullet(\operatorname{qgr} A)$ where $A$ is an elliptic quadratic Artin–Schelter regular $\mathbb{Z}$-algebra describing a noncommutative plane we need to compute the algebraic group $\operatorname{Aut}_{\mathcal{L}_0,\mathcal{L}_1}(C)$, where $(C, \mathcal{L}_0, \mathcal{L}_1)$ is the elliptic triple associated to $A$ by proposition 4.13. This description can be found in [50, table 1] and is recalled in table 4.4. Reading off the dimension of the algebraic groups leads to table 4.1.

The main observation is the cohomology jump, arising from the fact that the commutative plane has more symmetries than any of the noncommutative planes. The generic quadratic Artin–Schelter regular algebra is of type $\mathrm{P}_1$, whose associated finite-dimensional algebra has only finitely many outer automorphisms, hence the associated noncommutative plane has no infinitesimal automorphisms. Allowing the elliptic curve to degenerate further and further will usually increase the size of the automorphism group.

**Remark 4.36.** A similar drop in the size of the automorphism group can be observed for the automorphisms of "the affine complement of the point scheme $C$". If $A$ is a Sklyanin algebra associated to a point of infinite order it is claimed in [170, proposition 2.10] that the automorphism group of this $k$-algebra is finite. Remark that in table 4.4 all Sklyanin algebras are of type $\mathrm{P}_1$, and that the order of the translation does not influence the outer automorphism group of the algebra.

**Remark 4.37.** Observe that we are mostly interested in automorphisms of the algebra $kQ/I$. These should somehow be related to automorphisms of $\operatorname{qgr} A$, but there is no result relating the automorphisms of an abelian category to its first Hochschild cohomology.

If $A$ is a quadratic Artin–Schelter regular $\mathbb{Z}$-algebra the correspondence from remark 4.10 can be extended to also include equivalences of the quotient category [1, corollary A.10], i.e. equivalences of $\operatorname{qgr} A$ can be lifted back to equivalences of $\operatorname{gr} A$. Such a result is not true for cubic Artin–Schelter regular algebras as we will explain in remark 4.42.

### 4.4.2 Noncommutative quadrics

The classification of three-dimensional cubic Artin–Schelter regular algebras from proposition 4.14 is described by the classification of elliptic (and linear) quadruples. We are mostly interested in the elliptic case here, as the linear case corresponds to the commutative quadric surface, for which one can appeal to Hochschild–Kostant–Rosenberg to compute the Hochschild cohomology as in example 4.5.

The notion of elliptic quadruple used in proposition 4.13 is defined as follows.

**Definition 4.38.** An *elliptic quadruple* is a quadruple $(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ where

1. $C$ is a curve;
2. $(\mathcal{L}_0, \mathcal{L}_1)$ and $(\mathcal{L}_1, \mathcal{L}_2)$ embed $C$ into $\mathbb{P}^1 \times \mathbb{P}^1$;

such that $\deg(\mathcal{L}_0|_{C_i}) = \deg(\mathcal{L}_2|_{C_i})$ for each irreducible component $C_i$ of $C$, and moreover $\mathcal{L}_0 \not\cong \mathcal{L}_2$.

**Remark 4.39.** In [226] these are called admissible quadruples, but to make the terminology consistent with [50] we have adapted the definition of ellipticity to already include regularity and non-prelinearity.

Whereas the full case-by-case description of elliptic triples for noncommutative planes is in the literature, the analogous description for elliptic quadruples is not. The curve $C$ in an elliptic quadruple is a $(2,2)$-divisor on $\mathbb{P}^1 \times \mathbb{P}^1$, for which we can use Segre symbols to distinguish these. Now to perform this case-by-case analysis, we will distinguish the four types divisors, using the notation from table 4.3.

| | reduced | nonreduced |
|---|---|---|
| irreducible | $Q_1, Q_2, Q_3$ | $Q_{11}$ |
| reducible | $Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}$ | $Q_{12}, Q_{13}$ |

Table 4.5: 4 types of divisors

**Reduced and irreducible** These cases (corresponding to cases $Q_1$, $Q_2$ and $Q_3$) are covered explicitly in [50] for the case of noncommutative planes, and the proof can be adapted immediately to the case of $\mathbb{P}^1 \times \mathbb{P}^1$, using 2-torsion rather than 3-torsion in $\operatorname{Pic}^0 C$.

**Reduced, but reducible** Cases $Q_4$ up to $Q_{10}$ are of this form.

For cases $Q_4$, $Q_6$, $Q_8$ and $Q_9$ we have that $\operatorname{Pic}^0 C \cong \mathbb{G}_m$ because the singularities are all nodal. We can interpret $\operatorname{Pic}^0 C$ as the gluing data for a line bundle.

**Remark 4.40.** Cases $Q_4$ and $Q_9$ are isomorphic as abstract schemes, but they differ by their embedding into $\mathbb{P}^3$ and hence by the properties of the line bundles $\mathcal{L}_i$.

In each case the normalisation consists of the appropriate number of copies of $\mathbb{P}^1$, and to compute the automorphism groups one has to fix the preimages of the singularities on each $\mathbb{P}^1$, so the automorphism group of each component is the subgroup $\mathbb{G}_m$ of $\mathrm{PGL}_2$. Then it is possible to either permute the components or permute the preimages of the singularities, which gives a semidirect product of the torus with a finite group.

We will explicitly describe the action of $\operatorname{Aut} C$ on $\operatorname{Pic} C$ for case $Q_4$, the others are similar. The finite group in this case is $\mathrm{Dih}_2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. One copy of $\mathbb{Z}/2\mathbb{Z}$ exchanges the components, the other switches the preimages (and therefore each component of the normalisation is flipped too). So the action on $\mathbb{G}_m^2$ of the first copy exchanges the factors, whilst the second copy inverts both $\mathbb{G}_m$'s.

The copies of $\mathbb{G}_m$ in $\operatorname{Aut} C$ act by rescaling $\operatorname{Pic}^0 C$ taking into account the partial degrees of the line bundle, i.e. $\lambda \in \mathbb{G}_m \subseteq \operatorname{Aut} C$ corresponding to component $i$ will rescale an element of $\operatorname{Pic}^0 C \oplus \mathbb{Z}^{\oplus 4}$ by $\lambda^{k_i}$, where $k_i$ is the partial degree on the component $i$.

Finally, the first copy of $\mathbb{Z}$ permutes the copies of $\mathbb{Z}$ in $\operatorname{Pic} C$ and inverts $\operatorname{Pic}^0 C$, whilst the other only inverts $\operatorname{Pic}^0 C$.

For cases $Q_5$, $Q_7$ and $Q_{10}$ have $\operatorname{Pic}^0 C \cong \mathbb{G}_a$, which can now be interpreted as the tangent space.

**Remark 4.41.** Cases $Q_5$ and $Q_{10}$ are isomorphic as abstract schemes, but their embeddings in $\mathbb{P}^1 \times \mathbb{P}^1$ are different.

The computation for $Q_5$ (and hence $Q_{10}$) is analogous to that of $P_7$, except that the line bundles $\mathcal{L}_i$ will be different. To describe $\operatorname{Aut} C$ we take the normalisation which has 2 $\mathbb{P}^1$'s, where 1 point is fixed on each copy. So the automorphisms of this configuration are $(\mathbb{G}_a \rtimes \mathbb{G}_m)^2 \rtimes \operatorname{Sym}_2$. Because the singularity is *not* an ordinary multiple point we have to take into account that an automorphism needs to act in the same way on the two tangent spaces at the common point.

Written in affine coordinates an element $(a_i, b_i) \in \mathbb{G}_a \rtimes \mathbb{G}_m$ acts on $x \in \mathbb{P}^1 \setminus \{\infty\}$ as

$$x \mapsto a_i x + b_i \tag{4.40}$$

where the point $\infty$ is the point being fixed. Then for the point at $\infty$ we have that the action is

$$x^{-1} \mapsto (a_i x + b_i)^{-1} = x^{-1}(a_i + b_i x^{-1})^{-1} = a_i x^{-1} - b_i a^{-1} x^{-2} \tag{4.41}$$

because infinitesimally we have that $x^{-2} = 0$. So the condition is that $a_1 = a_2$.

For case $Q_5$, as in the case of $P_7$ we get that fixing the first line bundle (whose $\operatorname{Pic}^0 C$ we take to be 0) implies taking the diagonal of the two copies of $\mathbb{G}_a$. By the choice of $\mathcal{L}_0$ the copy of $\mathbb{G}_m$ is preserved. Fixing $\mathcal{L}_1$ removes this copy. The copy of $\operatorname{Sym}_2$ is preserved throughout.

For case $Q_{10}$ on the other hand, we have that the degrees of the line bundles are $(2, 0)$ (resp. $(1, 1)$). Taking line bundle of degree $(1, 1)$ the first step is the same as in the previous paragraph, but the copy of $\operatorname{Sym}_2$ will disappear in the second step.

The remaining case $Q_7$ is similar to $P_5$. Again the singularity is not an ordinary multiple point, so to describe the automorphism group we need to take the tangent space into account. If we let $v_i$ be the vectors that span the tangent space at the singularity for each component, then the kernel is described as $v_1 + v_2 + v_3 = 0$. We computed the action of $\mathbb{G}_a \rtimes \mathbb{G}_m$ on the tangent space in the description of case $Q_5$, so it acts by rescaling by $a_i$, so to preserve the kernel we need $a_1 = a_2 = a_3$. The total automorphism group is therefore $(\mathbb{G}_a^3 \rtimes \mathbb{G}_m) \rtimes \operatorname{Sym}_3$.

If we take the Picard group of the conic to be the first term in the Néron–Severi group, then the degrees of $\mathcal{L}_i$ are of the form $(1, 1, 0)$ or $(1, 0, 1)$. Hence we will reduce $\operatorname{Sym}_3$ to $\mathbb{Z}/2\mathbb{Z}$ and then to the trivial group. For the action on $\operatorname{Pic}^0 C$, we replace $\mathbb{G}_a^3$ by the $\mathbb{G}_a^2$ given by the kernel of the summation map, and $\mathbb{G}_m$ acts non-trivially on each non-zero element of $\mathrm{Pic}^0 C$.

**Nonreduced, but irreducible** Only case $Q_{11}$ is of this form. The automorphism group of the double conic is described in [12, proposition 8.22], and $\mathrm{Pic}^0 C \cong \mathbb{G}_a$. Because $\mathrm{PGL}_2$ cannot act non-trivially on $\mathrm{Pic}^0 C$ we only need to describe the action of $\mathrm{Aut}^0 C$, the automorphisms of $C$ which induce the identity on the reduced subscheme, defined by the ideal sheaf $\mathcal{I}$.

We can interpret the factor $\mathbb{G}_m$ as automorphisms of the ideal sheaf. Hence these act by rescaling $\mathrm{H}^1(C,\mathcal{I}) \cong \mathrm{H}^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1})$. The factor $\mathbb{G}_a$ on the other hand coming from $\mathrm{Aut}^1(C)$ in the notation of loc. cit. will take the degree into account, i.e. for $\beta \in \mathrm{Aut}^1(C)$ and $(n,\lambda) \in \mathbb{Z} \oplus \mathrm{Pic}^0 C$ we have

$$\beta\dot{(}n,\lambda) = (n, \lambda + n\beta). \tag{4.42}$$

The result now follows.

**Nonreduced and reducible** Cases $Q_{12}$ and $Q_{13}$ are of this form. To describe the automorphism group we use the technique from [12, proposition 8.22]. For case $Q_{12}$ the ideal sheaf is the line bundle $\mathcal{O}_C(-1,-1)$, hence the automorphisms which are the identity on the reduced subscheme are again $\mathbb{G}_a \rtimes \mathbb{G}_m$. For case $Q_{13}$ this changes: the ideal sheaf is only supported on the double line, and there it is isomorphic to $\mathcal{O}(-2)$. The automorphisms which are the identity on the reduced subscheme are therefore only $\mathbb{G}_m$.

Unlike case $Q_{11}$ where $\mathrm{PGL}_2$ acts trivially on $\mathrm{Pic}^0 C$ we have a nontrivial action of the $\mathbb{G}_m$-components coming from automorphisms of the underlying curve. To see this, observe that we can describe $\mathrm{Pic}^0 C$ using the long exact sequence coming from the "partial normalisation" which is the morphism from two disjoint double lines to two double lines. We get a long exact sequence

$$\begin{aligned} &0 \to k^\times \to (k[x]/(x^2))^\times \oplus (k[y]/(y^2))^\times \to (k[x,y]/(x^2,y^2))^2 \to \dots \\ &\to \mathrm{Pic}\, C \to \mathrm{Pic}\, C_1 \oplus \mathrm{Pic}\, C_2 \to 0. \end{aligned} \tag{4.43}$$

We realise $\mathrm{Pic}^0 C$ as the cokernel of the appropriate map, which is the vector space spanned by $xy$. To compute the action on $\mathrm{Pic}^0 C$ by the component of automorphisms coming from the reduced subscheme we again use the description as for case $Q_5$, and we see that to fix a non-zero element of $\mathrm{Pic}^0 C$ we need the $\mathbb{G}_m$'s to cancel their action, because the generator $xy$ gets sent to $a_1 a_2 \cdot xy$, where $a_i \in \mathbb{G}_m$.

The action of $\mathbb{G}_a \rtimes \mathbb{G}_m$ coming from the automorphisms fixing the reduced subscheme is as in the previous case. Hence we see that in the first step 1 copy of $\mathbb{G}_a$ is removed, whilst in the second step we remove the copy of $\mathbb{G}_m$ coming from the automorphisms of the ideal sheaf and we replace $\mathbb{G}_m^2$ by $\mathbb{G}_m$.

For case $Q_{12}$ this means that fixing $\mathcal{L}_0$ removes the automorphisms which leave the curve and the ideal fixed. In the second step the action of all $\mathbb{G}_m$'s is nontrivial, and the diagonal subgroup of $\mathbb{G}_m^2$ is considered. In the third step nothing changes.

For case $Q_{13}$ on the other hand the degree of the line bundles will play a role. The automorphisms of the ideal sheaf will disappear in the second step as before. Regarding the other components, the factor $\mathrm{Sym}_2$ disappears in the first step by considering the partial degrees of the line bundle. In the second and third step the $\mathbb{G}_{\mathrm{m}}$-components are paired together in two different ways, so that only a single $\mathbb{G}_{\mathrm{m}}$ remains.

**Remark 4.42.** In remark 4.37 it was explained that there is a correspondence between equivalences of $\mathrm{gr}\,A$ and $\mathrm{qgr}\,A$ where $A$ is a quadratic Artin–Schelter regular $\mathbb{Z}$-algebra. For *cubic* Artin–Schelter regular $\mathbb{Z}$-algebras such a result is *not* true: by the translation principle there exist non-isomorphic $\mathbb{Z}$-algebras whose qgr's are equivalent [226, §6].

This does not influence the method of proof taken in this chapter, but it is interesting to observe that only a discrete amount of information is added, which is irrelevant when taking Lie algebras. It is expected that this is the only new possibility for such a Morita equivalence [201, §11].

| | divisor | $\dim_k \mathrm{HH}^1_{\mathrm{ab}}(\mathrm{qgr}\, A)$ | $\dim_k \mathrm{HH}^2_{\mathrm{ab}}(\mathrm{qgr}\, A)$ |
|---|---|---|---|
| 1. | elliptic curve | 0 | 2 |
| 2. | cuspidal curve | 0 | 2 |
| 3. | nodal curve | 0 | 2 |
| 4. | three lines in general position | 2 | 4 |
| 5. | three lines through a point | 2 | 4 |
| 6. | conic and line in general position | 1 | 3 |
| 7. | conic and tangent line | 1 | 3 |
| 8. | triple line | 5 | 7 |
| 9. | double line and line | 3 | 5 |
| | commutative plane | 8 | 10 |

Table 4.1: Hochschild cohomology of noncommutative planes

| | divisor | $\dim_k \mathrm{HH}^1_{\mathrm{ab}}(\mathrm{qgr}\, A)$ | $\dim_k \mathrm{HH}^2_{\mathrm{ab}}(\mathrm{qgr}\, A)$ |
|---|---|---|---|
| 1. | elliptic curve | 0 | 3 |
| 2. | cuspidal curve | 0 | 3 |
| 3. | nodal curve | 0 | 3 |
| 4. | two conics in general position | 1 | 4 |
| 5. | two tangent conics | 1 | 4 |
| 6. | conic and two lines in a triangle | 1 | 4 |
| 7. | conic and two lines through a point | 2 | 5 |
| 8. | quadrangle | 2 | 5 |
| 9. | twisted cubic and a bisecant | 1 | 4 |
| 10. | twisted cubic and a tangent line | 1 | 4 |
| 11. | double conic | 3 | 6 |
| 12. | two double lines | 3 | 6 |
| 13. | double line and two lines in general position | 3 | 6 |
| | commutative quadric | 6 | 9 |

Table 4.2: Hochschild cohomology of noncommutative quadrics

| | divisor | Segre symbol |
|---|---|---|
| $Q_1$ | elliptic curve | $[1,1,1,1]$ |
| $Q_2$ | cuspidal curve | $[3,1]$ |
| $Q_3$ | nodal curve | $[2,1,1]$ |
| $Q_4$ | two conics in general position | $[(1,1),1,1]$ |
| $Q_5$ | two tangent conics | $[(2,1),1]$ |
| $Q_6$ | a conic and two lines in a triangle | $[2,(1,1)]$ |
| $Q_7$ | a conic and two lines intersecting in one point | $[(3,1)]$ |
| $Q_8$ | quadrangle | $[(1,1),(1,1)]$ |
| $Q_9$ | twisted cubic and a bisecant | $[2,2]$ |
| $Q_{10}$ | twisted cubic and a tangent line | $[4]$ |
| $Q_{11}$ | double conic | $[(1,1,1),1]$ |
| $Q_{12}$ | two double lines | $[(2,1,1)]$ |
| $Q_{13}$ | a double line and two lines | $[(2,2)]$ |

Table 4.3: Segre symbols for $(2,2)$-divisors on $\mathbb{P}^1 \times \mathbb{P}^1$

| | divisor | $\mathrm{Pic}^0 C$ | $\mathrm{Aut}_{\mathcal{L}_0,\mathcal{L}_1}(C)$ |
|---|---|---|---|
| $P_1$ | elliptic curve | $C$ | $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ |
| $P_2$ | cuspidal cubic | $\mathbb{G}_a$ | $1$ |
| $P_3$ | nodal cubic | $\mathbb{G}_m$ | $\mathbb{Z}/3\mathbb{Z}$ |
| $P_4$ | three lines in general position | $\mathbb{G}_m$ | $\mathbb{G}_m^2 \rtimes \mathrm{Cyc}_3$ |
| $P_5$ | three concurrent lines | $\mathbb{G}_a$ | $\mathbb{G}_a^2 \rtimes \mathrm{Sym}_3$ |
| $P_6$ | conic and a line | $\mathbb{G}_m$ | $\mathbb{G}_m$ |
| $P_7$ | conic and a tangent line | $\mathbb{G}_a$ | $\mathbb{G}_a$ |
| $P_8$ | triple line | $\mathbb{G}_a$ | $\mathbb{G}_a^2 \rtimes \mathrm{SL}_2$ |
| $P_9$ | double line and a line | $\mathbb{G}_a$ | $\mathbb{G}_a \times (\mathbb{G}_a \rtimes \mathbb{G}_m)$ |

Table 4.4: Automorphism groups of elliptic triples

| | decomposition | Segre symbol | $\mathrm{Pic}^0 C$ | $\mathrm{Aut}\, C$ |
|---|---|---|---|---|
| $Q_1$ | $(2,2)$ | $[1,1,1,1]$ | $C$ | $C \rtimes \mathrm{Aut}_p(C)$ |
| $Q_2$ | $(2,2)$ | $[3,1]$ | $\mathbb{G}_a$ | $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \rtimes \mathrm{Aut}_p(C)$ |
| $Q_3$ | $(2,2)$ | $[2,1,1]$ | $\mathbb{G}_m$ | $\mathbb{G}_m \rtimes \mathrm{Sym}_2$ |
| $Q_4$ | $(1,1)+(1,1)$ | $[(1,1),1,1]$ | $\mathbb{G}_m$ | $\mathbb{G}_m^2 \rtimes \mathrm{Dih}_2$ |
| $Q_5$ | $(1,1)+(1,1)$ | $[(2,1),1]$ | $\mathbb{G}_a$ | $(\mathbb{G}_a^2 \rtimes \mathbb{G}_m) \rtimes \mathrm{Sym}_2$ |
| $Q_6$ | $(1,1)+(1,0)+(0,1)$ | $[2,(1,1)]$ | $\mathbb{G}_m$ | $\mathbb{G}_m^3 \rtimes \mathrm{Sym}_3$ |
| $Q_7$ | $(1,1)+(1,0)+(0,1)$ | $[(3,1)]$ | $\mathbb{G}_a$ | $(\mathbb{G}_a^3 \rtimes \mathbb{G}_m) \rtimes \mathrm{Sym}_3$ |
| $Q_8$ | $(1,0)+(1,0)+(0,1)+(0,1)$ | $[(1,1),(1,1)]$ | $\mathbb{G}_m$ | $\mathbb{G}_m^4 \rtimes \mathrm{Dih}_4$ |
| $Q_9$ | $(2,1)+(0,1)$ | $[2,2]$ | $\mathbb{G}_m$ | $\mathbb{G}_m^2 \rtimes \mathrm{Dih}_2$ |
| $Q_{10}$ | $(2,1)+(0,1)$ | $[4]$ | $\mathbb{G}_a$ | $(\mathbb{G}_a^2 \rtimes \mathbb{G}_m) \rtimes \mathrm{Sym}_2$ |
| $Q_{11}$ | $2(1,1)$ | $[(1,1,1),1]$ | $\mathbb{G}_a$ | $(\mathbb{G}_a \rtimes \mathbb{G}_m) \times \mathrm{PGL}_2$ |
| $Q_{12}$ | $2(1,0)+2(0,1)$ | $[(2,1,1)]$ | $\mathbb{G}_a$ | $(\mathbb{G}_a \rtimes \mathbb{G}_m) \times (\mathbb{G}_a \rtimes \mathbb{G}_m)^2 \rtimes \mathrm{Sym}_2$ |
| $Q_{13}$ | $2(1,0)+(0,1)+(0,1)$ | $[(2,2)]$ | $\mathbb{G}_a$ | $\mathbb{G}_m \times \left(\mathbb{G}_m \times (\mathbb{G}_a \rtimes \mathbb{G}_m)^2\right) \rtimes \mathrm{Sym}_2$ |

Table 4.6: Properties of elliptic quadruples

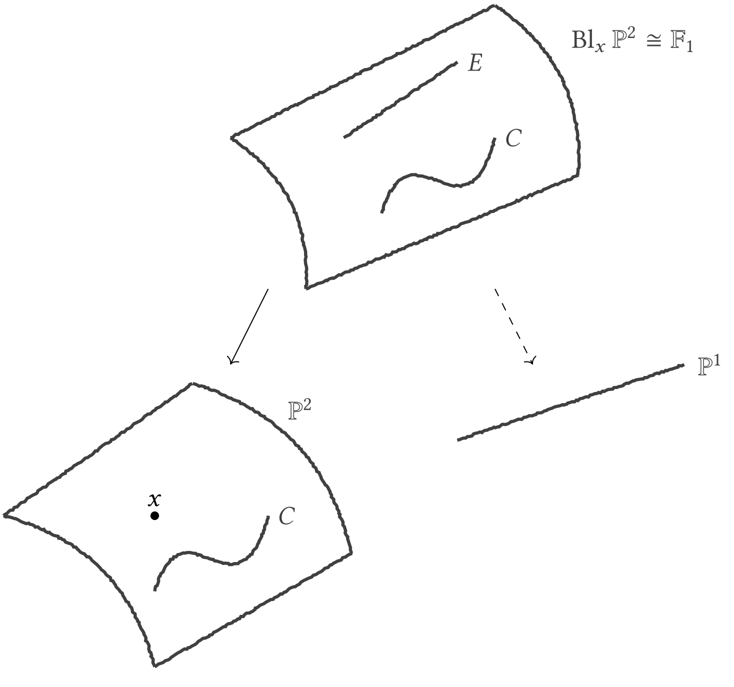
| | $\mathrm{Aut}_{\mathcal{L}_0}(C)$ | $\mathrm{Aut}_{\mathcal{L}_0,\mathcal{L}_1}(C)$ | $\mathrm{Aut}_{\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2}(C)$ |
|---|---|---|---|
| $Q_1$ | $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \rtimes \mathrm{Aut}_p(C)$ | $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ | $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ |
| $Q_2$ | $\mathbb{G}_m$ | 1 | 1 |
| $Q_3$ | $V_4$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $Q_4$ | $\mathbb{G}_m \rtimes \mathrm{Dih}_2$ | $\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}$ |
| $Q_5$ | $(\mathbb{G}_a \rtimes \mathbb{G}_m) \times \mathrm{Sym}_2$ | $\mathbb{G}_a \times \mathrm{Sym}_2$ | $\mathbb{G}_a \times \mathrm{Sym}_2$ |
| $Q_6$ | $\mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{G}_m$ | $\mathbb{G}_m$ |
| $Q_7$ | $(\mathbb{G}_a^2 \rtimes \mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{G}_a^2$ | $\mathbb{G}_a^2$ |
| $Q_8$ | $\mathbb{G}_m^3 \rtimes \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{G}_m^2 \rtimes Z/2\mathbb{Z}$ | $\mathbb{G}_m^2 \rtimes Z/2\mathbb{Z}$ |
| $Q_9$ | $\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ or[a] $\mathbb{G}_m \rtimes \mathrm{Dih}_2$ | $\mathbb{G}_m$ | $\mathbb{G}_m$ |
| $Q_{10}$ | $\mathbb{G}_a \rtimes \mathbb{G}_m$ or $(\mathbb{G}_a \rtimes \mathbb{G}_m) \times \mathrm{Sym}_2$ | $\mathbb{G}_a$ | $\mathbb{G}_a$ |
| $Q_{11}$ | $\mathbb{G}_m \times \mathrm{PGL}_2$ | $\mathrm{PGL}_2$ | $\mathrm{PGL}_2$ |
| $Q_{12}$ | $\mathbb{G}_m \times (\mathbb{G}_a \rtimes \mathbb{G}_m)^2 \rtimes \mathrm{Sym}_2$ | $\mathbb{G}_a^2 \rtimes (\mathbb{G}_m \times \mathrm{Sym}_2)$ | $\mathbb{G}_a^2 \rtimes (\mathbb{G}_m \times \mathrm{Sym}_2)$ |
| $Q_{13}$ | $\mathbb{G}_m \times \left(\mathbb{G}_m \times (\mathbb{G}_a \rtimes \mathbb{G}_m)^2\right)$ | $(\mathbb{G}_a \rtimes \mathbb{G}_m)^2$ | $\mathbb{G}_a^2 \rtimes \mathbb{G}_m$ |

Table 4.7: Automorphism groups of elliptic quadruples

[a] Depending on whether you take $\mathcal{L}_0$ of degree (2, 0) or (1, 1).

*Chapter 5*

# Constructing noncommutative surfaces of rank 4



## 5.1 Introduction

In a recent paper de Thanhoffer de Völcsey and Van den Bergh provide a *numerical* classification of possibly noncommutative surfaces with an exceptional sequence of length 4 [212]. Their classification describes the possible bilinear forms on a free abelian group of rank 4 mimicking the properties of the numerical Grothendieck group and Euler form on a smooth projective surface.

Those properties are described as follows: for a finitely generated free abelian group $\Lambda$ taking on the role of $\mathrm{K}_0$, a nondegenerate bilinear form $\langle -,-\rangle\colon \Lambda\times\Lambda\to\mathbb{Z}$ describing the Euler pairing and an automorphism $s\in\mathrm{Aut}(\Lambda)$ for the Serre functor we will ask that

**Serre automorphism** $\langle x, s(y)\rangle = \langle y, x\rangle$ for $x, y \in \Lambda$;

**unipotency** $s - \mathrm{id}_\Lambda$ is nilpotent;

**rank** $\mathrm{rk}(s-\mathrm{id}_\Lambda) = 2$;

By the nondegeneracy we know that if we choose a basis for $\Lambda$, and express the bilinear form (resp. the Serre automorphism) as the Cartan or Gram matrix $M$ (resp. the Coxeter matrix $C$), we have the relation

$$C = -M^{-1}M^{\mathrm{t}}, \tag{5.1}$$

so it suffices to specify the bilinear form.

The reason for considering these properties is explained in [212]: it can be shown that the action of the Serre functor on the Grothendieck group for all smooth projective surfaces has the extra property that $s-\mathrm{id}_\Lambda$ has rank precisely 2, whilst the unipotency of the Serre functor holds in complete generality [50, lemma 3.1].

Before giving the classification, recall that mutation and shifting of exceptional collections gives an action of the signed braid group $\Sigma\mathrm{B}_n$, and we will only be interested in the classification (of bilinear forms) up to this action.

For rank 3 the analogous problem is classical and is described by the Markov equation. In that case the only solution is given by $\mathbb{P}^2$, and its noncommutative analogues. The structure of the numerical Grothendieck group in this case can be read off from examples 5.7 and 5.8.

For rank 4 there are more solutions, which are described by de Thanhoffer de Völcsey and Van den Bergh in their main theorem [212, theorem A]. They show that, up to the action of the signed braid group, the matrices

$$\text{(A)}\quad \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$(\mathrm{B}_m)\quad \begin{pmatrix} 1 & m & 2m & m \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $m \in \mathbb{N}$, describe all possible bilinear forms $\langle -, - \rangle$ on $\mathbb{Z}^4$ satisfying the properties.

The case (A) corresponds to the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ (and its noncommutative analogues [226]). Using proposition 5.12 the cases ($\mathrm{B}_m$) with $m = 0$ (resp. 1) correspond to the disjoint union of $\mathbb{P}^2$ (and its noncommutative analogues [13, 50]) with a point, resp. the blowup of $\mathbb{P}^2$ in a point (and its noncommutative analogues [222]).

For ($\mathrm{B}_m$) with $m = 2$ de Thanhoffer de Völcsey–Presotto constructed families of noncommutative $\mathbb{P}^1$-bundles on $\mathbb{P}^1$ of rank $(1, 4)$, and showed that these give the correct Euler form [214].

In this chapter we give a streamlined construction of families of noncommutative surfaces with Grothendieck group $\mathbb{Z}^4$ and Euler form ($\mathrm{B}_m$) for all $n \geq 2$ using completely different methods. This is achieved by constructing a sheaf of maximal orders on $\mathrm{Bl}_x \mathbb{P}^2$, as done in section 5.3. The main technique here is a noncommutative generalisation of Orlov's blowup formula, as given in theorem 5.28, and which is probably of independent interest.

The result of the constructions in this chapter can be summarised as follows.

**Theorem 5.1.** For every $m \geq 2$ there exist maximal orders on $\mathrm{Bl}_x \mathbb{P}^2 \cong \mathbb{F}_1$ whose Euler form is of type ($\mathrm{B}_m$).

In particular, we provide an actual geometric construction for the numerical blowups of [212]. There are three degrees of freedom in our construction, which is the expected number of degrees of freedom, but we do not give a complete classification of noncommutative surfaces of rank 4.

In section 5.4 we give some properties of the orders we have constructed in the context of the minimal model program for orders. Especially for $m = 2$ they turn out to be interesting, as we get interesting new examples of so called *half ruled orders*. These are also the only ones which are del Pezzo, which is somewhat unexpected as the analogous construction for blowing up a point on the ramification locus always gives a del Pezzo order.

For the case of $m = 2$ there are now two constructions of an abelian category with the prescribed properties: we have that it is possible to view the abelian category both as a blowup, and a $\mathbb{P}^1$-bundle, just like it is possible for $m = 1$ to view $\mathrm{Bl}_x \mathbb{P}^2$ as the Hirzebruch surface $\mathbb{F}_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and vice versa. In chapter 6 we give a comparison of the two constructions.

## 5.2 Preliminaries

### 5.2.1 Artin–Schelter regular algebras

Artin and Schelter introduced in [11] a class of graded algebras that were to serve as the noncommutative analogues of the polynomial ring $k[x_0, \ldots, x_d]$. They are defined as follows.

**Definition 5.2.** Let $A$ be a connected graded $k$-algebra. Then we say $A$ is *Artin–Schelter regular* of dimension $d$ if

1. $\operatorname{gldim} A = d$;
2. $\operatorname{GKdim} A = d$;

3. $A$ is Gorenstein (with respect to the integer $d$), i.e. there exists $l \in \mathbb{Z}$ such that

$$\operatorname{Ext}^i_{\operatorname{Gr} A}(k_A, A) \cong \begin{cases} {}_A k(l) & i = d, \\ 0 & i \neq d. \end{cases} \tag{5.2}$$

Of particular importance in the study of noncommutative surfaces is the case where $d = 3$, and we will restrict ourselves to this case. Moreover we only consider 3-dimensional AS-regular algebras which are generated in degree 1. For these algebras, it turns out there are two possible Hilbert series [11, theorem 1.5(i)], one of which is precisely that of $k[x, y, z]$. These algebras are referred to as quadratic AS-regular algebras and the associated abelian category $\operatorname{qgr} A$ is called a *noncommutative plane*. The other class of algebras is related to $\mathbb{P}^1 \times \mathbb{P}^1$, and is of no role here.

Quadratic AS-regular algebras were classified in terms of triples of geometric data [13, definition 4.5].

**Definition 5.3.** An *elliptic triple* is a triple $(C, \sigma, \mathcal{L})$ where

1. $C$ is a divisor of degree 3 in $\mathbb{P}^2$;
2. $\sigma \in \operatorname{Aut}(C)$;
3. $\mathcal{L}$ is a very ample line bundle of degree 3 on $C$.

We say that it is *regular* if moreover

$$\mathcal{L} \otimes (\sigma^* \circ \sigma^*(\mathcal{L})) \cong \sigma^*(\mathcal{L}) \otimes \sigma^*(\mathcal{L}). \tag{5.3}$$

The classification of noncommutative planes using regular triples is originally due to Artin–Tate–Van den Bergh [13, §4], and later done using different techniques by Bondal–Polishchuk [50].

**Example 5.4.** The generic case in the classification is given by the *Sklyanin algebra*. The geometric data in this case is an elliptic curve, and the automorphism is given by a translation.

It is well known that a Sklyanin algebra can be written as the quotient of $k\langle x, y, z\rangle$ by the ideal generated by

$$\begin{cases} axy + byx + cz^2 = 0 \\ ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \end{cases} \tag{5.4}$$

where $[a : b : c] \in \mathbb{P}^2 \setminus S$ and $S$ is an explicitly known finite set of 12 points.

We will be particularly interested in the case where the Sklyanin algebra $A$ is finite over its center. This property is visible in the geometric data [12, theorem 7.1].

**Theorem 5.5** (Artin–Tate–Van den Bergh)**.** The algebra $A$ is finite over its center if and only if the automorphism in the associated elliptic triple is of finite order.

**Remark 5.6.** By inspecting the automorphism groups of cubic curves it can be seen that there are only 4 (out of 9) point schemes that are allowed in order for the automorphism to be of finite order. These are the elliptic curves, the nodal cubic, the triangle of lines and a conic and line in general position. For an alternative approach using the classification of maximal orders, see remark 5.19.

We will use the following properties of the algebras which are finite over their center:

1. we can obtain the category $\operatorname{qgr} A$ as the category $\operatorname{coh} \mathcal{S}$, where $\mathcal{S}$ is a sheaf of maximal orders on $\mathbb{P}^2$, as explained in lemma 5.20;

2. there exist so called fat points in $\operatorname{qgr} A$, as explained in section 5.2.3.

These will allow us to construct a sheaf of maximal orders on $\mathbb{F}_1 \cong \operatorname{Bl}_x \mathbb{P}^2$ with the desired properties in section 5.3.

The derived category of $\operatorname{qgr} A$ can be described explicitly using a full and strong exceptional collection. One is referred to section 1.2 for more information.

The main property of semiorthogonal decompositions that we will use in this chapter is that they are sent to direct sums by so called additive invariants. In particular for the Grothendieck group we get that

$$\mathrm{K}_0(\mathcal{T}) \cong \bigoplus_{i=1}^{n} \mathrm{K}_0(\mathcal{S}_i) \tag{5.5}$$

for a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{S}_1, \ldots, \mathcal{S}_n \rangle$. In particular, if $\mathcal{T}$ has a full exceptional collection of length $n$, then

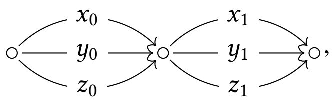
$$\mathrm{K}_0(\mathcal{T}) \cong \mathbb{Z}^{\oplus n}. \tag{5.6}$$

The main example of a full and strong exceptional collection, and also the example that motivated the construction in section 5.3 is Beilinson's collection on $\mathbb{P}^2$ [27].

**Example 5.7.** The derived category of $\mathbb{P}^2$ has a full and strong exceptional collection

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^2) = \langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2) \rangle \tag{5.7}$$

whose quiver is

(5.8) 


and the relations are

$$\begin{cases} x_0 y_1 = y_0 x_1 \\ x_0 z_1 = z_0 x_1 \\ y_0 z_1 = z_0 y_1. \end{cases} \tag{5.9}$$

In particular, this means that $K_0(\mathbf{D}^b(\mathbb{P}^2)) \cong \mathbb{Z}^{\oplus 3}$, and we can read off that the Gram (or Cartan) matrix is

$$(5.10)\quad M = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix},$$

whilst the Coxeter matrix is

$$(5.11)\quad C = \begin{pmatrix} -10 & -6 & -3 \\ 15 & 8 & 3 \\ -6 & -3 & -1 \end{pmatrix}.$$

This example can be generalised to noncommutative $\mathbb{P}^2$'s in the following way.

**Example 5.8.** The derived category of $\operatorname{qgr} A$, where $A$ is a quadratic 3-dimensional Artin–Schelter regular algebra, has a well-known full and strong exceptional collection mimicking that of Beilinson for $\mathbb{P}^2$, given by

$$(5.12)\quad \mathbf{D}^b(\operatorname{qgr} A) = \langle \pi A, \pi A(1), \pi A(2) \rangle$$

where $\pi$ denotes the quotient functor $\pi\colon \operatorname{gr} A \to \operatorname{qgr} A$, and $A(i)$ denotes the grading shift of $A$. More details can be found in [1, theorem 7.1].

The quiver has the same shape as (5.8), and the relations can be read off from the presentation of $A$ as a quotient of $k\langle x, y, z\rangle$ by 3 quadratic relations as in (5.9), see also lemma 5.21.

For instance in the case of the Sklyanin algebra of example 5.4 they are

$$(5.13)\quad \begin{cases} ax_0y_1 + by_0x_1 + cz_0z_1 = 0 \\ ay_0z_1 + bz_0y_1 + cx_0x_1 = 0 \\ az_0x_1 + bx_0z_1 + cy_0y_1 = 0 \end{cases}$$

The Cartan and Coxeter matrices $A$ and $C$ describing the Euler form and the Serre functor only depend on the structure of the quiver with relations, and this stays the same, so we obtain the matrices from (5.10) and (5.11). In [212] it is explained how up to the signed braid group action introduction in section 5.2.2 this is the only solution of rank 3.

### 5.2.2 Mutation

We quickly recall the theory of mutations of exceptional sequences.

**Definition 5.9.** Let $\mathcal{T}$ be an Ext-finite triangulated category and let $(E, F)$ be an exceptional pair of objects in $\mathcal{T}$. We define the *left mutation* $\mathrm{L}_E F$ as the cone of the morphism (since the pair is exceptional, the cone is unique up to *unique* isomorphism).

$$(5.14)\quad \operatorname{Hom}_{\mathcal{T}}(E, F) \otimes E \to F \to \mathrm{L}_E F$$

If $\mathbb{E} := (E_1, \dots E_n)$ is an exceptional collection in $\mathcal{T}$ we define the *mutation at i* to be the exceptional collection $(E_1, \dots, \mathrm{L}_{E_i} E_{i+1}, E_i, \dots, E_n)$.

These mutations can be interpreted as an action of the braid group on $n$ strings, denoted $\mathrm{B}_n$, on the set of all exceptional collections. To see this, let $\sigma_1, \dots, \sigma_{n-1}$ be the standard generators for $\mathrm{B}_n$, then $\sigma_i$ acts on an exceptional collection via mutation at $i$, i.e.

$$\sigma_i(\mathbb{E}) := (E_1, \dots, \mathrm{L}_{E_i} E_{i+1}, E_i, \dots, E_n). \tag{5.15}$$

**Remark 5.10.** By a celebrated theorem by Kuleshov and Orlov [134] we know that for a del Pezzo surface $X$ the braid group $\mathrm{B}_m$ (where $m = \operatorname{rk} \mathrm{K}_0(X)$) acts transitively on the set of exceptional collections in $\mathbf{D}^{\mathrm{b}}(X)$.

Inspired by [212] we also consider the action of the *signed* braid group, which also takes shifting into account.

**Definition 5.11.** The *signed braid group* $\Sigma\mathrm{B}_n$ is the semidirect product $\mathrm{B}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$, where $(\mathbb{Z}/2\mathbb{Z})^n$ acts on $\mathrm{B}_n$ by considering the quotient $\mathrm{B}_n \twoheadrightarrow \mathrm{Sym}_n$. As such, the signed braid group has $2n-1$ generators:

- $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$, as for the braid group $\mathrm{B}_n$,
- $n$ generators $\epsilon_1, \dots, \epsilon_n$, as for $(\mathbb{Z}/2\mathbb{Z})^n$.

These generators satisfy the relations

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & i = 1, \dots, n-2 \\ \epsilon_i^2 = 1 & \\ \epsilon_i\epsilon_j = \epsilon_j\epsilon_i & \\ \epsilon_i\sigma_i\epsilon_{i+1} = \sigma_i & i = 1, \dots, n-1. \end{cases} \tag{5.16}$$

In [212, §4] the rules for computing the action of $\Sigma\mathrm{B}_n$ on a bilinear form $\langle -, - \rangle$ are given. These rules naturally generalise the induced action of $\mathrm{B}_n$ on the Euler form on the Grothendieck group $\mathrm{K}_0(\mathcal{T})$. We will construct an abelian category of "geometric origin" which has a Grothendieck group that is in the same orbit as the matrix $(\mathrm{B}_m)$. To do so we will use the representative of the equivalence class found in the next proposition.

**Proposition 5.12.** The matrices $(\mathrm{B}_m)$ are mutation equivalent to the matrices

$$(\mathrm{B}'_m) \quad \begin{pmatrix} 1 & 3 & 6 & m \\ 0 & 1 & 3 & m \\ 0 & 0 & 1 & m \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $n \in \mathbb{N}$.

*Proof.* We have that $\epsilon_1\epsilon_3\sigma_3\sigma_1\sigma_2\sigma_3$ sends $(\mathrm{B}'_m)$ to $(\mathrm{B}_m)$. The mutations $\sigma_1\sigma_2\sigma_3$ provide a shift in the helix, whilst the mutation $\sigma_3$ at that point corresponds to the mutation that sends $(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$ to $(\mathcal{O}_{\mathbb{P}^2}, \mathrm{T}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}(1))$.

The intermediate steps are

$$\sigma_3(\mathrm{B}'_m) = \begin{pmatrix} 1 & 3 & -5m & 6 \\ 0 & 1 & -2m & 3 \\ 0 & 0 & 1 & -m \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.17}$$

$$\sigma_2\sigma_3(\mathrm{B}'_m) = \begin{pmatrix} 1 & m & 3 & 6 \\ 0 & 1 & 2m & 5m \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.18}$$

$$\sigma_1\sigma_2\sigma_3(\mathrm{B}'_m) = \begin{pmatrix} 1 & -m & -m & -m \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.19}$$

$$\sigma_3\sigma_1\sigma_2\sigma_3(\mathrm{B}'_m) = \begin{pmatrix} 1 & -m & -2m & -m \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.20}$$

$$\epsilon_3\sigma_3\sigma_1\sigma_2\sigma_3(\mathrm{B}'_m) = \begin{pmatrix} 1 & -m & -2m & -m \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.21}$$

□

### 5.2.3 Fat point modules

According to [197, §7] we take:

**Definition 5.13.** A *fat point module* for $A$ is a graded module $F$ satisfying the following properties:

1. $F$ is generated by $F_0$,
2. the Hilbert function $\dim_k F_n$ is a constant $\geq 2$, which is called the *multiplicity*,
3. $F$ has no nonzero finite-dimensional submodules,
4. $\pi F \in \operatorname{qgr} A$ is simple.

A *fat point* is the isomorphism class of a fat point module in $\operatorname{qgr} A$.

The following result tells us that fat point modules for quadratic Artin–Schelter regular algebras which are finite over their center behave particularly well.

By the classification of those algebras which are finite over their center we know that such an algebra is described by an elliptic triple $(E,\sigma,\mathcal{L})$ where $\sigma$ is an automorphism of finite order. We will denote

(5.22) $s := \min\{k \mid \sigma^{k,*}(\mathcal{L}) \cong \mathcal{L}\}.$

It is this integer, and not the order of $\sigma$ (which will be denoted $n$) that is the important invariant of the triple $(E,\sigma,\mathcal{L})$. Observe that we have $s \mid n$. The following proposition then describes the exact value of $s$, which depends on the behaviour of the normal element $g$ which lives in degree 3.

**Proposition 5.14.** If $(E,\sigma,\mathcal{L})$ is the regular triple associated to an Artin–Schelter regular algebra $A$ which is finite over its center, then the multiplicity of the fat point modules of $A$ is

(5.23) $s = \begin{cases} n & \gcd(n,3) = 1 \\ n/3 & \gcd(n,3) = 3. \end{cases}$

*Proof.* In [12, theorem 7.3] it is shown that $A[g^{-1}]$ is Azumaya of degree $s$. Now by [9, lemma 5.5.5(i)] we have that $s$ is the order of the automorphism $\eta$ introduced in [12, §5]. By [9, theorem 5.3.6] we have that $\eta = \sigma^3$, hence $s$ is $n$ or $n/3$ depending on $\gcd(n,3)$.

Moreover, by [9, lemma 5.5.5(ii)] we have that all fat point modules are of multiplicity $s$. □

**Remark 5.15.** It can be shown that a Sklyanin algebra associated to a translation of order 3 has the property that $\operatorname{qgr} A \cong \operatorname{coh}\mathbb{P}^2$. This case is not considered in the remainder of this chapter.

We will also need the following two facts about fat point modules.

**Proposition 5.16.** Fat point modules are $g$-torsion free.

*Proof.* Let $M$ be a simple graded $A$-module. Then there exists an index $n \in \mathbb{Z}$ such that $M_i = 0$ for all $i \neq n$. To see this, note that if $M_i \neq 0$ and $M_n \neq 0$ for some $i > n$, then the truncation $M_{\geq n+1}$ is a non-trivial submodule of $M$.

In particular, let $F$ be a fat point module and $M \subset F$ a simple graded submodule. By the above and definition 5.13(2). $M$ is finite-dimensional, implying $M = 0$ by definition 5.13(3).

Hence $F$ has a trivial socle. As such we can apply [12, proposition 7.7(ii)] from which the lemma follows because $F$ cannot be an extension of point modules as we assumed $\pi F \in \operatorname{qgr} A$ to be simple. □

**Lemma 5.17.** The fat point module $F$ is invariant under triple degree shifting, i.e. there exists an isomorphism

(5.24) $F \cong (F(3))_{\geq 0}$

in $\operatorname{gr} A$

*Proof.* This is direct corollary of proposition 5.16: the isomorphism is given by multiplication by $g$. □

## 5.3 Construction

### 5.3.1 Noncommutative planes finite over their center

Consider a 3-dimensional quadratic Artin–Schelter-regular algebra $A$ which is finite over its center $\mathrm{Z}(A)$. Using theorem 5.5 we know that this is the case precisely when the automorphism in the associated elliptic triple is of finite order.

In this case we can consider

$$X \coloneqq \operatorname{Proj} \mathrm{Z}(A), \tag{5.25}$$

and the sheafification $\mathcal{R}$ of $A$ over $X$. This is a sheaf of noncommutative $\mathcal{O}_X$-algebras, coherent as $\mathcal{O}_X$-module. Often, but not always, we have that $X \cong \mathbb{P}^2$ [8, theorem 5.2]. It is possible to improve this situation by considering a finite cover of $X$.

The center $\mathrm{Z}(\mathcal{R})$, which is not necessarily $\mathcal{O}_X$, is a (coherent) sheaf of commutative $\mathcal{O}_X$-algebras, hence we can consider

$$f\colon Y \coloneqq \mathbf{Spec}_X \mathrm{Z}(\mathcal{R}) \to X. \tag{5.26}$$

Because $\mathrm{Z}(\mathcal{R})$ is coherent as an $\mathcal{O}_X$-module the projection map $f$ is finite.

The main result about $Y$, for any Artin–Schelter regular algebra finite over its center, is that $Y$ is isomorphic to $\mathbb{P}^2$. This was proven:

1. by Artin for Sklyanin algebras associated to points of order coprime to 3, where $X \cong Y$, as mentioned before,
2. by Smith–Tate for all Sklyanin algebras [198],
3. by Mori for algebras of type $\mathrm{S}_1$ [162] (these have a triangle of $\mathbb{P}^1$'s as their point scheme),
4. and finally by Van Gastel in complete generality [229], with an analogous proof in [9, theorem 5.3.7].

We will denote by $\mathcal{S}$ the sheaf of algebras on $Y$ induced by $\mathcal{R}$, so the situation is described as follows.

$$\begin{array}{ccc} \mathcal{S} & & \mathcal{R} = f_*(\mathcal{S}) \\ \vdots & & \vdots \\ \mathbb{P}^2 \cong Y \coloneqq \mathbf{Spec}_X \mathrm{Z}(\mathcal{R}) & \xrightarrow{\quad f \quad} & X. \end{array} \tag{5.27}$$

The sheaf of algebras $\mathcal{S}$ has many pleasant properties and will be used in the construction.

**Lemma 5.18.** $\mathcal{S}$ is a sheaf of maximal orders on $\mathbb{P}^2$ of rank $s^2$, with $s$ as in proposition 5.14.

*Proof.* By the discussion above we have $Y \cong \mathbb{P}^2$, so that $\mathcal{S}$ is a maximal order follows from [149, proposition 1]. Observe that the notation in the statement of loc. cit. is somewhat unfortunate, and should be taken as in (5.27).

It is locally free because it is a reflexive sheaf over a regular scheme of dimension 2, and the statement on the rank follows from [12, theorem 7.3]. □

Using this we can decompose $\mathbb{P}^2$ into a *ramification divisor* $C$ and its complement, the *Azumaya locus*.

**Remark 5.19.** It is also possible to classify the curves that can appear as ramification divisors for a maximal order on $\mathbb{P}^2$ using the Artin–Mumford sequence [10], as explained in [223, lemma 1.1(2)]. This gives the same result as remark 5.6, taking care of the distinction between the point scheme and the ramification divisor. The details for this can be found in section 5.A.

The noncommutative plane associated to the algebra $A$ is the category $\operatorname{qgr} A$. In the case where $A$ is finite over its center we have a second interpretation for this category, namely as the category of coherent $\mathcal{R}$-and $\mathcal{S}$-modules.

**Lemma 5.20.** There are equivalences of categories

$$\operatorname{qgr} A \cong \operatorname{coh} \mathcal{R} \cong \operatorname{coh} \mathcal{S}. \tag{5.28}$$

*Proof.* The equivalence $\operatorname{QGr} A \cong \operatorname{Qcoh} \mathcal{R}$ is given by the restriction of the equivalence $\widetilde{(-)}\colon \operatorname{QGr} \mathrm{Z}(A) \to \operatorname{Qcoh} X$.

Similarly there is an equivalence of categories $\operatorname{Qcoh} \mathrm{Z}(\mathcal{R}) \cong \operatorname{Qcoh} Y$, and one easily checks that this restricts to $\operatorname{Qcoh} \mathcal{R} \cong \operatorname{Qcoh} \mathcal{S}$ (see for example [55, proposition 3.5]). This equivalence also restricts to noetherian objects. □

### 5.3.2 Description of the exceptional sequence

We will use the notation $\mathcal{S}_i \in \operatorname{coh} \mathcal{S}$ for the images of $\pi A(i) \in \operatorname{qgr} A$ under the above equivalences. Similarly we fix a fat point module $F$ and let $\mathcal{F} \in \operatorname{coh} \mathcal{S}$ be its image. The collection

$$(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{F}) \tag{5.29}$$

in $\mathbf{D}^{\mathrm{b}}(\mathcal{S})$ is the noncommutative analogue of $\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}, k(x)$, where $k(x)$ is the skyscraper in a closed point $x$. This is not an exceptional collection for $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^2)$: we have that $\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^2}(i), k(x)) \neq 0$ for all $i$, but it will become one after blowing up at $p$.

The point we wish to blow up is the support of the fat point module, considered as an object in $\operatorname{coh} \mathcal{S}$. This corresponds precisely with a point of $\mathbb{P}^2 \setminus C$, where $C$ is the ramification divisor of $\mathcal{S}$. This will give us a new exceptional object, with the appropriate number of morphisms towards it.

We can perform the analogues construction in the noncommutative situation. Let $x \in \mathbb{P}^2 \setminus C$ be the unique closed point in the support of $\mathcal{F}$, where $C$ is the ramification locus of $\mathcal{S}$.

Consider the blowup square

$$\begin{array}{ccc} E = \mathbb{P}^1 & \xrightarrow{j} & Z = \mathbb{F}_1 \\ \downarrow q & & \downarrow p \\ x & \xrightarrow{i} & Y = \mathbb{P}^2. \end{array} \tag{5.30}$$

As in section 5.3.3 we will use the notation $p_{\mathcal{S}}^*$ for the inverse image functor obtained from the morphism of ringed spaces $(\mathbb{F}_1, p^*(\mathcal{S})) \to (\mathbb{P}^2, \mathcal{S})$.

As explained in example 5.8 the structure of $\mathbf{D}^{\mathrm{b}}(\mathcal{S})$ is obtained by changing the relations in the quiver according to the generators and relations for the Artin–Schelter regular algebra. This is a well known result, and a noncommutative analogue of Serre's description of the sheaf cohomology of $\mathcal{O}_{\mathbb{P}^n}(i)$.

**Lemma 5.21.** Let $A$ be a quadratic 3-dimensional Artin–Schelter regular algebra. Then there is a full and strong exceptional collection

$$\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}\, A) = \langle \pi A, \pi A(1), \pi A(2) \rangle\,, \tag{5.31}$$

such that

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}\, A)}(\pi A(i), \pi A(i+1)) &\cong A_1 \\ \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}\, A)}(\pi A(i), \pi A(i+2)) &\cong A_2 \end{aligned} \tag{5.32}$$

and the composition law in the quiver is given by the multiplication law $A_1 \otimes_k A_1 \twoheadrightarrow A_2$.

*Proof.* By [14, theorem 8.1] we have

$$\mathrm{Ext}^m_{\mathrm{qgr}\, A}(\pi A, \pi A(j-i)) \cong \begin{cases} A_{j-i} & m = 0 \\ A^{\vee}_{i-j-3} & m = 2 \\ 0 & m \neq 0, 2 \end{cases}. \tag{5.33}$$

Moreover by assumption the algebra is generated in degree 1, hence we have a complete description of the structure of the exceptional collection. That it is full is proven in [1, theorem 7.1]. □

Using lemma 5.20 this gives a description for the derived category $\mathcal{S}$ as

$$\mathbf{D}^{\mathrm{b}}(\mathcal{S}) = \langle \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2 \rangle\,. \tag{5.34}$$

**Lemma 5.22.** Let $F$ be a normalised fat point module for the algebra $A$ and let $s$ be as in proposition 5.14. Then

$$\dim_k \left(\mathrm{Hom}_{\mathrm{qgr}\, A}(\pi A(j), \pi F)\right) = s, \tag{5.35}$$

and

$$\mathrm{Ext}^k_{\mathrm{qgr}\, A}(\pi A(j), \pi F) = 0 \tag{5.36}$$

for $k \geq 1$.

*Proof.* Using the identities

$$(5.37)\quad \begin{aligned} \mathrm{Hom}_{\mathrm{qgr}\,A}(\pi A(j),\pi F) &\cong \mathrm{Hom}_{\mathrm{qgr}\,A}(\pi A,\pi F(-j)) \\ F(-j) &\cong F(-j+3) \qquad \text{(see lemma 5.17)} \end{aligned}$$

we can assume, without loss of generality, that $j \leq 0$.

Recall that

$$(5.38)\quad \mathrm{Hom}_{\mathrm{qgr}\,A}(\pi A,\pi F(-j)) \cong \lim_{i\to\infty} \mathrm{Hom}_{\mathrm{gr}\,A}(A_{\geq i},F(-j)).$$

We now claim

$$(5.39)\quad \lim_{i\to\infty} \mathrm{Hom}_{\mathrm{gr}\,A}(A_{\geq i},F(-j)) \cong \mathrm{Hom}_{\mathrm{gr}\,A}(A,F(-j)) \cong F_{-j}.$$

The lemma follows by combining this claim with proposition 5.14.

To prove the claim, note that one can compute the limit by restricting to the directed subsystem $3\mathbb{N}$. It then suffices to prove that the natural map

$$(5.40)\quad \alpha_i\colon \mathrm{Hom}_{\mathrm{gr}\,A}(A_{\geq 3i},F(-j)) \to \mathrm{Hom}_{\mathrm{gr}\,A}(A_{\geq 3i+3},F(-j))$$

is an isomorphism for all $i \geq 0$. But this follows as its inverse is given by

$$(5.41)\quad \beta_i\colon \mathrm{Hom}_{\mathrm{gr}\,A}(A_{\geq 3i+3},F(-j)) \to \mathrm{Hom}_{\mathrm{gr}\,A}(A_{\geq 3i},F(-j)) : \beta_i(\varphi)(x) = g^{-1}\varphi(gx)$$

where $g^{-1}\colon F_n \to F_{n-3}$ is the inverse of the isomorphism as in (5.24).

To see that the Ext vanish, we use that $F$ (resp. $F(-j)$) is finitely generated (by the degree zero part) and has Gelfand–Kirillov dimension 1. By [12, theorem 4.1] we can conclude

$$(5.42)\quad \mathrm{Ext}^i_{\mathrm{gr}\,A}(F,A(j)) = 0 \text{ for all } j \text{ and } i = 0,1$$

and [14, theorem 8.1] implies

$$(5.43)\quad \mathrm{Ext}^i_{\mathrm{qgr}\,A}(\pi F,\pi A(j)) = \mathrm{Ext}^i_{\mathrm{gr}\,A}(F,A(j)) = 0 \text{ for all } j \text{ and } i = 0,1$$

It was proven in [70, theorem 2.9.1] that $\mathrm{qgr}\,A$ satisfies the following version of non-commutative Serre duality:

$$(5.44)\quad \mathrm{Ext}^i_{\mathrm{qgr}\,A}(\pi F,\pi A(j)) \cong \mathrm{Ext}^{2-i}_{\mathrm{qgr}\,A}(\pi A(j),\pi F(-3))^\vee.$$

But the latter is isomorphic to $\mathrm{Ext}^{2-i}_{\mathrm{qgr}\,A}(\pi A(j),\pi F)^\vee$ by lemma 5.17, which implies the vanishing of Ext. ◻

We will also use the following lemma in checking that the exceptional collection is indeed strong, and of the prescribed form.

**Lemma 5.23.** There exists an isomorphism

$$(5.45)\quad \mathbf{R}p_* \circ p^*(\mathcal{F}) \cong \mathcal{F}.$$

*Proof.* Consider the divisor short exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_1}(-E) \to \mathcal{O}_{\mathbb{F}_1} \to j_*(\mathcal{O}_E) \to 0. \tag{5.46}$$

By applying the exact functor $p^*(\mathcal{S}) \otimes_{\mathcal{O}_{\mathbb{F}_1}} -$ to it we get a short exact sequene of left $p^*(\mathcal{S})$-modules

$$0 \to p^*(\mathcal{S}) \otimes_{\mathcal{O}_{\mathbb{F}_1}} \mathcal{O}_{\mathbb{F}_1}(-E) \to p^*(\mathcal{S}) \to p^*(\mathcal{S}) \otimes_{\mathcal{O}_{\mathbb{F}_1}} j_*(\mathcal{O}_E) \to 0. \tag{5.47}$$

We have the chain of isomorphisms

$$\begin{aligned} & p^*(\mathcal{S}) \otimes_{\mathcal{O}_{\mathbb{F}_1}} j_*(\mathcal{O}_E) \\ &\cong j_* \circ j^* \circ p^*(\mathcal{S}) && \text{projection formula} \\ &\cong j_* \circ q^* \circ i^*(\mathcal{S}) && \text{functoriality} \\ &\cong p^* \circ i_* \circ i^*(\mathcal{S}) && \text{base change for affine morphisms} \\ &\cong p^*(\mathcal{F}^{\oplus n}) \end{aligned} \tag{5.48}$$

where the last step uses that $i^*(\mathcal{S}) \cong \mathrm{Mat}_n(k)$ is the direct sum of the $n$-dimensional representation corresponding to the fat point $F$.

Because the first two terms in (5.47) are $p_*$-acyclic, so is the third and its direct summands $p^*(\mathcal{F})$. Hence we can use the projection formula

$$p_*(p^*(\mathcal{S}) \otimes_{p^*(\mathcal{S})} p^*(\mathcal{F})) \cong p_* \circ p^*(\mathcal{S}) \otimes_{\mathcal{S}} \mathcal{F} \cong \mathcal{F}. \tag{5.49}$$

□

We can now prove the main theorem of this chapter, which gives a construction of noncommutative surfaces with prescribed Grothendieck group. It uses a semiorthogonal decomposition that generalises Orlov's blowup formula, and which is proved in some generality in section 5.3.3.

**Theorem 5.24.** Let $A$ be a quadratic 3-dimensional Artin–Schelter regular algebra, finite over its center. Let $F$ be a fat point module of $A$. Let $p\colon \mathrm{Bl}_x \mathbb{P}^2 \to \mathbb{P}^2$ be the blowup in the point $x$ which is the support of the $\mathcal{S}$-module $\mathcal{F}$ as a sheaf on $\mathbb{P}^2$. Let $s$ be the integer as in (5.23). Then

$$\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S}) = \left\langle \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_0, \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_1, \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_2, p_{\mathcal{S}}^*\mathcal{F} \right\rangle \tag{5.50}$$

is a full and strong exceptional collection, whose Gram matrix is of type ($\mathrm{B}'_m$), where we take $m = s$.

*Proof.* By theorem 5.28 we obtain that the collection is indeed a full and strong exceptional collection, where we use that $\mathcal{F}$ can be considered as the (noncommutative) skyscraper sheaf for $\mathcal{S}$, because we are in the Azumaya locus.

The structure of $\langle \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_0, \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_1, \mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_2 \rangle$ is described in lemma 5.21 using the fully faithfulness of $\mathbf{L}p_{\mathcal{S}}^*$ from lemma 5.31.

Finally using lemma 5.23 we get that

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S})}(\mathbf{L}p_{\mathcal{S}}^*\mathcal{S}_i, p^*\mathcal{F}) &\cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{S})}(\mathcal{S}_i, \mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathcal{S}}(\mathcal{S}_i, \mathcal{F}) \end{aligned} \tag{5.51}$$

which is $s$-dimensional by lemma 5.22, and similarly we get that there are no forward Ext's to conclude that the collection is indeed strong. □

**Remark 5.25.** There are three degrees of freedom in this construction: generically the point scheme is an elliptic curve for which we have the $j$-line as moduli space, for each curve there are only finitely many torsion automorphisms, and then there is the choice of a point in $\mathbb{P}^2 \setminus C$. There are only finitely many automorphisms of $\mathbb{P}^2$ that preserve $C$, so we get three degrees of freedom. This is the expected number, using the methods of chapter 4, where a formula for $\dim_k \mathrm{HH}^2 - \dim_k \mathrm{HH}^1$ is given in terms of the number of exceptional objects.

**Remark 5.26.** The derived category $\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S})$ comes with its standard t-structure. By [228] we have that Serre duality takes on the form that it does for ordinary smooth and projective varieties. This means that the Serre functor is compatible with the t-structure, as required in Bondal's definition of geometric t-structure [45]. In particular, it is an example of a noncommutative variety of dimension 2 in this sense.

**Remark 5.27.** Using [178] and the full and strong exceptional collection from theorem 5.24 there exists an embedding of $\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S})$ into the derived category of a smooth projective variety. Now in the spirit of [82, 132, 176] and chapter 3 it is an interesting question whether there exists a natural embedding, i.e. where the smooth projective variety is associated to $p^*\mathcal{S}$ in a natural way. An obvious candidate would be the Brauer–Severi scheme of the maximal order, and indeed in the case where the automorphism is of order 2 there exists a fully faithful embedding into $\mathbf{D}^{\mathrm{b}}(\mathrm{BS}(p^*\mathcal{S}))$ by [60, §6] and [139], as the maximal order is the even part of a sheaf of Clifford algebras. In this case the result is even a Fano variety. What happens for the more general case and the study of the derived category of the Brauer–Severi scheme in this situation, is left for future work.

### 5.3.3 Orlov's blowup formula for orders

The main ingredient in the construction of theorem 5.24 is the observation that it is possible to generalise Orlov's blowup formula [177, theorem 4.3] to a sufficiently nice noncommutative setting where we blow up a point on the underlying variety, and pull back the sheaf of algebras to the blown up variety. This is a result of independent interest.

**Theorem 5.28.** Let $X$ be a smooth quasiprojective variety. Let $\mathcal{A}$ be a locally free sheaf of orders of degree $n$ on $X$ such that $\operatorname{gldim} \mathcal{A} < +\infty$. Let $Y$ be a smooth closed subvariety such that it does not meet the ramification locus of $\mathcal{A}$. Assume moreover that $\mathcal{A}|_Y \cong \mathrm{Mat}_n(\mathcal{O}_Y)$. Consider the blowup square

$$\begin{array}{ccc} E := \mathbb{P}(\mathrm{N}_Y X) & \xrightarrow{j} & Z := \mathrm{Bl}_Y X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & X \end{array} \tag{5.52}$$

and its noncommutative analogue obtained by pulling back the sheaf of algebras $\mathcal{A}$

$$\begin{array}{ccc} (E, \mathrm{Mat}_n(\mathcal{O}_E)) & \xrightarrow{j_{\mathcal{A}}} & (Z, p^*\mathcal{A}) \\ \downarrow q_{\mathcal{A}} & & \downarrow p_{\mathcal{A}} \\ (Y, \mathrm{Mat}_n(\mathcal{O}_Y) & \xrightarrow{i_{\mathcal{A}}} & (X, \mathcal{A}) \end{array} . \tag{5.53}$$

Then we have a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(Z, p^*(\mathcal{A})) = \left\langle \mathbf{D}^{\mathrm{b}}(X, \mathcal{A}), \mathbf{D}^{\mathrm{b}}(Y), \ldots, \mathbf{D}^{\mathrm{b}}(Y) \right\rangle \tag{5.54}$$

where the first component is embedded using the functor $\mathbf{L}p_{\mathcal{A}}^*$, and the subsequent components by $j_{\mathcal{A},*}(q_{\mathcal{A}}^*(-) \otimes \mathcal{O}_E(kE))$, for $k = 0, \ldots, \operatorname{codim}_X Y - 2$.

**Remark 5.29.** The case where $\mathcal{A} = \mathcal{O}_X$ is Orlov's blowup formula. Observe that his proof works verbatim for a smooth quasiprojective variety as all morphisms are projective, so the bounded derived category is preserved throughout. We will use in the proof of theorem 5.28.

**Remark 5.30.** One easily show that $p^*\mathcal{A}$ is again of finite global dimension.

We can prove this result by bootstrapping the original proof. To do so we will need generalisations of some standard results in algebraic geometry such as the adjunction between (derived) pullback and direct image, or the projection formula. A reference for these in the setting of Azumaya algebras can be found in [144, §10]. We will only need results that do not depend on the algebras being Azumaya, hence in the words of remark 10.5 of op. cit. we are working with noncommutative finite flat (and not étale) coverings.

**Lemma 5.31.** The functor $\mathbf{L}p_{\mathcal{A}}^*$ is fully faithful.

*Proof.* In the commutative setting this is proven using the derived projection formula and the fact that $\mathbf{R}p_* \circ \mathbf{L}p^*(\mathcal{O}_X) \cong \mathcal{O}_X$. In the noncommutative setting the appropriate projection formula is given as the first isomorphism in [144, lemma 10.12] taking into account that we have a bimodule structure, whilst the isomorphism

$$\mathbf{R}p_{\mathcal{A},*} \circ \mathbf{L}p_{\mathcal{A}}^*(\mathcal{A}) \cong \mathcal{A} \tag{5.55}$$

follows from the third isomorphism in loc. cit. □

By the assumption that $\mathcal{A}|_Y \cong \mathrm{Mat}_n(\mathcal{O}_Y)$ and the projection formula as given in the third isomorphism of loc. cit. we have that [177, lemma 4.2] goes through as stated. In particular we only need to check that the semiorthogonal decomposition is indeed full.

*Proof of theorem 5.28.* We can mimick the proof of the first part of [177, theorem 4.3]. Consider an object in the right orthogonal of (5.54), in particular it is right orthogonal to $\mathbf{L}p_{\mathcal{A}}^*(\mathbf{D}^{\mathrm{b}}(X, \mathcal{A}))$. As Serre duality for sheaves of maximal orders takes on the expected form by [228, corollary 2] we get that $\mathbf{R}p_*$ of this object is indeed zero, and therefore that it is contained in the minimal full subcategory containing the image of $\mathbf{D}(E, \mathrm{Mat}_n(\mathcal{O}_E))$.

Now use that blowups commute with flat base change, in particular we can take an étale neighbourhood of the exceptional divisor that splits $\mathcal{A}$ and such that its image in $X$ and the ramification divisor are disjoint. Then we are in the usual setting of Orlov's blowup formula (up to Morita equivalence), and we can use the usual proof to conclude that the object is indeed zero. □

Two remarks are in order, which are already important in the case of blowing up a point on a surface.

**Remark 5.32.** If we were to blow up a point *on the ramification divisor*, then the resulting algebra is not necessarily of finite global dimension. Considering the case of a Sklyanin algebra associated to a point of order 2, we have that the complete local structure of this algebra at the point on the intersection of the exceptional divisor and the ramification locus is given by

$$\begin{pmatrix} R & R \\ (xy) & R \end{pmatrix} \tag{5.56}$$

where $R = k[[x, y]]$. One then checks that the module $\begin{pmatrix} 0 \\ R/(x) \end{pmatrix}$ has a periodic minimal projective resolution of the form

$$\ldots \to \begin{pmatrix} (xy) \\ (xy) \end{pmatrix} \oplus \begin{pmatrix} (x) \\ (x^2y) \end{pmatrix} \xrightarrow{\psi'} \begin{pmatrix} (x) \\ (x) \end{pmatrix} \oplus \begin{pmatrix} R \\ (xy) \end{pmatrix} \xrightarrow{\varphi} \begin{pmatrix} R \\ R \end{pmatrix} \xrightarrow{\psi} \begin{pmatrix} 0 \\ R/(x) \end{pmatrix} \to 0 \tag{5.57}$$

To see that this resolution is in fact periodic, note that $\ker(\psi') \cong \ker(\psi)$ as

$$\ker(\psi) = \begin{pmatrix} R \\ (x) \end{pmatrix} \text{ and } \ker(\psi') \cong \begin{pmatrix} (xy) \\ (x^2y) \end{pmatrix} = xy \begin{pmatrix} R \\ (x) \end{pmatrix}. \tag{5.58}$$

This description also shows that the result is no longer a maximal order, and there is a choice of embedding, as explained in [62, §4]. Without the embedding in a maximal order one does not expect a meaningful semiorthogonal decomposition. In this chapter we do not need to take a maximal order containing the pullback as it is already maximal using the Auslander–Goldman criterion.

**Remark 5.33.** In the construction of [222] a point *on the point scheme* is blown up. For an algebra finite over its center this is not the same as the ramification curve, these two curves are only isogeneous. In the context of the previous remark, the difference is measured by the choice of a maximal order containing the pullback.

## 5.4 Properties of the maximal orders

In the minimal model program for orders on surfaces as studied in [9, 62] there exists the notion of del Pezzo orders, and (half-)ruled orders. The orders we have constructed are obviously not minimal, but they give rise to interesting examples in the study of maximal orders.

In the commutative case the surface $\mathrm{Bl}_x \mathbb{P}^2 = \mathbb{F}_1$ is both del Pezzo and ruled. We are considering orders *on* this surface, and for the value of $m = 2$ in the classification we obtain that it is both *del Pezzo* and *half ruled*, as explained in proposition 5.37 and proposition 5.43.

This latter notion is introduced by Artin, to describe a class of orders which is not ruled, but whose cohomological properties mimick those of ruled orders.

### 5.4.1 The case $m = 2$ is del Pezzo

In this section we quickly recall the notion of del Pezzo order, and show that the intuition from numerical blowups agrees with the a priori independent notion of del Pezzo order, introduced in [63, §3].

Throughout we let $\mathcal{A}$ be a maximal order on a smooth projective surface $S$.

**Definition 5.34.** The *canonical sheaf of* $\mathcal{A}$ is the $\mathcal{A}$-bimodule

$$\omega_{\mathcal{A}} := \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathcal{A}, \omega_S). \tag{5.59}$$

Now denote $\omega_{\mathcal{A}}^* := \mathcal{H}\mathrm{om}_{\mathcal{A}}(\omega_{\mathcal{A}}, \mathcal{A})$, whereas $\mathcal{F}^\vee$ is used for $\mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$, hence the reflexive hull is denoted $\mathcal{F}^{\vee\vee}$.

**Definition 5.35.** Let $\mathcal{L}$ be an invertible $\mathcal{A}$-bimodule, which is moreover $\mathbb{Q}$-Cartier, i.e. $(\mathcal{L}^{\otimes n})^{\vee\vee}$ is again invertible for some $n$. Then $\mathcal{L}$ is *ample* if

$$\mathrm{R}^q\,\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, (\mathcal{L}^{\otimes k})^{\vee\vee} \otimes \mathcal{F}) \cong \mathrm{H}^q(S, (\mathcal{L}^{\otimes k})^{\vee\vee} \otimes \mathcal{F}) \tag{5.60}$$

is zero for $q \geq 1$ and $k \gg 0$, where the isomorphism is induced by applying the forgetful functor.

Then analogous to the commutative situation we define

**Definition 5.36.** The maximal order $\mathcal{A}$ is *del Pezzo* if $\omega_{\mathcal{A}}^\vee$ is ample.

In particular, by [63, lemma 8] it suffices to check that the divisor

$$\mathrm{K}_{\mathcal{A}} = \mathrm{K}_S + \sum_{i=1}^n \left(1 - \frac{1}{e_i}\right) C_i \tag{5.61}$$

is anti-ample: the del Pezzoness only depends on the center and the ramification data.

**Proposition 5.37.** Let $\mathcal{A}$ be the pullback of a maximal order of degree $m$ on $\mathbb{P}^2$ ramified on a cubic curve along the blowup $\mathbb{F}_1 \to \mathbb{P}^2$ in a point outside the ramification locus. Then $\mathcal{A}$ is del Pezzo if and only if $m = 2$.

*Proof.* If we denote $\mathrm{Pic}\,\mathbb{F}_1 = \mathbb{Z}H \oplus \mathbb{Z}E$, such that $H^2 = 1$, $H \cdot E = 0$ and $E^2 = -1$, then

$$\begin{aligned} \mathrm{K}_{\mathcal{A}} &= -3p^*(H) + E + \left(1 - \frac{1}{m}\right) 3p^*(H) \\ &= \frac{-3}{m} p^*(H) + E \end{aligned} \tag{5.62}$$

because the ramification data for the pullback is the pullback of the ramification data, which is a cyclic cover of degree $m \geq 2$ of a cubic curve.

By the Kleiman criterion for ampleness we need to check that $-\mathrm{K}_{\mathcal{A}} \cdot C \geq 0$, for $C$ in the Mori–Kleiman cone of $\mathbb{F}_1$. This cone is spanned by a fibre $f$ and the section $C_0$

of the projection $\mathbb{F}_1 \to \mathbb{P}^1$. We have that $p^*(H) = C_0 + f$ and $E = C_0$ in the translation between the canonical bases for $\mathrm{Pic}(\mathbb{F}_1)$ and $\mathrm{Pic}(\mathrm{Bl}_x \mathbb{P}^2)$.

Using the description of the intersection form on a ruled surface we obtain

$$\begin{aligned} -\mathrm{K}_{\mathcal{A}} \cdot f &= \frac{3}{m} - 1, \\ -\mathrm{K}_{\mathcal{A}} \cdot C_0 &= 1. \end{aligned} \tag{5.63}$$

The first intersection number is positive if and only if $m = 2$. The second intersection number is always positive. □

**Remark 5.38.** The computation for the del Pezzoness of the numerical blowup reduces to the same equation (up to multiplication by $m^2$).

**Remark 5.39.** For the case $m = 2$ the equations for a Sklyanin algebra from example 5.4 take on a particularly easy form

$$\begin{cases} xy + yx + cz^2 = 0 \\ yz + zy + cx^2 = 0 \\ zx + xz + cy^2 = 0 \end{cases} \tag{5.64}$$

where $c^3 \neq 0, 1, -8$, [68, theorem 3.1]. The case $c = 0$ corresponds to an order associated to a so called skew polynomial algebra, and the ramification curve is a triangle of $\mathbb{P}^1$'s. If $c^3 = 1, -8$ the ramification curve is the union of a conic and a line in general position. We will come back to this situation in chapter 6.

### 5.4.2 The case $m = 2$ is half ruled

In the following definition, the curve of genus 0 will be a curve over the function field of the base curve of a ruled surface, i.e. if we consider $\pi\colon S \to C$ over the field $k$, then $K$ will be the function field of $\mathbb{P}^1_{k(C)}$.

**Definition 5.40.** Let $\mathcal{A}$ be a maximal order in a central simple algebra of degree 2 over the function field $K$ of a curve $X$ of genus 0. If $\mathcal{A}$ is ramified in 3 points, with ramification degree 2 in each point, then we say that $\mathcal{A}$ is *half ruled*.

**Remark 5.41.** The case where the ramification is of type $(e, e)$ is the *ruled* case.

It is shown in [9, proposition 4.2.4] that being (half-)ruled is equivalent to the Euler characteristic $\chi(X, \mathcal{A})$ of the coherent sheaf $\mathcal{A}$ being positive.

The following definition seems to be missing as such from the literature, but it is used implicitly in [9].

**Definition 5.42.** Let $\mathcal{A}$ be a maximal order on a ruled surface $S \to C$. Then we say that it is *half ruled* if the fiber of the order over the generic point of $C$ is half ruled.

**Proposition 5.43.** The sheaf of maximal orders constructed for the case $m = 2$ is half ruled.

*Proof.* The ramification divisor on $\mathbb{P}^2$ being a cubic curve we get that the generic intersection of the fibre of the ruling with the inverse image of the ramification divisor in $\mathbb{F}_1$ is 3, which proves the claim. □

### 5.4.3 The case $m = 3$ is elliptic

We will reuse the notation of section 5.4.2.

**Definition 5.44.** Let $\mathcal{A}$ be a maximal order in a central simple algebra of degree 3 over the function field $K$ of a curve $X$ of genus 0. If $\mathcal{A}$ is ramified in 3 points, with ramification degree 3 in each point, then we say that $\mathcal{A}$ is *elliptic*.

The following proposition is proven in the same way as proposition 5.43.

**Proposition 5.45.** The sheaf of maximal orders constructed for the case $m = 3$ is elliptic.

## 5.A Ramification for maximal orders on $\mathbb{P}^2$

The following is left as an exercise in [9, §5.1] or [223, lemma 1.1].

**Lemma 5.46.** Let $D$ be a central division algebra over $k(\mathbb{P}^2_k)$ ramified on a cubic curve $C$. Then $C$ is one of the following

1. an elliptic curve;
2. a nodal cubic;
3. three lines in general position;
4. a line and a conic in general position.

*Proof.* The Artin–Mumford sequence in this case reads

$$0 \to \mathrm{Br}(k(\mathbb{P}^2)) \to \bigoplus_{C \hookrightarrow \mathbb{P}^2_k} \mathrm{H}^1_{\text{ét}}(k(C), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{p \in \mathbb{P}^2_k} \mu^{-1} \to \mu^{-1} \to 0 \tag{5.65}$$

because $\mathrm{Br}(\mathbb{P}^2_k) \cong \mathrm{Br}(k) \cong 0$ as $k$ is algebraically closed, where $C$ runs over all irreducible curves and $p$ over all closed points.

Now if $[D] \in \mathrm{Br}(k(\mathbb{P}^2))$ is the Brauer class associated to $D$, then it defines the *primary ramification* in the second term of the exact sequence. For a given ramification data to correspond to a Brauer class $[D]$ it is necessary and sufficient that the *secondary ramification* vanishes, by exactness of the sequence. So we want to consider all possible ramification data on a cubic curve (which may be reducible), and check when it is possible for the secondary ramification to vanish.

There are 9 cubic curves (see also chapter 4), but we can ignore the two non-reduced curves.

We discuss the remaining cases, and argue why they can(not) occur. The following are the ones which *can* arise as the ramification divisor.

**an elliptic curve** There exist isogenies of any degree, for which there is no ramification, hence there exist Brauer classes with an elliptic curve as its primary ramification.

**nodal cubic** Generalising the construction of [95, exercise III.10.6] there exist étale covers of the nodal cubic of any degree, so we can use these to have covers with vanishing secondary ramification.

**three lines in general position** Because the étale fundamental group of $\mathbb{P}^1$ is trivial by Riemann–Hurwitz, any cyclic extension of each $k(\mathbb{P}^1)$ necessarily ramifies, and each such cover of $\mathbb{P}^1$ must necessarily ramify (again by Riemann–Hurwitz) in $\geq 2$ points. By choosing the ramification points to be the two intersections on each $\mathbb{P}^1$ we can cancel the secondary ramification contributions by each irreducible component.

**a line and a conic in general position** Exactly as in the previous case.

The following cases *cannot* arise as the ramification divisor.

**cuspidal cubic** The normalisation morphism $\mathbb{P}^1 \to C$ where $C$ is the cuspidal cubic is a universal homeomorphism, hence the étale sites agree, and so does the étale fundamental group. Hence there are no unramified covers, and a ramified cover necessarily ramifies in $\geq 2$ points, which cannot be cancelled in the second ramification because it is irreducible.

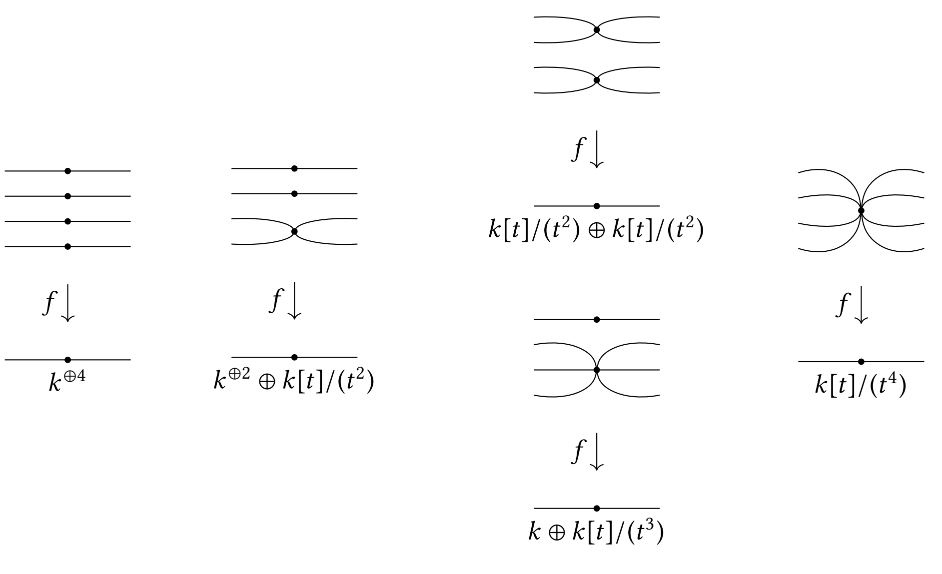
**three concurrent lines** As before each cyclic cover of a component $\mathbb{P}^1$ ramifies. Hence there is at least one ramification point on each curve besides the intersection, and which cannot be cancelled by contributions from the other curves.

**a conic and a tangent line** Exactly as in the previous case.

□

*Chapter 6*

# Comparing two constructions of noncommutative del Pezzo surfaces of rank 4



## 6.1 Introduction

In chapter 5 we have constructed a class of noncommutative surfaces with Euler forms of type ($\mathrm{B}_m$), for $m \geq 2$. In [214] a suitable adaptation of the theory of noncommutative $\mathbb{P}^1$-bundles [225] was used to construct a noncommutative surface of type ($\mathrm{B}_2$). Hence there are two geometric constructions of an abelian category which behaves like the category of coherent sheaves on a smooth projective surface: it is of global dimension 2, and its derived category has a Serre functor which is compatible with the standard t-structure, just like in algebraic geometry.

The goal of this chapter is to compare these two constructions, i.e. compare the abelian category arising as a noncommutative $\mathbb{P}^1$-bundle on $\mathbb{P}^1$ to the category of coherent sheaves on a maximal order on $\mathrm{Bl}_x \mathbb{P}^2$. It turns out that they can be explicitly compared using the geometry of the linear systems used in their construction. The idea is to write both categories in terms of (relative) Clifford algebras, which in turn reduces the problem to comparing the data defining these algebras. This allows us to explicitly relate the input for the noncommutative $\mathbb{P}^1$-bundle to the input for the maximal order construction.

In section 6.2 we recall various notions of Clifford algebra which are needed for the results in this chapter. Most of these are well-known, but we also introduce a comparison result which might be of independent interest, relating the Clifford algebra with values in a relatively ample line bundles to the total Clifford algebra. In section 6.3 we consider the noncommutative $\mathbb{P}^1$-bundles constructed in [214], and explain how we can use the formalism of generalised preprojective algebras to describe these abelian categories using Clifford algebras. In section 6.4 we obtain a similar description of the blowups of maximal orders on $\mathbb{P}^2$ constructed in chapter 5 for the special case where the fat point modules have multiplicity 2. In op. cit. it is also shown that these are del Pezzo orders in the sense of [63], and moreover that they are half ruled in the sense of [59].

We compare these constructions in section 6.5. The essential ingredient in the proof is a comparison between basepoint-free pencils of binary quartics and basepoint-free nets of conics together with the choice of a smooth conic in the net. The correspondence on the geometric level is not complete at the moment, but it is sufficient to give a comparison of the two constructions.

The comparison of the two constructions can be seen as a suitable noncommutative analogue of the isomorphism of the first Hirzebruch surface with the blowup of $\mathbb{P}^2$ in a point. The formalism of Clifford algebras will allow us to compare Clifford algebras relative to the two projective morphisms in

$$\begin{array}{ccc} & \mathbb{F}_1 \cong \mathrm{Bl}_x \mathbb{P}^2 & \\ {}^{p}\swarrow & & \searrow^{\pi} \\ \mathbb{P}^2 & & \mathbb{P}^1. \end{array} \tag{6.1}$$

There is also a different (and still conjectural) noncommutative analogue of the isomorphism $\mathbb{F}_1 \cong \mathrm{Bl}_x \mathbb{P}^2$, which is actually a deformation of the commutative isomorphism. The two constructions in this case are the noncommutative $\mathbb{P}^1$-bundles from [225] versus the notion of blowup from [222]. The main difference between the notions of bundles is that the one considered in this chapter is of rank $(1, 4)$, where originally they were of rank $(2, 2)$. The main difference between the notions of blowing up is that we consider a point outside the point scheme (or rather the ramification), whereas the original blowup considers a point on the point scheme. We come back to the comparison of these constructions in section 6.5.5.

**Conventions** The more general parts of this chapter are valid for any scheme on which 2 is invertible. For the actual comparison we will work with varieties over an algebraically closed field $k$ not of characteristic 2 or 3, whilst the comparison involving the classifications due to Wall in in characteristic 0.

## 6.2 Clifford algebras

The main tool in comparing the constructions from [214] and chapter 5 is the formalism of Clifford algebras. It turns out that it is possible to write both abelian categories as a category associated to a certain Clifford algebra. This is done in section 6.3 and section 6.4. It then becomes possible to compare the linear algebra data describing the quadratic forms. In this way we can set up an explicit correspondence between the two models, and explicitly relate the input data for the $\mathbb{P}^1$-bundle model to the input data for the blowup model. This is done in section 6.5.

In section 6.2.1 we recall the notion of the Clifford algebra associated to a quadratic form on a scheme. This is a coherent $\mathbb{Z}/2\mathbb{Z}$-graded sheaf of algebras. In section 6.2.2 we recall the more general version where the quadratic form is allowed to take values in a line bundle [40, 56]. In this case we cannot obtain a $\mathbb{Z}/2\mathbb{Z}$-algebra, but the even part of the Clifford algebra is nevertheless well-defined.

In section 6.2.3 we recall the notion of the graded Clifford algebra associated to a linear system of quadrics [16, 149]. Provided the linear system is basepoint-free we get an Artin–Schelter regular algebra which is finite over its center.

Finally, we can generalise the notion of a graded Clifford algebra to the relative setting for projective morphisms. This is done in section 6.2.4, by combining section 6.2.2 and section 6.2.3. The classical case of a graded Clifford algebra then corresponds to the morphism $\operatorname{Proj}\operatorname{Sym}_k(E) \to \operatorname{Spec} k$, where $E$ is the vector space spanned by the elements of degree 1.

### 6.2.1 Clifford algebras

Classically the Clifford algebra is a finite-dimensional $k$-algebra associated to a quadratic form $q\colon E \to k$ on a vector space $E$. It is defined as the quotient of the tensor algebra $\mathrm{T}(E)$ by the two-sided ideal generated by

$$v \otimes v - q(v) \tag{6.2}$$

for $v \in E$. We will use the notation $\mathcal{C}\ell_k(E, q)$ for the Clifford algebra. Under the standing assumption that the characteristic of $k$ is not two we can also consider the symmetric bilinear form $b_q$ associated to $q$, and rewrite the relation as

$$u \otimes v + v \otimes u - 2b_q(u, v). \tag{6.3}$$

The correspondence between symmetric bilinear forms and quadratic forms is an instance of the isomorphism

$$\operatorname{Sym}^2 E \cong \operatorname{Sym}_2 E \tag{6.4}$$

Here $\operatorname{Sym}_2 E \subset E \otimes E$ is the subspace of all symmetric tensors, i.e. as a vectorspace it is generated by elements of the form $u \otimes v + v \otimes u$. Conversely $\operatorname{Sym}^2 E$ is the quotient of $E \otimes E$ by all relations of the form $u \otimes v - v \otimes u$. The isomorphism in (6.4) is then given by

$$\overline{u \otimes v} \mapsto \frac{u \otimes v + v \otimes u}{2}$$

This isomorphism will be used in a more general form on schemes later.

One can similarly define a Clifford algebras over arbitrary commutative rings

**Definition 6.1.** Let $A$ be a commutative algebra and $E$ a finitely generated projective $A$-module. Moreover let $q\colon E \to A$ be a quadratic form, i.e. (up to the identification of $q$ and $b_q$) $q$ is a symmetric $A$-linear bilinear form

$$q\colon \operatorname{Sym}^2_A(E) \longrightarrow A \tag{6.5}$$

Then the Clifford algebra $\mathcal{C}\ell_A(E, q)$ is the quotient of $\mathrm{T}_A(E)$ by the relations

$$v \otimes w + w \otimes v - q(vw) \tag{6.6}$$

Observe that the obvious $\mathbb{Z}$-grading on $\mathrm{T}(V)$ equips $\mathcal{C}\ell(V, q)$ with a $\mathbb{Z}/2\mathbb{Z}$-grading. The even degree part forms a subalgebra which we will denote $\mathcal{C}\ell(V, q)_0$, and the odd degree part forms the $\mathcal{C}\ell(V, q)_0$-bimodule $\mathcal{C}\ell(V, q)_1$.

We want to globalise these constructions to quadratic forms over schemes. First we have to say what we mean by a quadratic form on a scheme $X$.

**Definition 6.2.** Let $X$ be a scheme such that 2 is invertible on $X$. A *quadratic form on $X$* is a pair $(\mathcal{E}, q)$ where $\mathcal{E}$ is a locally free sheaf and $q\colon \operatorname{Sym}_2(\mathcal{E}) \to \mathcal{O}_X$ is a morphism of $\mathcal{O}_X$-modules, and $\operatorname{Sym}_2 \mathcal{E}$ denotes the submodule of symmetric tensors inside $\mathrm{T}^2(\mathcal{E})$.

Let us denote the symmetric square as $\operatorname{Sym}^2 \mathcal{E}$, i.e. this is the quotient of $\mathrm{T}^2 \mathcal{E}$ by the relation $v \otimes w - w \otimes v$. As in (6.4), using the assumption that 2 is invertible on $X$, there exists an isomorphism $\operatorname{Sym}_2 \mathcal{E} \cong \operatorname{Sym}^2 \mathcal{E}$, relating quadratic forms to symmetric bilinear forms. We will from now on identify both sheaves and consider quadratic forms as morphisms from $\operatorname{Sym}^2 \mathcal{E}$.

We can then define what we mean by a Clifford algebra associated to such a quadratic form.

**Definition 6.3.** Let $X$ be a scheme. Let $(\mathcal{E}, q)$ be a quadratic form over $X$. The *Clifford algebra* $\mathcal{C}\ell_X(\mathcal{E}, q)$ is the sheafification of

$$U \mapsto \mathrm{T}_{\mathcal{O}_X(U)}(\mathcal{E}(U))/(v \otimes v - q(v)1 \mid v \in \mathrm{H}^0(U, \mathcal{E})). \tag{6.7}$$

From the grading on the tensor algebra we see that $\mathcal{C}\ell_X(\mathcal{E},q)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, as was the case for $\mathcal{C}\ell(V,q)$. In order to be compatible with the notation from section 6.2.2 we will sometimes denote this Clifford algebra as $\mathcal{C}\ell_X(\mathcal{E},q,\mathcal{O}_X)$.

**Remark 6.4.** Now in the case that $q$ is nondegenerate we get that $\mathcal{C}\ell_X(\mathcal{E},q)$ is *strongly* $\mathbb{Z}/2\mathbb{Z}$-graded, and $\mathcal{C}\ell_X(\mathcal{E},q)_1$ is an invertible sheaf of $\mathcal{C}\ell_X(\mathcal{E},q)_0$-bimodules. See also [98, example 1.1.24].

Similar to definition 6.1 we can extend this definition by replacing $\mathcal{O}_X$ by a sheaf of commutative algebras $\mathcal{A}$ as follows.

**Definition 6.5.** Let $X$ be a scheme and $\mathcal{A}$ a sheaf of commutative algebras on $X$. Let $\mathcal{E}$ be a locally free $\mathcal{A}$-module and let $q\colon \operatorname{Sym}^2_{\mathcal{A}}(\mathcal{E}) \to \mathcal{A}$ be a morphism of $\mathcal{A}$-modules. The *Clifford algebra* $\mathcal{C}\ell_{\mathcal{A}}(\mathcal{E},q)$ is the sheafification of

$$(6.8)\quad U \mapsto \mathrm{T}_{\mathcal{A}(U)}(\mathcal{E}(U))/(v \otimes_{\mathcal{A}(U)} v - q(v)1 \mid v \in \mathrm{H}^0(U,\mathcal{E})).$$

We will use this construction in remark 6.9 and section 6.2.4, where $\mathcal{A}$ will be a sheaf of *graded* algebras concentrated in even degree. In this case we give $\mathcal{E}$ degree 1, and then $\mathcal{C}\ell_{\mathcal{A}}(\mathcal{E},q)$ will be a $\mathbb{Z}$-graded algebra.

### 6.2.2 Clifford algebras with values in line bundles

We would also like to consider morphisms of the form $\operatorname{Sym}^2(\mathcal{E}) \to \mathcal{L}$, where $\mathcal{L} \ncong \mathcal{O}_X$ is a line bundle on $X$. In this case we cannot mimick the construction of section 6.2.1 to produce a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford algebra for this more general situation. But an analogue of the even part of the Clifford algebra *can* be defined, together with a bimodule over this algebra [40, 56] which generalises the odd part of the usual Clifford algebra.

The generalisation of quadratic forms taking values in a line bundle is the following.

**Definition 6.6.** Let $X$ be a scheme such that 2 is invertible on $X$. Let $\mathcal{L}$ be an invertible sheaf. An *$\mathcal{L}$-valued quadratic form on $X$* is a triple $(\mathcal{E},q,\mathcal{L})$ where $\mathcal{E}$ is a locally free sheaf and $q\colon \operatorname{Sym}^2(\mathcal{E}) \to \mathcal{L}$ is a morphism of $\mathcal{O}_X$-modules.

Let $X$ be a scheme. Let $(\mathcal{E},q,\mathcal{L})$ be an $\mathcal{L}$-valued quadratic form on $X$. We can consider the total space of $\mathcal{L}$ as

$$(6.9)\quad p\colon Y := \mathbf{Spec}_X\left(\bigoplus_{n\in\mathbb{N}} \mathcal{L}^{\otimes n}\right) \to X.$$

Because $p^*\mathcal{L} \cong \mathcal{O}_Y$ the quadratic form $(p^*\mathcal{E}, p^*q)$ takes values in $\mathcal{O}_Y$, so we are in the situation of the previous section.

**Definition 6.7.** The sheaf of algebras $p_*(\mathcal{C}\ell_Y(p^*\mathcal{E},p^*q))$ on $X$ is called the *generalised Clifford algebra* or *total Clifford algebra*. As $p$ is an affine morphism, $p_*$ is just the forgetful functor, and the structure as $\mathcal{O}_Y$-module induces a $\mathbb{Z}$-grading on the total Clifford algebra.

The *even Clifford algebra* $\mathcal{C}\ell_X(\mathcal{E},q,\mathcal{L})_0$ is the degree 0 subalgebra of the total Clifford algebra $p_*(\mathcal{C}\ell_Y(p^*\mathcal{E},p^*q))$.

The *Clifford module* $\mathcal{C}\ell_X(\mathcal{E}, q, \mathcal{L})_1$ is the degree 1 submodule of the total Clifford algebra considered as a bimodule over $\mathcal{C}\ell_X(\mathcal{E}, q, \mathcal{L})_0$.

It is not possible to combine these two pieces into a $\mathbb{Z}/2\mathbb{Z}$-graded algebra. Rather we get that the bimodule structure gives rise to a multiplication map

(6.10) $\mathcal{C}\ell_X(\mathcal{E}, q, \mathcal{L})_1 \otimes_{\mathcal{C}\ell_X(\mathcal{E},q,\mathcal{L})_0} \mathcal{C}\ell_X(\mathcal{E}, q, \mathcal{L})_1 \to \mathcal{C}\ell(\mathcal{E}, q, \mathcal{L})_0 \otimes_{\mathcal{O}_X} \mathcal{L}.$

These construction satisfy the same pleasant properties as the Clifford algebra from section 6.2.1. In particular we will use that it is compatible with base change [40, lemma 3.4].

### 6.2.3 Graded Clifford algebras

In [40, 56] there is a construction of connected graded algebras based on linear algebra input. For sufficiently general choices it gives rise to an interesting class of Artin–Schelter regular algebras of arbitrary dimension which are finite over their center.

**Definition 6.8.** Let $M_1, \ldots, M_n$ be symmetric matrices in $\mathrm{Mat}_n(k)$. The *graded Clifford algebra* associated to $(M_i)_{i=1}^n$ is the quotient of the graded free $k$-algebra

(6.11) $k\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$, where $|x_i| = 1$ and $|y_i| = 2$,

by the relations

1. $x_i x_j + x_j x_i = \sum_{m=1}^n (M_m)_{i,j} y_m$, where $i, j = 1, \ldots, n$
2. $[x_i, y_j] = 0$ and $[y_i, y_j] = 0$, where $i, j = 1, \ldots, n$, i.e. $y_i$ is central.

To understand the properties of this graded Clifford algebra, we have to interpret the matrices $M_1, \ldots, M_n$ as quadratic forms $Q_i$ (in the variables $y_i$). In this way we obtain $n$ quadric hypersurfaces in $\mathbb{P}^{n-1}$, which span a linear system. There are many different ways in which the geometry of the linear system of quadrics influences the algebraic and homological properties of these Clifford algebras.

In particular, the graded Clifford algebra is Artin–Schelter regular if and only if the linear system of quadrics is basepoint-free [149, proposition 7]. It is also possible to describe the (fat) point modules [149, proposition 9]. We can consider the matrix $M = \sum_{i=1}^n M_i y_i$, and at a point $p \in \mathbb{P}^{n-1}_{y_1,\ldots,y_n}$ we can consider the rank of the matrix $M(p)$. Generically it is of full rank, and the point modules correspond to those points for which the rank is 1 or 2. We are interested in the case where $n = 3$, so the point modules are given by the determinant of $M$, which describes a cubic curve inside $\mathbb{P}^2$. We get back to this in section 6.4.2.

**Remark 6.9.** Although definition 6.8 is at first sight quite different from definition 6.1, we would like to mention that every graded Clifford algebra is in fact a Clifford algebra. To see this, let $\mathcal{C}\ell(M)$ be the graded Clifford algebra defined by $n$ symmetric $n \times n$ matrices $M_m$. Then there is an isomorphism

(6.12) $\mathcal{C}\ell(M) \cong \mathcal{C}\ell_A(E, q)$

by setting

$$
\begin{aligned}
&A = k[y_1, \dots, y_n] \\
(6.13)\quad &F = kx_1 \oplus \dots \oplus kx_n \\
&E = F \otimes_k A
\end{aligned}
$$

and considering the quadratic form

$$
(6.14)\quad
\begin{aligned}
q_F \colon \operatorname{Sym}^2_k F &\to ky_1 \oplus \dots \oplus ky_n \\
x_i x_j + x_j x_i &\mapsto \sum_{m=1}^{n} (M_m)_{i,j} y_m
\end{aligned}
$$

and extending it as

$$
(6.15)\quad q \colon \operatorname{Sym}^2_A E = \operatorname{Sym}^2_k F \otimes_k A \xrightarrow{q_F \otimes \mathrm{id}_A} (ky_1 \oplus \dots \oplus ky_n) \otimes_k A = A_2 \otimes_k A \to A
$$

In the next section we continue this idea of interpreting Clifford algebras in two ways.

### 6.2.4 Clifford algebras with values in ample line bundles

Consider a projective morphism $f \colon Y \to X$, and let $\mathcal{L}$ be an $f$-relatively ample line bundle on $Y$. In this case we can consider the graded $\mathcal{O}_X$-algebra

$$
(6.16)\quad \mathcal{A} := \bigoplus_{n \geq 0} f_*(\mathcal{L}^{\otimes n})
$$

and we have that the relative Proj recovers $Y$, i.e. $Y \cong \mathbf{Proj}_X \mathcal{A}$.

Let $\mathcal{E}$ be a vector bundle on $X$. Consider a quadratic form

$$
(6.17)\quad q \colon \operatorname{Sym}^2_Y(f^*\mathcal{E}) \to \mathcal{L}
$$

We can associate two Clifford algebras to this quadratic form:

1. Using section 6.2.2 there is the sheaf of even Clifford algebras $\mathcal{C}\ell_0(f^*(\mathcal{E}, q, \mathcal{L})$, which is a coherent sheaf of algebras on $Y$.

2. On the other hand, we can use the algebra $\mathcal{A}$ from (6.16) and consider it with a doubled grading, i.e.

   $$
   (6.18)\quad \mathcal{A}_n = \begin{cases} f_*(\mathcal{L}^{\otimes m}) & n = 2m \\ 0 & n = 2m+1. \end{cases}
   $$

   Changing the grading in this way does not change the property that $Y$ is the relative Proj of $\mathcal{A}$.

   Because $f^*$ is monoidal and using the adjunction $f^* \dashv f_*$ we can consider the quadratic form $q$ as a morphism $\operatorname{Sym}^2_X \mathcal{E} \to f_*(\mathcal{L}) = \mathcal{A}_2$. We can then extend this morphism to the quadratic form

   $$
   (6.19)\quad Q \colon \operatorname{Sym}^2_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}) \to \mathcal{A}.
   $$

Using section 6.2.1 we can define the Clifford algebra $\mathcal{C}\ell_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}, Q)$, which we can consider as a sheaf of graded $\mathcal{A}$-algebras on $X$.

We can compare these two constructions as follows.

**Proposition 6.10.** With the notation from above, the coherent sheaf of algebras on $Y$ associated to $\mathcal{C}\ell_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}, Q)$ is isomorphic to the sheaf of even Clifford algebras $\mathcal{C}\ell_0(f^*\mathcal{E}, q, \mathcal{L})$.

Moreover, the coherent sheaf on $Y$ associated to $\mathcal{C}\ell_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}, Q)(1)$ is isomorphic to the Clifford bimodule $\mathcal{C}\ell_1(f^*\mathcal{E}, q, \mathcal{L})$.

*Proof.* We have by the relative version of Serre's theorem that $\operatorname{Qcoh} Y \cong \operatorname{QGr}_X \mathcal{A}$. Then $f^*\mathcal{E}$ on $Y$ is given by the graded $\mathcal{A}$-module $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}$. It now suffices to observe that by the characterisation of relatively ample line bundles [200, tag 01VJ] we can compare the two constructions of Clifford algebras, using [40, lemmas 3.1 and 3.2].

The statement about the Clifford module follows from the fact that $\mathcal{A}$ is concentrated in even degree, and [40, lemma 3.1]. □

**Example 6.11.** As a special case of this construction we recover the graded Clifford algebra from section 6.2.3. To see this, we consider the graded Clifford algebra as an algebra over $\mathbb{C}[y_1, \ldots, y_n]$. If we denote $E$ the vector space spanned by the $y_i$'s, we have a morphism $\mathbb{P}(E) \to \operatorname{Spec} k$, and the graded Clifford algebra is nothing but the Clifford algebra associated to $(E \otimes_k \mathcal{O}_{\mathbb{P}(E)}, q, \mathcal{O}_{\mathbb{P}(E)}(1))$, where $q$ describes the linear system of quadrics as symmetric matrices in the global sections $E \otimes E^\vee$.

## 6.3 Noncommutative $\mathbb{P}^1$-bundles as Clifford algebras

As mentioned in the introduction the goal of this chapter is to compare two constructions of noncommutative surfaces of (numerical) type $m = 2$ of ($\mathrm{B}_m$) which was described above. We will prove that both categories are equivalent to (a Serre quotient of) the module category over a sheaf of graded Clifford algebras on $\mathbb{P}^1$. In this section we describe how the noncommutative $\mathbb{P}^1$-bundle obtained in [214, theorem 5.1] is equivalent to the module category over a sheaf of graded Clifford algebras on $\mathbb{P}^1$.

In section 6.3.1 we quickly recall this construction as a noncommutative $\mathbb{P}^1$-bundle, or rather the associated module category $\operatorname{QGr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$.

In section 6.3.4 we recall the technicalities in the construction of noncommutative $\mathbb{P}^1$-bundles as in [214] in more detail. We also recall the notion of *twisting* for sheaf-$\mathbb{Z}$-algebras, an operation which induces equivalences of categories at the level of Gr and QGr. We use this twist operation to show that $\operatorname{QGr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ is equivalent to $\operatorname{QGr}(\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1))$ for a sheaf of graded algebras $\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)$ associated to the finite morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$.

In section 6.3.5 we then show that $\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)$ is a *symmetric sheaf of graded algebras*. In the particular case that $f$ has degree 4, this implies using lemma 6.39 that $\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)$ is isomorphic to a sheaf of graded Clifford algebras on $\mathbb{P}^1$.

The proofs of both lemma 6.37 and lemma 6.39 are based on local computations: it was shown in [214, §3] that noncommutative $\mathbb{P}^1$-bundles can be studied locally using the theory of generalised preprojective algebras $\Pi_C(D)$ associated to a relative Frobenius pair $D/C$ of finite rank as in [213]. We recall this theory in section 6.3.3. In this section we also show that $\mathrm{QGr}(\Pi_C(D))$ is equivalent to $\mathrm{QGr}(\mathrm{H}(D/C))$ where $\mathrm{H}(D/C)$ is a graded algebra with symmetric relations. The theory in section 6.3.2 then shows that $\mathrm{H}(D/C)$ maps surjectively onto a graded Clifford algebra $\mathcal{C}\ell_A(E,q)$. In the particular case of a relative Frobenius pair $D/C$ of rank 4 (these are the only pairs we encounter locally when $f$ has degree 4) this map is in fact an isomorphism by proposition 6.22.

We can summarise the steps in the comparison as follows.

$$\begin{array}{cr}
\operatorname{qgr} \mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}) & \\
\big\downarrow & \text{lemma 6.32} \\
\operatorname{qgr} \Pi(Y/X) & \\
\big\downarrow & \\
\operatorname{qgr} \Pi(Y/X)^{(2)} & \\
\big\downarrow & (6.75) \\
\operatorname{qgr} \mathrm{H}(Y/X) & \\
\big\downarrow & \text{corollary 6.44} \\
\operatorname{qgr}\left(\mathcal{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}\otimes_{\mathcal{O}_{\mathbb{P}^1}}\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)),q\right)\right) & \\
\big\downarrow & \text{remark 6.45} \\
\operatorname{qgr}\left(\mathcal{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(E\otimes_k\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)),q\right)\right) &
\end{array}$$

### 6.3.1 Construction of the surface as a noncommutative bundle

In [225] a notion of noncommutative $\mathbb{P}^1$-bundles was introduced, in order to describe noncommutative Hirzebruch surfaces. This is done by defining a suitable notion of a noncommutative symmetric algebra for a locally free sheaf, where the left and right structures do not necessarily agree but where the rank is 2 on both sides.

In [214] this construction was modified to define a noncommutative symmetric algebra where the rank is 1 on the left and 4 on the right. In particular it provides noncommutative $\mathbb{P}^1$-bundles on $\mathbb{P}^1$ as categories $\mathrm{QGr}(\mathbb{S}({}_f\mathcal{L}_{\mathrm{id}}))$ where

1. $\mathcal{L}$ is a line bundle on $\mathbb{P}^1$,
2. $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a finite morphism of degree 4,
3. $\mathbb{S}({}_f\mathcal{L}_{\mathrm{id}})$ is the *symmetric sheaf* $\mathbb{Z}$*-algebra* associated to the bimodule ${}_f\mathcal{L}_{\mathrm{id}}$

One is referred to section 6.3.4 for the exact definition of this object.

Such a noncommutative $\mathbb{P}^1$-bundle on $\mathbb{P}^1$ has "morphisms" to two copies of $\mathbb{P}^1$ (see [214, §4]). This means that there are functors $\Pi_{0,*}, \Pi_{1,*}\colon\ \mathrm{QGr}(\mathbb{S}({}_f\mathcal{L}_{\mathrm{id}})) \to \mathrm{Qcoh}(\mathbb{P}^1)$ with left adjoints denoted by $\Pi_0^*, \Pi_1^*$. It is then shown in [214, theorem 5.1] that one can use these functors to lift the Beilinson exceptional sequence on $\mathbb{P}^1$ to a full and strong exceptional sequence

(6.20) $\mathbf{D}^{\mathrm{b}}(\operatorname{qgr}(\mathbb{S}({}_f\mathcal{L}_{\mathrm{id}}))) = \langle \Pi_1^*\mathcal{O}_{\mathbb{P}^1}, \Pi_1^*\mathcal{O}_{\mathbb{P}^1}(1), \Pi_0^*\mathcal{O}_{\mathbb{P}^1}, \Pi_0^*\mathcal{O}_{\mathbb{P}^1}(1)\rangle$

in $\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(\mathbb{S}({}_f\mathcal{L}_{\mathrm{id}}))$. Moreover there is an explicit formula ([214, theorem 4.1]) for computing Ext-spaces of the form $\mathrm{Ext}^n(\Pi_i^*\mathcal{F}, \Pi_j^*\mathcal{G})$. In particular for $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}$ one finds that the Gram matrix for the above exceptional sequence is given by

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.21}$$

It was shown in [212, §6] that $\mathrm{B}_2$ is mutation equivalent to this matrix.

### 6.3.2 Graded algebras with symmetric relations

Let $C$ be a commutative ring in which 2 is invertible. Let $F$ be a finitely generated, projective $C$-module and let $R$ be a direct summand of $\mathrm{Sym}^2_C(F)$. As we assumed $2 \in C$ to be invertible, we can use the isomorphism in (6.4) to view $R$ as a submodule of $\mathrm{Sym}_2(F) \subset \mathrm{T}^2_C(F)$. As such we can consider the algebra $\mathrm{T}_C\, F/(R)$. The goal of this section is to show that this algebra is isomorphic to a Clifford algebra.

We construct the Clifford algebra $\mathcal{C}\ell_A(E, q)$ as in definition 6.3. For this let

$$Q = \mathrm{Sym}^2_C(F)/R,\ A = \mathrm{Sym}_C Q \tag{6.22}$$

and

$$E = F \otimes_C A. \tag{6.23}$$

The quotient map $\tilde{q}\colon \mathrm{Sym}^2_C(F) \to Q$ induces a map:

$$q\colon \mathrm{Sym}^2_A(E) = \mathrm{Sym}^2_C(F) \otimes_C A \to Q \otimes_C A \xrightarrow{\mu} A \tag{6.24}$$

where $\mu$ is given by multiplication in $A$. Moreover $A$ can be considered as a $\mathbb{Z}$-graded algebra by giving $Q$ degree 2.

**Remark 6.12.** Giving $A$ degree 0 and $E$ degree 1 equips $\mathcal{C}\ell_A(E, q)$ with a filtration such that

$$\mathrm{gr}^\bullet\, \mathcal{C}\ell_A(E, q) \cong \bigwedge\nolimits^\bullet E \tag{6.25}$$

as graded $A$-modules.

**Lemma 6.13.** With the above notation there exists a morphism

$$\varphi\colon \mathrm{T}_C(F)/(R) \to \mathcal{C}\ell_A(E, q). \tag{6.26}$$

*Proof.* The inclusions $C \hookrightarrow A$ and $F \hookrightarrow E$ induce a morphism

$$\mathrm{T}_C(F) \to \mathrm{T}_A(E) \to \mathcal{C}\ell_A(E, q). \tag{6.27}$$

It hence suffices to show that this morphism factors through $\mathrm{T}_C(F)/(R)$, i.e. that the above map sends $R$ to 0. For this note that $R$ is generated by elements of the form $v \otimes w + w \otimes v$. The image of such an element in $\mathcal{C}\ell_A(E, q)$ is given by $q(vw)$. And $q(vw) = 0$ for $v \otimes w + w \otimes v \in R$ because $q$ was obtained as an $A$-linear extension of the quotient map $\tilde{q}$. ∎

**Proposition 6.14.** The morphism (6.26) is always an epimorphism.

*Proof.* It suffices to show that $A \subset \mathcal{C}\ell_A(E, q)$ and $E \subset \mathcal{C}\ell_A(E, q)$ lie in the image of $\varphi$. By construction $C$ and $F$ lie in the image of $\varphi$ and $E = F \otimes_C A$. In order to prove surjectivity of $\varphi$, we hence only need to show that $A$ lies in the image of $\varphi$. Finally, as $A$ is $\mathbb{N}$-graded and generated in degree 2, it suffices to show that $A_0$ and $A_2$ lie in the image of $\varphi$. For $A_0$ this is obvious as $A_0 = C$. For $A_2$ this follows by noticing that $A_2 = q(F \otimes F)$.

□

### 6.3.3 Generalised preprojective algebras

In this section we introduce generalised preprojective algebras as in [213]. First we introduce some short hand notation.

**Definition 6.15.** Let $C$ be a commutative ring and $D$ a commutative $C$-algebra, then we define the tensor algebra $\mathrm{T}(D/C)$ via

(6.28) $\mathrm{T}(D/C) \coloneqq \mathrm{T}_{C\oplus D}({}_C D_D \oplus {}_D D_C)$

where ${}_C D_D$ is a $C \oplus D$-bimodule by letting $C$ act on the left, $D$ act on the right and all other actions being trivial. Similarly ${}_D D_C$ is a $C \oplus D$-bimodule. In particular ${}_C D_D \otimes_{C\oplus D} {}_C D_D = {}_D D_C \otimes_{C\oplus D} {}_D D_C = 0$.

**Definition 6.16.** Let $C$ be a commutative ring and $D$ a commutative $C$-algebra. Then the morphism $D/C$ is called a *relative Frobenius extension of rank $n$* if the following conditions are satified:

- $D$ is a free $C$-module of rank $n$,
- there exists an isomorphism $\varphi\colon \mathrm{Hom}_C(D, C) \to D$ of $D$-(bi)modules.

**Remark 6.17.** As mentioned in [214, remark 1.2] one can also define relative Frobenius pairs $D/C$ where $D$ is projective over $C$. Most of the results in op. cit. can be lifted to the projective setting by suitably localizing $C$. As we will only need the case where $C$ is a local ring, this extra generality is not necessary for our purposes.

**Definition 6.18.** Let $D/C$ be a relative Frobenius extension of rank $n$ and let $\varphi$ denote the isomorphism $\mathrm{Hom}_C(D, C) \cong D$. The *generalised preprojective algebra* $\Pi_C(D)$ is defined as the quotient of $\mathrm{T}(D/C)$ with the relations given by the images of the following morphisms

- the structure morphism $i\colon C \to {}_C D_C = {}_C D_D \otimes_D D_C$
- the $D$-bimodule morphism $r\colon D \to {}_D D_C \otimes_C D_D : 1_D \mapsto r_\varphi$, where $r_\varphi$ is the element in ${}_D D_C \otimes_C D_D$ corresponding to $\varphi \in \mathrm{Hom}_D(\mathrm{Hom}_C(D, C), D)$ under the identification

  (6.29) $\mathrm{Hom}_D(\mathrm{Hom}_C(D, C), D) \cong {}_D D_C \otimes_C D_D.$

**Remark 6.19.** It was mentioned in [213, definition 1.3 and remark 1.4] that the morphism $r$ as above can also be described in a different way. For this one fixes a morphism $\Lambda\colon C \to D$ which generates $\mathrm{Hom}_C(D, C)$ as a $D$-module, i.e. $\mathrm{Hom}_C(D, C) = \Lambda D$. Every basis $\{e_1, \ldots, e_n\}$ for $D$ as a $C$-module admits a dual basis $\{f_1, \ldots, f_n\}$ for $D$ in the sense that

$$\Lambda(e_i f_j) = \delta_{i,j}. \tag{6.30}$$

The element $r_\varphi \in {}_D D_C \otimes_C D_D$ is then given by $\sum_{i=1}^n e_i \otimes f_i$. The fact that this defines a $D$-bimodule morphism follows from [214, lemma 3.24]

**Definition 6.20.** Let $D/C$ be a relative Frobenius extension of rank $n$ and let $\Pi_C(D)$ be as above. Then we define

$$\mathrm{H}(D/C) := ((1_C, 0)\Pi_C(D)(1_C, 0))^{(2)} \tag{6.31}$$

It is immediate that $\mathrm{H}(D/C)$ can be written as a quotient of $\mathrm{T}_C(D/i(C))$.

**Lemma 6.21.** There exists a Clifford algebra $\mathcal{C}\ell_A(E, q)$ and a surjective morphism

$$\varphi\colon \mathrm{H}(D/C) \to \mathcal{C}\ell_A(E, q). \tag{6.32}$$

*Proof.* By proposition 6.14 it suffices to prove that the relations in $\mathrm{H}(D/C)$ as a quotient of $\mathrm{T}_C(D/i(C))$ are symmetric. Note that

$$\mathrm{H}(D/C) = (\mathrm{T}_C(D/i(C)))\,/(R) \tag{6.33}$$

where $R$ is the image of

$$D \xrightarrow{r} D \otimes_C D \to D/i(C) \otimes_C D/i(C) \tag{6.34}$$

We now make the following three observations:

- The image of $r$ is described by $r_\varphi$, which can be written as $\sum_{i=1}^n e_i \otimes f_i$ where we take $\{e_1, \ldots, e_n\}$ a basis for $D$ as a $C$-module and $\{f_1, \ldots, f_n\}$ is a basis dual to this one.
- As $r_\varphi$ can be defined without the choice of a basis for $D$ as $C$-module, it does not depend on the choice of basis $\{e_1, \ldots, e_n\}$.
- If $\Lambda(e_i f_j) = \delta_{i,j}$ then $\Lambda(f_i e_j) = \delta_{i,j}$ as well. In particular if $\{f_1, \ldots, f_n\}$ is the dual basis for $\{e_1, \ldots, e_n\}$, then $\{e_1, \ldots, e_n\}$ is the dual basis for $\{f_1, \ldots, f_n\}$ as well.

These 3 observations together imply

$$\sum_{i=1}^n e_i \otimes f_i = r_\varphi = \sum_{i=1}^n f_i \otimes e_i \tag{6.35}$$

such that $R \subset \mathrm{Sym}_{C,2}(D/i(C)) \cong \mathrm{Sym}^2_C(D/i(C))$ as required. □

**Proposition 6.22.** The morphism in (6.32) is an isomorphism provided that $n = 4$.

*Proof.* Surjectivity is immediate by the construction as in proposition 6.14. Hence it suffices to prove injectivity.

We now claim that $\mathrm{H}(D/C)$ and $\mathcal{C}\ell_A(E,q)$ have a structure as graded $C$-algebra such that $\gamma$ becomes a graded $C$-algebra morphism and such that the graded pieces $\mathrm{H}(D/C)_n$ and $(\mathcal{C}\ell_A(E,q))_n$ are free $C$-modules of the same rank for each $n$. The proposition then follows.

As $(1_C\Pi_C(D))^{(2)} = (1_C\Pi_C(D)1_C)^{(2)} = \mathrm{H}(D/C)$ we can use [214, lemma 3.10] to conclude that $\mathrm{H}(D/C)$ is a graded $C$-algebra such that $\mathrm{H}(D/C)_n$ is a free $C$-module of rank $n+1$.

Next note that $A$ has a structure of a graded $C$-algebra. Moreover, as before, we let $Q \subset A$ have degree 2. Hence letting $F \subset E$ have degree 1, the relations

$$u \otimes v + v \otimes u = q(uv) \tag{6.36}$$

become homogeneous of degree 2 for each $u, v \in F$ and there is a unique induced grading on $\mathcal{C}\ell_A(E,q)$. Using remark 6.12 we find isomorphisms of $C$-modules:

$$\begin{aligned} \mathcal{C}\ell_A(E,q)_{2n} &\cong A_{2n} \oplus (E \wedge E)_{2n-2}, \\ \mathcal{C}\ell_A(E,q)_{2n+1} &\cong E_{2n} \oplus (E \wedge E \wedge E)_{2n-2}. \end{aligned} \tag{6.37}$$

As $E$, $E \wedge E$ and $E \wedge E \wedge E$ are free $A$-modules of ranks 3, 3 and 1 respectively, we can calculate the rank of $\mathcal{C}\ell_A(E,q)_{2n(+1)}$ using the rank of $A_n$. Recall that $A = \mathrm{Sym}_C(Q)$ and that $Q$ is a free $C$-module of rank 2. As such $h_A(2n) = n+1$. We hence find that $\mathcal{C}\ell_A(E,q)_{2n}$ is a free $C$-module of rank $n+1+3n = 4n+1 = 2(2n)+1$ and that $\mathcal{C}\ell_A(E,q)_{2n+1}$ is a free $C$-module of rank $3(n+1)+n = 4n+3 = 2(2n+1)+1$.

Finally it is immediate from the construction that $\gamma\colon \mathrm{H}(D/C) \to \mathcal{C}\ell_A(E,q)$ is a graded $C$-algebra morphism. The claim and the result follow. □

For use below we also introduce the following $\mathbb{Z}$-algebra:

**Definition 6.23.** Let $D/C$ be a relative Frobenius extension of rank $n$. We define the tensor-$\mathbb{Z}$-algebra $\mathrm{T}_{\mathbb{Z}}(D/C)$ via

$$\mathrm{T}_{\mathbb{Z}}(D/C)_{(j,j+1)} = \begin{cases} {}_C D_D & \text{if } j \text{ is even} \\ {}_D D_C & \text{if } j \text{ is odd} \end{cases} \tag{6.38}$$

and we denote $\Pi_{\mathbb{Z}}(D/C)$ for the quotient of $\mathrm{T}_{\mathbb{Z}}(D/C)$ with the relations given by the images of $i$ and $r$ as in definition 6.18.

**Remark 6.24.** The generalised preprojective algebra $\Pi_D(C)$ is related to the $\mathbb{Z}$-algebra $\Pi_{\mathbb{Z}}(D/C)$ as follows:

- $\Pi_D(C) = \overline{\Pi_{\mathbb{Z}}(D/C)}$ using the notation of [214, §3.3].
- $\widetilde{\mathrm{H}(D/C)} := \left((1_C,0)\widetilde{\Pi_C(D)}(1_C,0)\right)^{(2)} \cong \Pi_{\mathbb{Z}}(D/C)^{(2)}$

### 6.3.4 Bimodule $\mathbb{Z}$-algebras and twisting

We start by recalling the definitions of bimodules and sheaf $\mathbb{Z}$-algebras as in [214, 225]. Whenever we use the notations $W$, $X$, $Y$, $X_n$, …, these schemes are in fact smooth $k$-varieties.

**Definition 6.25.** A coherent $X$-$Y$ bimodule $\mathcal{E}$ is a coherent $\mathcal{O}_{X\times Y}$-module such that the support of $\mathcal{E}$ is finite over $X$ and $Y$. We denote the corresponding category by $\mathrm{bimod}(X\text{-}Y)$.

More generally an $X$-$Y$-bimodule is a quasicoherent $\mathcal{O}_{X\times Y}$-module which is a filtered direct limit of objects in $\mathrm{bimod}(X\text{-}Y)$. The category of $X$-$Y$-bimodules is denoted $\mathrm{BiMod}(X\text{-}Y)$.

Most bimodules we encounter in this chapter are of the form ${}_u\mathcal{U}_v$ where $u, v$ are finite maps and $\mathcal{U}$ is a quasicoherent sheaf on some variety $W$. More fomally we have the following.

**Definition 6.26.** Consider finite morphisms $u\colon W \to X$ and $v\colon W \to Y$.

If $\mathcal{U} \in \mathrm{Qcoh}(W)$, then we denote $(u,v)_*\mathcal{U} \in \mathrm{BiMod}(X\text{-}Y)$ as ${}_u\mathcal{U}_v$. One easily checks that

$$(6.39)\quad -\otimes {}_u\mathcal{U}_v = v_*(u^*(-)\otimes_W \mathcal{U})$$

Moreover it was shown in [225] that (under some technical conditions which will always be fulfilled in this chapter) every bimodule $\mathcal{E} \in \mathrm{bimod}(X\text{-}Y)$ has a unique *right dual* $\mathcal{E}^* \in \mathrm{bimod}(Y\text{-}X)$ which is defined by requiring that the functor

$$(6.40)\quad -\otimes_Y \mathcal{E}^*\colon\ \mathrm{Qcoh}(Y) \to \mathrm{Qcoh}(X)$$

is right adjoint to the functor $-\otimes_X \mathcal{E}$. The dual notion leads to the *left dual*: an object ${}^*\mathcal{E} \in \mathrm{bimod}(Y\text{-}X)$. By Yoneda's lemma we have

$$(6.41)\quad \mathcal{E} =^* (\mathcal{E}^*) = ({}^*\mathcal{E})^*$$

Hence one can use the following notation

$$(6.42)\quad \mathcal{E}^{*n} = \begin{cases} \mathcal{E}^{\overbrace{*\cdots*}^{n}} & n \geq 0 \\ {}^{\overbrace{*\cdots*}^{-n}}\mathcal{E} & n < 0 \end{cases}$$

and there are unit and counit morphisms

$$(6.43)\quad \begin{aligned} i_n &\colon {}_{\mathrm{id}}(\mathcal{O}_{X_n})_{\mathrm{id}} \to \mathcal{E}^{*n} \otimes \mathcal{E}^{*n+1} \\ j_n &\colon \mathcal{E}^{*n} \otimes \mathcal{E}^{*n-1} \to {}_{\mathrm{id}}(\mathcal{O}_{X_n})_{\mathrm{id}} \end{aligned}$$

where $X_{2n} = X$ and $X_{2n+1} = Y$ for all $n$.

**Example 6.27.** Let $f\colon Y \to X$ be a finite morphism, where $Y$ is a smooth variety over $\mathrm{Spec}\, k$, and let $\mathcal{E}$ be the bimodule ${}_f(\mathcal{O}_Y)_{\mathrm{id}}$. One can use the explicit formula for $\mathcal{E}^*$ as in the discussion following [225, proposition 3.1.6] to obtain $\mathcal{E}^* = {}_{\mathrm{id}}(\mathcal{O}_Y)_f$. Using (6.39) we find

(6.44) $- \otimes_X \mathcal{E} = f^*(-)$ and $- \otimes_Y \mathcal{E}^* = f_*(-)$.

As such the adjunction $- \otimes_X \mathcal{E} \dashv - \otimes_Y \mathcal{E}^*$ is nothing but the usual adjunction $f^* \dashv f_*$. Using Grothendieck duality we know that $f_*$ has a right adjoint, which is given by $f^!(-) = f^*(-) \otimes \omega_{X/Y}$ because $f$ is finite and flat [152], where $\omega_{X/Y} \in \mathrm{Qcoh}(Y)$ is defined by

(6.45) $f_*\omega_{X/Y} = \mathcal{H}om(f_*\mathcal{O}_Y, \mathcal{O}_X)$.

In particular we find

(6.46) $\mathcal{E}^{**} = {}_f(\omega_{X/Y})_{\mathrm{id}} = \mathcal{E} \otimes \omega_{X/Y}$

with associated unit morphism

(6.47) ${}_{\mathrm{id}}(\mathcal{O}_Y)_{\mathrm{id}} \to \mathcal{E}^* \otimes \mathcal{E} \otimes {}_{\mathrm{id}}(\omega_{X/Y})_{\mathrm{id}}$

Moreover by induction we have that

(6.48) $$\begin{aligned} \mathcal{E}^{(2n)*} &= {}_f(\omega_{X/Y}^n)_{\mathrm{id}} \\ \mathcal{E}^{(2n+1)*} &= {}_{\mathrm{id}}(\omega_{X/Y}^{-n})_f \end{aligned}$$

The tensor product of $\mathcal{O}_{W \times X \times Y}$-modules induces a tensor product

(6.49) $\mathrm{BiMod}(W\text{-}X) \otimes \mathrm{BiMod}(X\text{-}Y) \to \mathrm{BiMod}(W\text{-}Y) : (\mathcal{E}, \mathcal{F}) \mapsto \mathcal{E} \otimes_X \mathcal{F}$

through the formula

(6.50) $\mathcal{E} \otimes \mathcal{F} := \pi_{W \times Y *}\Big(\pi_{W \times X}^*(\mathcal{E}) \otimes_{W \times X \times Y} \pi_{X \times Y}^*(\mathcal{F})\Big)$.

For each $\mathcal{E} \in \mathrm{BiMod}(W\text{-}X)$ this defines a functor

(6.51) $- \otimes_X \mathcal{E} \colon \mathrm{Qcoh}(W) \to \mathrm{Qcoh}(X) : \mathcal{M} \mapsto \mathcal{M} \otimes_X \mathcal{E} := \pi_{X*}\Big(\pi_W^*(\mathcal{M}) \otimes_{W \times X} \mathcal{E}\Big)$

analogous to the notion of a Fourier–Mukai transform, which is right exact in general and exact if $\mathcal{E}$ is locally free on the left. We mention that [225, lemma 3.1.1] shows that this functor determines the bimodule $\mathcal{E}$ uniquely. As such the category $\mathrm{BiMod}(W\text{-}X)$ is embedded in the more abstract categories $\mathrm{BiMod}(W\text{-}X)$ and $\mathrm{BiMod}(W\text{-}X)$ as in op. cit.

The above tensor product turns $\mathrm{BiMod}(X\text{-}X)$ into a monoidal category. As such it is possible to define algebra objects in this category. More general one can construct $\mathbb{Z}$-algebra objects with respect to a collection of categories $\mathrm{BiMod}(X_i\text{-}X_j)$ as follows.

**Definition 6.28.** Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of smooth varieties. A *sheaf $\mathbb{Z}$-algebra* $\mathcal{A}$ is a collection of $X_i$-$X_j$-bimodules $\mathcal{A}_{i,j}$ equipped with multiplication and identity maps

$$\begin{aligned} \mu_{i,j,k}\colon \mathcal{A}_{i,j}\otimes\mathcal{A}_{j,k} &\longrightarrow \mathcal{A}_{i,k} \\ u_i\colon \mathcal{O}_{X_i} &\longrightarrow \mathcal{A}_{i,i} \end{aligned} \tag{6.52}$$

such that the associativity

$$\begin{array}{ccc} \mathcal{A}_{i,j}\otimes\mathcal{A}_{j,k}\otimes\mathcal{A}_{k,l} & \xrightarrow{\mu_{i,j,k}\otimes 1} & \mathcal{A}_{i,k}\otimes\mathcal{A}_{k,l} \\ {\scriptstyle 1\otimes\mu_{j,k,l}}\downarrow & & \downarrow{\scriptstyle \mu_{i,k,l}} \\ \mathcal{A}_{i,j}\otimes\mathcal{A}_{j,l} & \xrightarrow{\mu_{i,j,l}} & \mathcal{A}_{i,l} \end{array} \tag{6.53}$$

and unit diagrams

$$\begin{array}{ccc} \mathcal{O}_{X_i}\otimes\mathcal{A}_{i,j} & \xrightarrow{u_i\otimes 1} & \mathcal{A}_{i,i}\otimes\mathcal{A}_{i,j} \\ & \searrow \quad \swarrow{\scriptstyle \mu_{i,i,j}} & \\ & \mathcal{A}_{i,j} & \end{array} \qquad \begin{array}{ccc} \mathcal{A}_{i,j}\otimes\mathcal{O}_{X_i} & \xrightarrow{1\otimes u_i} & \mathcal{A}_{i,j}\otimes\mathcal{A}_{j,j} \\ & \searrow \quad \swarrow{\scriptstyle \mu_{i,j,j}} & \\ & \mathcal{A}_{i,j} & \end{array} \tag{6.54}$$

commute.

A *graded $\mathcal{A}$-module* is a sequence of quasicoherent $\mathcal{O}_{X_i}$-modules $\mathcal{M}_i$ together with maps

$$\mu_{\mathcal{M},i,j}\colon \mathcal{M}_i\otimes\mathcal{A}_{i,j}\longrightarrow\mathcal{M}_j \tag{6.55}$$

compatible with the multiplication and identity maps on $\mathcal{A}$ in the usual sense.

A *morphism of graded $\mathcal{A}$-modules* $f\colon\mathcal{M}\longrightarrow\mathcal{N}$ is a collection of $X_i$-module morphisms $f_i\colon\mathcal{M}_i\longrightarrow\mathcal{N}_i$ such that the obvious diagrams commute. The associated category is denoted $\operatorname{Gr}\mathcal{A}$.

An $\mathcal{A}$-module is *right bounded* if $\mathcal{M}_i = 0$ for $i\gg 0$. An $\mathcal{A}$-module is called *torsion* if it is a filtered colimit of right bounded modules. Let $\operatorname{Tors}(\mathcal{A})$ be the subcategory of $\operatorname{Gr}\mathcal{A}$ consisting of torsion modules. If $\operatorname{Gr}\mathcal{A}$ is a locally noetherian category (which is always the case for our applications, see for example [214, theorem 3.1]), then $\operatorname{Tors}\mathcal{A}$ is a localizing subcategory and the corresponding quotient category is denoted by $\operatorname{QGr}\mathcal{A}$.

**Remark 6.29.** Let $\mathcal{R}$ be a sheaf of graded algebras on $X$, then $\mathcal{R}$ induces a sheaf-$\mathbb{Z}$-algebra $\check{\mathcal{R}}$ on $(X_i)_{i\in\mathbb{Z}}$ with $X_i = X$ for all $i$ via

$$\check{\mathcal{R}}_{i,j} = \mathcal{R}_{j-i}. \tag{6.56}$$

It is well known that there are induced equivalences of categories

$$\operatorname{Gr}(\check{\mathcal{R}})\cong\operatorname{Gr}\mathcal{R} \text{ and } \operatorname{QGr}\check{\mathcal{R}}\cong\operatorname{QGr}\mathcal{R}. \tag{6.57}$$

Of particular interest are so-called *symmetric* sheaf-$\mathbb{Z}$-algebras, which are defined in [225] and [214].

**Definition 6.30.** Let $X$ and $Y$ be smooth varieties over $k$ and let $\mathcal{E}$ be an $X$-$Y$-bimodules for which all duals $\mathcal{E}^{n*}$ exist. Then the *tensor sheaf $\mathbb{Z}$-algebra* $\mathrm{T}(\mathcal{E})$ is the sheaf-$\mathbb{Z}$-algebra generated by the $\mathcal{E}^{n*}$, more precisely

$$(6.58)\quad \mathrm{T}(\mathcal{E})_{m,n} = \begin{cases} 0 & n < m \\ {}_{\mathrm{id}}\left(\mathcal{O}_X\right)_{\mathrm{id}} & \text{if } n = m \text{ is even} \\ {}_{\mathrm{id}}\left(\mathcal{O}_Y\right)_{\mathrm{id}} & \text{if } n = m \text{ is odd} \\ \mathcal{E}^{m*} \otimes \ldots \otimes \mathcal{E}^{(n-1)*} & n > m \end{cases}$$

The *symmetric sheaf-$\mathbb{Z}$-algebra* $\mathbb{S}(\mathcal{E})$ is the quotient of $\mathrm{T}(\mathcal{E})$ by the relations defined by (6.43) above. More precisely, $\mathbb{S}(\mathcal{E})_{m,n}$ is defined as

(6.59)

$$\begin{cases} \mathrm{T}(\mathcal{E}_i)_{m,n} & n \le m+1 \\ \mathrm{T}(\mathcal{E})_{m,n}/\Big( \mathrm{im}(i_m) \otimes \ldots) + (\mathcal{E}^{m*} \otimes \mathrm{im}(i_{m+1}) \otimes \ldots) + \ldots + (\ldots \otimes \mathrm{im}(i_{m-2}))\Big) & n \ge m+2. \end{cases}$$

A fundamental operation in the context of sheaf-$\mathbb{Z}$-algebras is that of twisting by a sequence of invertible bimodules. Invertible bimodules are defined in [214] or [225], for our applications it suffices to realise that bimodules of the form ${}_{\mathrm{id}}\mathcal{L}_{\mathrm{id}}$ are invertible. As the next theorem shows, this operation induces equivalences of categories at the level of Gr and QGr.

**Theorem 6.31.** (see for example [214, theorem 2.14]) Let $(X_i)_i$ and $(Y_i)_i$ be sequences of smooth varieties over $k$ and $\mathcal{A}$ a sheaf-$\mathbb{Z}$-algebra on $(X_i)_i$. Given a collection of invertible $X_i$-$Y_i$-bimodules $(\mathcal{T}_i)_i$ one can construct a sheaf-$\mathbb{Z}$-algebra $\mathcal{B}$ by

$$(6.60)\quad \mathcal{B}_{i,j} := \mathcal{T}_i^{-1} \otimes \mathcal{A}_{i,j} \otimes \mathcal{T}_j$$

called the *twist* of $\mathcal{A}$ by $(\mathcal{T}_i)_i$.

There is an equivalence of categories given by the functor

$$(6.61)\quad \mathcal{T}\colon\ \mathrm{Gr}\,\mathcal{A} \cong \mathrm{Gr}\,\mathcal{B} : \mathcal{M}_i \longrightarrow \mathcal{M}_i \otimes \mathcal{T}_i.$$

If moreover $\mathcal{A}$ and $\mathcal{B}$ are noetherian, then this also induces an equivalence at the level of QGr.

We will apply this theorem to the symmetric sheaf-$\mathbb{Z}$-algebras as defined in [214, definition 2.13] or [225, §1].

**Lemma 6.32.** Let $f : Y \to X$ be a finite morphism between smooth projective curves and let $\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})$ be the associated symmetric sheaf-$\mathbb{Z}$-algebra.

Moreover denote $\Pi_{\mathbb{Z}}(Y/X)$ for the sheaf-$\mathbb{Z}$-algebra (over the collection $(X_i)_{i\in\mathbb{Z}}$ with $X_{2j} = X$ and $X_{2j+1} = Y$) defined as follows:

- $\Pi_{\mathbb{Z}}(Y/X)_{m,n} = 0$ whenever $m > n$;
- $\Pi_{\mathbb{Z}}(Y/X)_{2j,2j} = {}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}}$ for all $j$;
- $\Pi_{\mathbb{Z}}(Y/X)_{2j+1,2j+1} = {}_{\mathrm{id}}(\mathcal{O}_Y)_{\mathrm{id}}$ for all $j$;

- $\Pi_{\mathbb{Z}}(Y/X)_{2j,2j+1} = {}_f(\mathcal{O}_Y)_{\mathrm{id}}$ for all $j$;
- $\Pi_{\mathbb{Z}}(Y/X)_{2j+1,2j+2} = {}_{\mathrm{id}}(\mathcal{O}_Y)_f$ for all $j$;
- $\Pi_{\mathbb{Z}}(Y/X)$ is freely generated by the $\Pi_{\mathbb{Z}}(Y/X)_{n,n+1}$ subject to the relations

$$\begin{aligned} {}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}} &\subset {}_{\mathrm{id}}(f_*\mathcal{O}_Y)_{\mathrm{id}} \\ &= {}_f(\mathcal{O}_X)_f \\ &= \Pi_{\mathbb{Z}}(Y/X)_{2j,2j+1} \otimes_Y \Pi_{\mathbb{Z}}(Y/X)_{2j+1,2j+2} \end{aligned} \tag{6.62}$$

and

$$r({}_{\mathrm{id}}(\omega_f^{-1})_{\mathrm{id}}) \subset \Pi_{\mathbb{Z}}(Y/X)_{2j-1,2j} \otimes_Y \Pi_{\mathbb{Z}}(Y/X)_{2j,2j+1} \tag{6.63}$$

where

$$r\colon {}_{\mathrm{id}}(\omega_f^{-1})_{\mathrm{id}} \to {}_{\mathrm{id}}(\mathcal{O}_Y)_f \otimes {}_f(\mathcal{O}_Y)_{\mathrm{id}} \tag{6.64}$$

is induced by $f_* \dashv f^! = f^* \otimes \omega_{X/Y}$ as in (6.47).

Then $\Pi_{\mathbb{Z}}(Y/X)$ is a twist of $\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})$.

*Proof.* Using (6.48) it follows immediately that

$$\Big(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})\Big)_{2j,2j+1} = (\Pi_{\mathbb{Z}}(Y/X))_{2j,2j+1} \otimes {}_{\mathrm{id}}(\omega^j_{X/Y})_{\mathrm{id}} \tag{6.65}$$

and

$$\Big(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})\Big)_{2j+1,2j+2} = {}_{\mathrm{id}}(\omega^{-j}_{X/Y})_{\mathrm{id}} \otimes (\Pi_{\mathbb{Z}}(Y/X))_{2j+1,2j+2}. \tag{6.66}$$

Now let $\mathcal{T}_i$ be defined by

$$\mathcal{T}_i := \begin{cases} {}_{\mathrm{id}}(\mathcal{O}_X)_{\mathrm{id}} & i = 2j \\ {}_{\mathrm{id}}(\omega^j_{X/Y})_{\mathrm{id}} & i = 2j+1. \end{cases} \tag{6.67}$$

We then claim that

$$\Big(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})\Big)_{m,n} \cong \mathcal{T}_m^{-1} \otimes \Pi_{\mathbb{Z}}(Y/X)_{m,n} \otimes \mathcal{T}_n \tag{6.68}$$

holds for all $m, n$.

As both algebras are generated in degree 1 and have quadratic relations, it suffices to check the claim for $n - m = 0, 1, 2$. By the above the claim holds for $n - m = 0, 1$.

For $n = m + 2 = 2j + 2$ we have

$$\Big(\mathbb{S}({}_f(\mathcal{O}_Y)_{\mathrm{id}})\Big)_{2j,2j+2} = {}_{\mathrm{id}}\,(f_*\mathcal{O}_Y/\mathcal{O}_X)_{\mathrm{id}} = (\Pi_{\mathbb{Z}}(Y/X))_{2j,2j+2} \tag{6.69}$$

To get the claim for $n = m + 2$, $m = 2j + 1$ it suffices to check that

$$\begin{aligned} i_{2j+1}({}_{\mathrm{id}}(\mathcal{O}_Y)_{\mathrm{id}}) &= {}_{\mathrm{id}}(\omega^{-j}_{X/Y})_{\mathrm{id}} \otimes r({}_{\mathrm{id}}(\omega_f^{-1})_{\mathrm{id}}) \otimes {}_{\mathrm{id}}(\omega^{j+1}_{X/Y})_{\mathrm{id}} \\ &= {}_{\mathrm{id}}(\omega^{-j}_{X/Y})_{\mathrm{id}} \otimes r({}_{\mathrm{id}}(\omega_f^{-1})_{\mathrm{id}}) \otimes {}_{\mathrm{id}}(\omega_{X/Y})_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\omega^j_{X/Y})_{\mathrm{id}} \end{aligned} \tag{6.70}$$

This in turn follows by using the fact that $i_{2j+1}$ and $r$ are defined as unit maps, the fact that the bimodules underlying these unit maps are twists of each other and the fact that

$$\left({}_{\mathrm{id}}(\omega_{X/Y}^{-j})_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\mathcal{O}_Y)_f\right)^* = \left({}_{\mathrm{id}}(\mathcal{O}_Y)_f\right)^* \otimes {}_{\mathrm{id}}(\omega_{X/Y}^{j})_{\mathrm{id}}. \tag{6.71}$$

□

**Remark 6.33.** The sheaf-$\mathbb{Z}$-algebra $\Pi_{\mathbb{Z}}(Y/X)$ as introduced above is 2-periodic, i.e. for each $i$ and $j$ there is an isomorphism

$$(\Pi_{\mathbb{Z}}(Y/X))_{i,j} \cong (\Pi_{\mathbb{Z}}(Y/X))_{i+2,j+2} \tag{6.72}$$

and these isomorphisms are compatible with the multiplication in $\Pi_{\mathbb{Z}}(Y/X)$.

In particular let $\mathrm{H}(Y/X)$ be the graded sheaf of algebras on $X$ defined as follows:

- $\mathrm{H}(Y/X)_0 = \mathcal{O}_X$;
- $\mathrm{H}(Y/X)_1 = f_*\mathcal{O}_Y/\mathcal{O}_X$;
- $\mathrm{H}(Y/X)$ is generated by $\mathrm{H}(Y/X)_1$ subject to the relation

  $$f_*\omega_{X/Y}^{-1} \subset \mathrm{H}(Y/X)_1 \otimes \mathrm{H}(Y/X)_1 = f_*\mathcal{O}_Y/\mathcal{O}_X \otimes f_*\mathcal{O}_Y/\mathcal{O}_X \tag{6.73}$$

  which is induced by

  $$\begin{aligned} {}_{\mathrm{id}}(f_*\omega_{X/Y}^{-1})_{\mathrm{id}} &= {}_f(\omega_{X/Y})_f \\ &= {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\omega_{X/Y}^{-1})_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\mathcal{O}_Y)_f \\ &\xrightarrow{r} {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\mathcal{O}_Y)_f \otimes {}_f(\mathcal{O}_Y)_{\mathrm{id}} \otimes {}_{\mathrm{id}}(\mathcal{O}_Y)_f \\ &= {}_{\mathrm{id}}(f_*\mathcal{O}_Y \otimes f_*\mathcal{O}_Y)_{\mathrm{id}} \\ &\to {}_{\mathrm{id}}(f_*\mathcal{O}_Y/\mathcal{O}_X \otimes f_*\mathcal{O}_Y/\mathcal{O}_X)_{\mathrm{id}}; \end{aligned} \tag{6.74}$$

then there is an isomorphism of sheaf-$\mathbb{Z}$-algebras:

$$\Pi_{\mathbb{Z}}(Y/X)^{(2)} \cong \check{\mathrm{H}}(Y/X) \tag{6.75}$$

Using theorem 6.31, lemma 6.32, remark 6.33 and remark 6.29 we have equivalences of categories

$$\begin{aligned} \mathrm{QGr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})) &\cong \mathrm{QGr}(\Pi_{\mathbb{Z}}(\mathbb{P}^1/\mathbb{P}^1)) \\ &\cong \mathrm{QGr}(\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)). \end{aligned} \tag{6.76}$$

As a last result of this section we mention that the above category is preserved by the $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-action on the set of degree 4 maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ (where the action is given by coordinate changes of on the domain and codomain).

**Lemma 6.34.** Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a finite map and let $\varphi, \psi \in \mathrm{Aut}(\mathbb{P}^1)$, then

$$\mathrm{QGr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})) \cong \mathrm{QGr}(\mathbb{S}({}_{\varphi\circ f\circ\psi}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})). \tag{6.77}$$

*Proof.* By theorem 6.31 it suffices to show that $\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})$ and $\mathbb{S}({}_{\varphi\circ f\circ\psi}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})$ are twists of each other. We claim that

$$\left(\mathbb{S}({}_{\varphi\circ f\circ\psi}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})\right)_{i,j} \cong \mathcal{T}_i^{-1} \otimes \left(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})\right)_{i,j} \otimes \mathcal{T}_j. \tag{6.78}$$

holds for all $i$ and $j$ if we choose

$$\mathcal{T}_i := \begin{cases} {}_{\mathrm{id}}\mathcal{O}_{\varphi} & \text{if } i \text{ is even} \\ {}_{\psi}\mathcal{O}_{\mathrm{id}} & \text{if } i \text{ is odd} \end{cases}. \tag{6.79}$$

For example we have that

$$\begin{aligned} \left(\mathbb{S}({}_{\varphi\circ f\circ\psi}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})\right)_{2,3} &= {}_{\varphi\circ f\circ\psi}\left(\omega_{\varphi\circ f\circ\psi}\right)_{\mathrm{id}} \\ &\cong {}_{\varphi\circ f\circ\psi}\left(\omega_{\psi} \otimes \psi^*\omega_{X/Y} \otimes \psi^* f^* \omega_{\varphi}\right)_{\mathrm{id}} \\ &\cong {}_{\varphi\circ f\circ\psi}\left(\mathcal{O}_{\mathbb{P}^1} \otimes \psi^*\omega_{X/Y} \otimes \psi^* f^* \mathcal{O}_{\mathbb{P}^1}\right)_{\mathrm{id}} \\ &\cong {}_{\varphi\circ f\circ\psi}\left(\psi^*\omega_{X/Y}\right)_{\mathrm{id}} \\ &\cong {}_{\varphi}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}} \otimes {}_f\left(\omega_{X/Y}\right)_{\mathrm{id}} \otimes {}_{\psi}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}} \\ &\cong \left({}_{\mathrm{id}}(\mathcal{O}_{\mathbb{P}^1})_{\varphi}\right)^{-1} \otimes \left(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})\right)_{2,3} \otimes {}_{\psi}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}. \end{aligned} \tag{6.80}$$

We leave it as an exercise to the reader to check that this twisting is compatible with the quadratic relations of $\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})$ and $\mathbb{S}({}_{\varphi\circ f\circ\psi}(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}})$. ◻

### 6.3.5 Symmetric sheaves of graded algebras

This section is devoted to generalizing lemma 6.13, proposition 6.14 and lemma 6.21 to the level of sheaves of algebras. Throughout this section, all schemes are assumed to have 2 invertible, and for ease of statement we will also assume that they are irreducible.

**Definition 6.35.** Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots$ be a graded sheaf of algebras on a scheme $X$. We say $\mathcal{H}$ is a *symmetric sheaf of graded algebras* if the following conditions hold:

- $\mathcal{H}_1$ is a locally free sheaf on $X$;
- there is a surjective morphism of sheaves of graded algebras

  $$\varphi\colon\ \mathrm{T}_{\mathcal{O}_X}(\mathcal{H}_1) \twoheadrightarrow \mathcal{H}; \tag{6.81}$$

- $\varphi$ is an isomorphism in degree 0 and 1;
- $\ker(\varphi)$ is isomorphic to $\mathcal{R} \otimes_{\mathcal{O}_X} \mathrm{T}_{\mathcal{O}_X}(\mathcal{H}_1)$ where $\mathcal{R}$ is a direct summand of the sheaf $\mathrm{Sym}_{\mathcal{O}_X,2}(\mathcal{H}_1) \subset \mathrm{T}_{\mathcal{O}_X}(\mathcal{H}_1)_2$.

The following lemma is immediate from the local nature of the construction of $\mathrm{Sym}_{\mathcal{O}_X,2}(-)$ and $\mathrm{T}_{\mathcal{O}_X}(-)$.

**Lemma 6.36.** Let $\mathcal{H}$ be a sheaf of graded algebras on $X$. Then the following are equivalent:

1. $\mathcal{H}$ is a symmetric sheaf of graded algebras on $X$.
2. for each point $p \in X$ we have that $\mathcal{H}_p$ is a graded ($\mathcal{O}_{X,p}$-)algebra with symmetric relations as in section 6.3.2.

We use this lemma to prove the following.

**Lemma 6.37.** Let $X$ and $Y$ be smooth varieties over a field $k$ of characteristic different from 2 and let $f\colon Y \to X$ be a finite morphism of degree $n$. Let $\mathrm{H}(Y/X)$ be the associated sheaf of graded algebras as in remark 6.33. Then $\mathrm{H}(Y/X)$ is a symmetric sheaf of graded algebras on $X$.

*Proof.* Using lemma 6.36 it suffices to prove that $\mathrm{H}(Y/X)_p$ is a graded algebra with symmetric relations as in section 6.3.2. For this we first use [214, lemma 3.2.7] which (among other things) shows that $(f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}$ is relative Frobenius of rank $n$. For example one has an isomorphism of $(f_*\mathcal{O}_Y)_p$-modules

$$(f_*\mathcal{O}_Y)_p \cong (\omega_{X/Y})_p \cong \mathrm{Hom}_{\mathcal{O}_{X,p}}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}). \tag{6.82}$$

We now claim that

$$\mathrm{H}(Y/X)_p = \mathrm{H}\Big((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}\Big) \tag{6.83}$$

where the right hand side was defined in (6.31). The lemma follows from this claim and the fact that $\mathrm{H}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p})$ is a graded algebra with symmetric relations (see for example the proof of lemma 6.21).

By comparing (6.33) with remark 6.33 we see that the claim holds in degree 0 and 1: both algebras can be written as quotients of the tensor algebra $\mathrm{T}_{\mathcal{O}_{X,p}}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p})$ subject to quadratic relations.

Recall that the quadratic relations in $\mathrm{H}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p})$ were obtained by composing the morphisms $r$ in (6.34) with the obvious morphism

$$\pi\colon (f_*\mathcal{O}_Y)_p \otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p \to (f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p} \otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}. \tag{6.84}$$

A closer examination of the morphisms $r$ in (6.34), see for example the proof of [214, lemma 3.4.2], shows that the forgetful functor $\mathrm{Mod}((f_*\mathcal{O}_Y)_p) \to \mathrm{Mod}(\mathcal{O}_{X,p})$ is left adjoint to the functor $- \otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p : \mathrm{Mod}(\mathcal{O}_{X,p}) \to \mathrm{Mod}((f_*\mathcal{O}_Y)_p)$ and that $r$ coincides with the associated unit morphism

$$(f_*\mathcal{O}_Y)_p \to (f_*\mathcal{O}_Y)_p \otimes_{\mathcal{O}_{X,p}} (f_*\mathcal{O}_Y)_p. \tag{6.85}$$

Similarly the quadratic relations in $\mathrm{H}(Y/X)_p$ were obtained as the image of $\pi \circ r'$ where $\pi$ is as in (6.84) and $r'$ is obtained by localizing the unit morphism in (6.64). Hence in order to prove the claim it suffices to show that the localization of the unit morphism in (6.64) coincides with the unit morphism in (6.85). For this we use the isomorphism $(f_*\mathcal{O}_Y)_p \cong (\omega_{X/Y})_p$ as in (6.82) and the commutativity of the following diagrams

$$\begin{array}{ccc} \mathrm{Qcoh}(\mathcal{O}_Y) & \xrightarrow{f_* = - \otimes_{\mathrm{id}} (\mathcal{O}_Y f} & \mathrm{Qcoh}(\mathcal{O}_X) \\ \Big\downarrow {\scriptstyle f_{*,p}} & & \Big\downarrow {\scriptstyle (-)_p} \\ \mathrm{Mod}((f_*\mathcal{O}_Y)_p) & \xrightarrow{(-)_{\mathcal{O}_{X,p}}} & \mathrm{Mod}(\mathcal{O}_{X,p}) \end{array} \tag{6.86}$$

and

$$\begin{array}{ccc} \mathrm{Qcoh}(\mathcal{O}_X) & \xrightarrow{f^*=-\otimes_f(\mathcal{O}_Y)_{\mathrm{id}}} & \mathrm{Qcoh}(\mathcal{O}_Y) \\ \downarrow (-)_p & & \downarrow f_{*,p} \\ \mathrm{Mod}(\mathcal{O}_{X,p}) & \xrightarrow{(-)_{\mathcal{O}_{X,p}}} & \mathrm{Mod}((f_*\mathcal{O}_Y)_p) \end{array} \tag{6.87}$$

where for the second diagram, commutativity follows from the projection formula. □

The local nature of the constructions in definition 6.5 allow us to generalise lemma 6.13 and proposition 6.14 as follows.

**Lemma 6.38.** Let $\mathcal{H} = \mathrm{T}_{\mathcal{O}_X}(\mathcal{H}_1)/\mathcal{R}$ be a symmetric sheaf of graded algebras on $X$. Let $\mathcal{Q} = \mathrm{Sym}^2_{\mathcal{O}_X}(\mathcal{H}_1))/\mathcal{R}$, $\mathcal{A} = \mathrm{Sym}_{\mathcal{O}_X}\mathcal{Q}$ and $\mathcal{E} = \mathcal{H}_1 \otimes_{\mathcal{O}_X} \mathcal{A}$.

Let $q\colon \mathrm{Sym}^2_{\mathcal{A}}(\mathcal{E}) \to \mathcal{A}$ be induced by the quotient map $\tilde{q}\colon \mathrm{Sym}^2_{\mathcal{O}_X}(\mathcal{H}_1)) \to \mathcal{Q}$; i.e.

$$q\colon \mathrm{Sym}^2_{\mathcal{A}}(\mathcal{E}) = \mathrm{Sym}^2_{\mathcal{O}_X}(\mathcal{H}_1) \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{A} \xrightarrow{\mu} \mathcal{A} \tag{6.88}$$

where $\mu$ is given by multiplication in $\mathcal{A}$.

Then the inclusions $\mathcal{O}_X \hookrightarrow \mathcal{A}$ and $\mathcal{H}_1 \hookrightarrow \mathcal{E}$ induce a epimorphism

$$\varphi\colon \mathcal{H} \twoheadrightarrow \mathcal{C}\ell_{\mathcal{A}}(\mathcal{E}, q). \tag{6.89}$$

We now combine lemma 6.38 with lemma 6.37 to obtain the following.

**Lemma 6.39.** Let $X$ and $Y$ be smooth varieties over a field $k$ of characteristic different from 2 and let $f\colon Y \to X$ be a finite morphism of degree 4. Let $\mathrm{H}(Y/X)$ and $\mathcal{C}\ell_{\mathcal{A}}(\mathcal{E}, q)$ be as above. Then there is an isomorphism

$$\mathrm{H}(Y/X) \cong \mathcal{C}\ell_{\mathcal{A}}(\mathcal{E}, q). \tag{6.90}$$

*Proof.* Combining lemma 6.38 and lemma 6.37 we already know that there is an epimorphism

$$\varphi\colon \mathrm{H}(Y/X) \twoheadrightarrow \mathcal{C}\ell_{\mathcal{A}}(\mathcal{E}, q). \tag{6.91}$$

It hence suffices to show that $\varphi$ is in fact an isomorphism, and this can be checked locally. Hence let $p$ be any point in $X$. We must show that

$$\varphi_p\colon \mathrm{H}(Y/X)_p = \mathrm{H}((f_*\mathcal{O}_Y)_p/\mathcal{O}_{X,p}) \twoheadrightarrow (\mathcal{C}\ell_{\mathcal{A}}(\mathcal{E}, q))_p = \mathcal{C}\ell_{\mathcal{A}_p}(\mathcal{E}_p, q_p) \tag{6.92}$$

is an isomorphism. This follows from proposition 6.22 by noticing that $\varphi_p$ coincides with (6.32). □

We are particularly interested in the case where $Y = X = \mathbb{P}^1$. Hence from now on fix a finite, degree 4 morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$. We continue this section by describing the sheaf of algebras $\mathcal{A}$ and the $\mathcal{A}$-module $\mathcal{E}$ in (6.90). We will use the following easy lemmas.

**Lemma 6.40.** We have for $n \in \mathbb{Z}$ and $i = 0, 1, 2, 3$ that

$$f_*(\mathcal{O}_{\mathbb{P}^1}(4n+i)) \cong \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus i+1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-1)^{3-i}. \tag{6.93}$$

*Proof.* Because $f$ is a finite morphism between regular schemes we have that it is necessarily flat. Hence $f_*(\mathcal{O}_{\mathbb{P}^1}(4n+i))$ must be locally free. By Grothendieck's splitting theorem it must split as a direct sum of line bundles. The exact splitting can be found using that

$$\begin{aligned} \mathrm{Hom}_{\mathbb{P}^1}(f_*(\mathcal{O}_{\mathbb{P}^1}(4n+i)), \mathcal{O}_{\mathbb{P}^1}(a)) &\cong \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(4n+i), f^*(\mathcal{O}_{\mathbb{P}^1}(a))) \\ &\cong \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(4n+i), \mathcal{O}_{\mathbb{P}^1}(4a)) \end{aligned} \tag{6.94}$$

which is $\max\{0, 4a-4n-i\}$-dimensional. □

**Lemma 6.41.** We have that

$$\mathcal{F}_{Y/X} := \mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)_1 \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}. \tag{6.95}$$

*Proof.* By lemma 6.40 we have that $f_*(\mathcal{O}_{\mathbb{P}^1}) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$. As the structure morphism $\mathcal{O}_{\mathbb{P}^1} \to f_*(\mathcal{O}_{\mathbb{P}^1})$ is nonzero we have that it is necessarily an isomorphism between $\mathcal{O}_{\mathbb{P}^1}$ and the summand $\mathcal{O}_{\mathbb{P}^1}$ of $f_*(\mathcal{O}_{\mathbb{P}^1})$. □

**Lemma 6.42.** We have that

$$\omega_{X/Y} \cong \mathcal{O}_{\mathbb{P}^1}(6). \tag{6.96}$$

*Proof.* By construction

$$\begin{aligned} f_*\omega_{X/Y} &\cong \mathcal{H}om(f_*\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \\ &\cong \mathcal{H}om(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}, \mathcal{O}_{\mathbb{P}^1}) \\ &\cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \\ &\cong f_*(\mathcal{O}_{\mathbb{P}^1}(6)) \end{aligned} \tag{6.97}$$

where we applied lemma 6.40 twice. The result now follows as $\mathcal{I}$ is an invertible sheaf and $f_*$ is faithful (because $f$ is affine). □

We can use these lemmas to obtain the following.

**Proposition 6.43.** Let $\mathcal{Q} := \mathrm{Sym}^2_{\mathbb{P}^1}\left(\mathcal{F}_{Y/X}/f_*(\omega^{-1}_{X/Y})\right)$ as above.
We have that

$$\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1). \tag{6.98}$$

*Proof.* We wish to prove that

1. the morphism $r\colon f_*(\omega^{-1}_{X/Y}) \to \mathrm{Sym}^2_{\mathbb{P}^1}(\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1)_1)$ is injective;
2. the quotient $\mathcal{Q}$ is a locally free sheaf.

Assuming that these hold there is a short exact sequence

(6.99) $0 \to \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-3) \to \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 6} \to \mathcal{Q} \to 0$

where we used lemma 6.41 to obtain $\mathrm{Sym}^2_{\mathbb{P}^1}(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 6}$ and lemma 6.42 together with lemma 6.40 to find $f_*(\omega^{-1}_{X/Y}) \cong f_*(\mathcal{O}_{\mathbb{P}^1}(-6)) \cong \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$. If $\mathcal{Q}$ is locally free it splits as a sum of two line bundles $\mathcal{O}_{\mathbb{P}^1}(i) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$ where $i \geq j \geq -2$. By [96, exercise II.5.16.d] we find

(6.100) $\mathcal{O}_{\mathbb{P}^1}(-12) \cong \mathcal{O}_{\mathbb{P}^1}(-9) \otimes \mathcal{O}_{\mathbb{P}^1}(i+j)$

with unique solution $i = -1, j = -2$. It hence remains to prove that $r$ is injective and that is cokernel is locally free.

The injectivity is checked locally. As in the proof of lemma 6.37, $r$ is locally given by

(6.101) $D \to {}_D D_C \otimes_C D_D : 1_D \mapsto \sum_{i=1}^{4} e_i \otimes f_i$

where $D = (f_*\mathcal{O}_Y)_p$, $C = \mathcal{O}_{X,p}$ and $e_1, e_2, e_3, e_4$ and $f_1, f_2, f_3, f_4$ form dual bases for $D$ as $C$-module. As $\mathrm{coker}(r) = (0, 1_D)\,((\Pi_C(D))_2)\,(0, 1_D)$ is a free $C$-module of rank 12 by [213, lemma 3.10], $r$ must necessarily be injective.

That the cokernel of $r$ is locally free can also be checked locally.

Hence let $Q = \mathrm{Sym}^2_C(D/C)/\,\mathrm{im}(r)$ for a relative Frobenius pair $D/C$. A computation similar to the one carried out in [213, §3] reduces to showing that $\dim_k(Q)$ does not depend on $D$ in the case where $C$ is an algebraically closed field $k$. This in turn follows from the equality

(6.102) $\dim_k(Q) = \dim_k(\mathrm{Sym}^2_k(D/k)) - \dim_k(D) = 6 - 4 = 2.$

□

We can now conclude the description in the following corollary.

**Corollary 6.44.** There exists an isomorphism

(6.103)
$$\mathrm{H}(\mathbb{P}^1/\mathbb{P}^1) \cong \mathcal{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))}\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \otimes \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)), q\right)$$

We can moreover twist everything in the construction by the appropriate $\mathcal{O}_{\mathbb{P}^1}(i)$'s to find and equivalence of categories

$$\begin{array}{c}\mathrm{qgr}\left(\mathcal{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))}\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \otimes \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)), q\right)\right) \\ \downarrow \cong \\ \mathrm{qgr}\left(\mathcal{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \otimes \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q\right)\right)\end{array}$$

**Remark 6.45.** The morphism

$$q\colon \operatorname{Sym}^2(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \tag{6.104}$$

in (6.32) is obtained from a surjective map $q\colon \operatorname{Sym}^2(E) \to V$ where $E := \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus 3})$ and $V := \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ because these sheaves are generated by their global sections, and the dimensions of the Hom's between the sheaves is the same as the dimension of the Hom's between their global sections.

## 6.4 Quaternion orders on $\mathbb{F}_1$ as Clifford algebras

We will now write the construction from chapter 5 for type $m = 2$ in $(\mathrm{B}_m)$ as a Clifford algebra. This allows us to compare it to the Clifford algebra obtained in section 6.3, which is done in section 6.5.

In section 6.4.1 we quickly recall the construction of chapter 5. In section 6.4.2 we describe how we can describe the Artin–Schelter regular algebras in the special case where the order of the automorphism is 2. In section 6.4.3 we can then use the description as a graded Clifford algebra before blowing up to give a description as a Clifford algebra with values in a line bundle after blowing up, which allows us to write the associated abelian category in the same way as was obtained in section 6.3. This can be summarised as follows.

$$\begin{array}{cl}
\operatorname{coh} p^*\mathcal{S} & \\
\downarrow & \text{lemma 6.57} \\
\operatorname{coh} p^*\Big(\mathcal{C}\ell_{\mathbb{P}^2}(\operatorname{Sym}^2(E)\otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0\Big) & \\
\downarrow & \text{section 6.4.3} \\
\operatorname{coh} \mathcal{C}\ell_{\mathbb{F}_1}(\operatorname{Sym}^2(E)\otimes_k \mathcal{O}_{\mathbb{F}_1}, q, \mathcal{O}_{\mathbb{F}_1}(1))_0 & \\
\downarrow & \S 6.2.2 \\
\operatorname{qgr} \mathcal{C}\ell_{\operatorname{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\Big(E\otimes_k \operatorname{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)), q\Big) &
\end{array}$$

### 6.4.1 Construction of the surface as a blowup

In this section we quickly recall the construction from chapter 5. The idea is to use the maximal order on $\mathbb{P}^2$ induced from a quadratic 3-dimensional Artin–Schelter regular algebra finite over its center, and blow up a point $p$ on $\mathbb{P}^2$ *outside* the ramification locus of the maximal order. The pullback of the maximal order is again a maximal order, now on $\mathrm{Bl}_x\,\mathbb{P}^2$. Using a generalisation of Orlov's blowup formula to the case of maximal orders on smooth projective varieties we can then construct derived categories with a full and strong exceptional collection whose Cartan matrix is of the form described in [212] for all $m \geq 2$. For the purpose of this chapter we are only interested in the case $m = 2$, where we wish to show that the construction from op. cit. is equivalent to the construction from [214].

The setup for this situation is as follows. Let $A$ be a 3-dimensional quadratic Artin–Schelter regular algebra, which is finite over its center. Then in [229] it was shown that the center of the sheaf of algebras induced on the Proj of the center of the graded algebra is always isomorphic to $\mathbb{P}^2$. Hence we always obtain a maximal order $\mathcal{S}$ on $\mathbb{P}^2$. Because $\operatorname{coh}\mathcal{S} \cong \operatorname{qgr} A$, the derived category $\mathbf{D}^{\mathrm{b}}(\operatorname{coh}\mathcal{S})$ has a full and strong exceptional collection, mimicking that of $\mathbb{P}^2$. This maximal order has a ramification divisor $C$ and an Azumaya locus $\mathbb{P}^2 \setminus C$. The divisor $C$ is a cubic curve, which is an isogeny of another cubic curve $E$ which is the point scheme used in the classification of these algebras.

The final step is to consider a point $x$ in the Azumaya locus. There is a fat point module of the graded algebra $A$ associated to this point, which defines a skyscraper sheaf supported at $x$ whose fibre is a simple module for $\mathcal{S}_p \otimes k(x)$. In chapter 5 it is shown how it is possible to generalise Orlov's blowup formula to this setting. Let us denote the morphisms involved in the blowup as follows.

$$\begin{array}{ccc} E = \mathbb{P}^1 & \longrightarrow & \operatorname{Bl}_x \mathbb{P}^2 \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ x & \longrightarrow & \mathbb{P}^2. \end{array} \tag{6.105}$$

In this situation there exist a semiorthogonal decomposition of $\mathbf{D}^{\mathrm{b}}(\operatorname{coh} p^*\mathcal{S})$ in terms of $\mathbf{D}^{\mathrm{b}}(\operatorname{coh}\mathcal{S})$ and an exceptional object which is the (noncommutative) structure sheaf of the exceptional fibre. This gives the existence of a full and strong exceptional collection in $\mathbf{D}^{\mathrm{b}}(\operatorname{coh} p^*\mathcal{S})$. We can summarise the result of chapter 5 in the case we are interested in as follows, where the unexplained terminology regarding elliptic triples will be introduced shortly.

**Theorem 6.46.** Let $A$ be a quadratic 3-dimensional Artin–Schelter regular algebra associated to an elliptic triple $(E, \sigma, \mathcal{L})$ for which the order of $\sigma$ is 2. With the sheaf of maximal orders $\mathcal{S}$ as above, and $p\colon \operatorname{Bl}_x \mathbb{P}^2 \to \mathbb{P}^2$ the blowup in a point $x \in \mathbb{P}^2 \setminus C$ in the Azumaya locus, there exists a full and strong exceptional collection

$$\mathbf{D}^{\mathrm{b}}(p^*\mathcal{S}) = \langle p^*\mathcal{S}_0, p^*\mathcal{S}_1, p^*\mathcal{S}_2, p^*\mathcal{F} \rangle \tag{6.106}$$

where $\mathcal{S}_i := \widetilde{A(i)}$ and $\mathcal{F}$ is the skyscraper associated to a fat point module, and whose Gram matrix is mutation equivalent to type $\mathrm{B}_2$.

The Gram matrix for the exceptional colletion in (6.106) is given by

$$\begin{pmatrix} 1 & 3 & 6 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.107}$$

It was shown in chapter 5 that $\mathrm{B}_2$ is mutation equivalent to this matrix.

### 6.4.2 Quaternionic noncommutative planes

We wish to understand all quadratic AS-regular algebras that can appear in theorem 6.46, and we want to write everything in terms of Clifford algebras. There exists a complete classification of 3-dimensional quadratic Artin–Schelter regular algebras, due to Artin–Tate–Van den Bergh [13] and Bondal–Polishchuk [50]. The Artin–Tate–Van den Bergh approach to the classification is in terms of triples of geometric data [13, definition 4.5].

**Definition 6.47.** An *elliptic triple* is a triple $(C, \sigma, \mathcal{L})$ where

1. $C$ is a divisor of degree 3 in $\mathbb{P}^2$;
2. $\sigma \in \mathrm{Aut}(C)$;
3. $\mathcal{L}$ is a very ample line bundle of degree 3 on $C$.

We say that it is a *regular triple* if moreover

$$\mathcal{L} \otimes (\sigma^* \circ \sigma^*(\mathcal{L})) \cong \sigma^*(\mathcal{L}) \otimes \sigma^*(\mathcal{L}). \tag{6.108}$$

We are only interested in algebras which are finite over their center, and by [12, theorem 7.1] we have that this is the case if and only if the order of $\sigma$ is finite.

**Remark 6.48.** The condition that $\sigma$ is of finite order gives a restriction on the curves which can appear in a regular triple: out of 9 possible point schemes, only 4 remain. These are

1. the elliptic curves,
2. the nodal cubic,
3. a conic and line in general position,
4. a triangle of lines.

This can be done explicitly by describing the automorphism groups of cubic curves, and then take into account the required compatibility between $\sigma$ and $\mathcal{L}$ to define a regular triple.

Another way of obtaining this restriction is by using the description of the ramification data for maximal orders on $\mathbb{P}^2$. In this case the Artin–Mumford sequence can be used to show that these 4 curves are the only possible ramification divisors.

A third way in which it is possible to see this restriction is in terms of the étale local structure of a maximal order, which shows that the singularities are necessarily at most nodal [229].

Finally, after proving that the specific algebras we are interested in (those for which $\sigma$ is not just finite, but of order 2) can be described using graded Clifford algebras, we will see how for those algebras there is yet another way of seeing these 4 curves, namely as discriminants of nets of conics without basepoints.

As the construction in [214] only concerns the case $m = 2$ of type $(\mathrm{B}_m)$ in the classification of [212], we can restrict the general construction from chapter 5 to the case where the order of $\sigma$ is 2. Then the fat points all have multiplicity 2. Let us introduce some terminology for these.

**Definition 6.49.** Let $A$ be a quadratic 3-dimensional Artin–Schelter regular algebra. We say that $A$ (resp. $\operatorname{qgr} A$) is *quaternionic* if the automorphism in the associated elliptic triple is of order 2.

Observe that Artin–Schelter regular algebras whose automorphism $\sigma$ has order 6 *also* have fat point modules of multiplicity 2 [12, theorem 7.3]. But using the proof of proposition 6.50 we will show in lemma 6.53 that these algebras are not new when considered up to Zhang twist.

We will now show that all quaternionic noncommutative planes can be written as a graded Clifford algebra, up to Zhang twist. This result does not appear as such in the literature, but it follows almost immediately by combining the classification of elliptic triples and the description of "Clifford quantum $\mathbb{P}^2$'s" in [204, corollary 4.8]. These are precisely the 3-dimensional graded Clifford algebras which are Artin–Schelter regular, i.e. which are associated to a basepoint-free net of conics. The following proposition can be seen as a converse to this result. Here the condition that the characteristic of $k$ is not 3 is important.

**Proposition 6.50.** Let $A = A(C, \sigma, \mathcal{L})$ be a quaternionic Artin–Schelter regular algebra. Then $A$ is the Zhang twist of a graded Clifford algebra.

*Proof.* Recall from [192, theorem 1.2] that $A$ is a Zhang twist of $A'$ if and only if their associated $\mathbb{Z}$-algebras $\check{A}$ and $\check{A}'$ are isomorphic. Now $\check{A} = A(C, \mathcal{L}, \sigma^*\mathcal{L})$ is a quadratic Artin–Schelter regular $\mathbb{Z}$-algebra in the sense of [226]. It is shown in [226, theorem 3.5] (based upon the results in [12]) that there is a morphism of algebraic groups $\eta\colon \operatorname{Pic}_0 C \to \operatorname{Aut} C$ where $\operatorname{Pic}_0 C$ consists of all line bundles having degree 0 on each component of $C$. Moreover we have by [226, theorem 4.2.2] that every quadratic Artin–Schelter regular $\mathbb{Z}$-algebra $A(C, \mathcal{L}_0, \mathcal{L}_1)$ is of the form $\check{A}'$ where $A' = A(C, \mathcal{L}_0, \sigma')$ and $\sigma' \in \operatorname{im}(\eta)$. In particular $A$ is a Zhang twist of a quadratic Artin–Schelter regular algebra[1] $A'$ whose associated $C$-automorphism lies in $\operatorname{im}(\eta)$. We claim that $A'$ is a graded Clifford algebra.

A closer investigation of [12, §5] shows that for every $\mathcal{G} \in \operatorname{Pic}_0 C$ we have that $\eta(\mathcal{G})$ is uniquely defined by the following property:

$$\mathcal{O}_C(\eta(\mathcal{G})(p)) \cong \mathcal{G}(p) \tag{6.109}$$

if $p$ is a nonsingular point of $C$ and

$$\eta(\mathcal{G})(p) = p \tag{6.110}$$

if $p$ is a singular point of $C$. In particular if all points of $C$ are singular (which only happens when $C$ is a triple line), the only finite order automorphism in $\operatorname{im}(\eta)$ is $\operatorname{id}_C$. However in this case $A'$ does not allow fat point modules. This is a contradiction, because the existence (and multiplicity) of fat point modules is an invariant of the category $\operatorname{QGr} A$ and by construction there are equivalences of categories

$$\operatorname{QGr} A \cong \operatorname{QGr}(\check{A}) \cong \operatorname{QGr}(\check{A}') \cong \operatorname{QGr} A'. \tag{6.111}$$

[1] It is customary to refer to a quadratic Artin–Schelter regular algebra $A = A(C, \mathcal{L}, \sigma)$ as a *translation algebra* whenever $\sigma \in \operatorname{im}(\eta)$.

Hence we can assume that $C$ contains a nonsingular point $p$.

Now assume that $\eta(\mathcal{G})^n = \mathrm{id}_C$, then by (6.109) we find

$$\mathcal{O}_C(p) \cong \mathcal{G}^{\otimes n}(p) \tag{6.112}$$

and hence

$$\mathcal{O}_C \cong \mathcal{G}^{\otimes n}. \tag{6.113}$$

This implies that $\eta$ preserves the order of an element in $\mathrm{Pic}_0\, C$. By (6.108) we know that $\mathcal{L}^{-1} \otimes \sigma^*\mathcal{L}$ has order 2 in $\mathrm{Pic}_0\, C$, and hence $\sigma' = \eta(\mathcal{L}^{-1} \otimes \sigma^*\mathcal{L})$ has order 2 as well. By the description of $\mathrm{Pic}_0\, C$ as e.g. in [50, table 1], we see that $\mathrm{Pic}_0\, C$ only admits order 2 elements when $C$ is an elliptic curve, a nodal cubic, the union of a conic and a line in general position or a triangle of lines, as explained in remark 6.48.

In the case of an elliptic curve, $A'$ is a quaternionic Sklyanin algebra and it is shown in example 6.54 that such an $A'$ is a graded Clifford algebra. For the other 3 options for the point scheme $C$ there is a unique order 2 element in $\mathrm{Pic}_0\, C$ and the associated algebra $A'$ is the quotient of $k\langle x, y, z\rangle$ by the relations

$$\begin{cases} xy + yx = 0 \\ yz + zy = c_1 x^2 \\ zx + xz = c_2 y^2 \end{cases}, \tag{6.114}$$

where $(c_1, c_2) = (1, 1), (0, 1), (0, 0)$ for the 3 cases respectively. These algebras are graded Clifford algebra by example 6.55. □

The classification of [204, corollary 4.8] is a converse to proposition 6.50: it turns out that quaternionic translation algebras are precisely the graded Clifford algebras, and that they describe all quaternionic noncommutative planes.

**Remark 6.51.** The algebra $A'$ constructed in the above proposition is a graded $k$-algebra with symmetric relations. As such lemma 6.13 and proposition 6.14 show the existence of an epimorphism $\varphi\colon A' \to \mathcal{C}\ell_k(E, q)$. Both $A'$ and the Clifford algebra $\mathcal{C}\ell_k(E, q)$ are graded $k$-algebras with Hilbert series $1/(1-t)^3$, hence $\varphi$ is an isomorphism. This gives an alternative proof for showing that $A'$ is a Clifford algebra, but it does not highlight the fact that $\mathcal{C}\ell_k(E, q)$ is a graded Clifford algebra.

**Example 6.52.** Not every quaternionic Artin–Schelter regular algebra is a graded Clifford algebra on the nose. By the proof of proposition 6.50 we can also say that not every Artin–Schelter algebra is a translation algebra. As an example we can consider

$$k\langle x, y, z\rangle/(xz - zx, yz - zy, xy + yx). \tag{6.115}$$

The point scheme $C$ is the triangle of lines defined by $xyz$, and the automorphism is given by rescaling by $-1$ in one component, and exchanging the two others. In a translation algebra the automorphism needs to preserve the components and rescale them in the same way. The prescribed Zhang twist in this case is induced from the automorphism of the degree 1 part which has $z \mapsto -z$, and we obtain the Clifford algebra associated to the net $N^{\mathrm{E}}$ in (6.121).

| type | discriminant | number of double lines |
|---|---|---|
| A | elliptic curve | 0 |
| B | nodal cubic | 1 |
| D | conic and line in general position | 2 |
| E | triangle of lines | 3 |

Table 6.1: Base-point free nets of conics

We now show why the algebras with an automorphism of order 6 (and hence fat points of multiplicity 2) can be ignored for the purposes of this chapter.

**Lemma 6.53.** Let $A = A(C, \sigma, \mathcal{L})$ be a an Artin–Schelter regular algebra such that the order of $\sigma$ is 6. Then $A$ is the Zhang twist of a quaternionic Artin–Schelter regular algebra.

*Proof.* We have that $\sigma = \sigma^3 \circ \sigma^{-2}$, where $\sigma^{-2}$ is of order 3. Moreover $\sigma^{-2}$ commutes with $\sigma$, and it extends to an automorphism of the ambient $\mathbb{P}^2$ in which $C$ is embedded using $\mathcal{L}$. Hence up to a Zhang twist as in the proof of proposition 6.50 we can also work with the elliptic triple $(C, \sigma^3, \mathcal{L})$, which is quaternionic. □

**Nets of conics** Because we are only interested in the associated category qgr $A$, we can restrict ourselves to considering graded Clifford algebras if we are only interested in those algebras which have fat point modules of multiplicity 2. We will now recall the algebras from [204, corollary 4.8].

In the case of $n = 3$ the linear system of quadrics is known as a net of conics, and there exists a complete classification of these [234]. Because we are only interested in basepoint-free nets of conics we can summarise the part of this classification with the notation of op. cit. as follows. Recall that the *discriminant* is the locus of the singular conics in the linear system. Such a singular conic is either two lines intersecting in a point or a double line. The double lines in the net of conics correspond precisely to the singularities of the discriminant for the basepoint-free nets. In table 6.1 the situation is summarised for the types which are relevant to us, for a complete classification one is referred to [234, table 2].

**Example 6.54** (Quaternionic Sklyanin algebras)**.** 3-dimensional Sklyanin algebras associated at points of order 2 are an interesting class of graded Clifford algebras. It can be shown that such an algebra has a presentation of the form

$$(6.116)\quad \begin{cases} xy + yx = cz^2 \\ yz + zy = cx^2 \\ zx + xz = cy^2 \end{cases},$$

where $c \neq 0$, $c^3 \neq 8$ and $c^3 \neq -1$ [147, example 3.5].

If we identify $x = x_1$, $y = x_2$ and $z = x_3$ these algebras are Clifford algebras associated to the net of conics given by

$$(6.117)\quad M_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & c \\ 0 & c & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & c \\ 0 & 2 & 0 \\ c & 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

or equivalently the symmetric matrix

$$(6.118)\quad M = \begin{pmatrix} 2x^2 & cz^2 & cy^2 \\ cz^2 & 2y^2 & cx^2 \\ cy^2 & cx^2 & 2z^2 \end{pmatrix}$$

In this case the central elements $y_i$ are $x^2$, $y^2$ and $z^2$ which are in turn the norms of the elements $x, y, z$ in degree 1. Computing the determinant of this matrix to describe the point modules we get the discriminant curve

$$(6.119)\quad (2c^3 + 8)x^2y^2z^2 - 2c^2(x^6 + y^6 + z^6).$$

**Example 6.55** (Special quaternionic algebras)**.** We have seen in proposition 6.50 that there are three other isomorphism classes of graded Clifford algebras which are relevant to us, besides the generic case of a quaternionic Sklyanin algebra. These are described in , and their corresponding nets of conics are given by

$$(6.120)\quad \begin{aligned} M^{\mathrm{B}} &= \begin{pmatrix} 2x^2 & 0 & y^2 \\ 0 & 2y & x^2 \\ y^2 & x^2 & 2z^2 \end{pmatrix}, \\ M^{\mathrm{D}} &= \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 2y^2 & x^2 \\ 0 & x^2 & 2z^2 \end{pmatrix}, \\ M^{\mathrm{E}} &= \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 2y^2 & 0 \\ 0 & 0 & 2z^2 \end{pmatrix} \end{aligned}$$

where B, D and E refer to the classification in table 6.1. The particular choice of basis is made to be compatible with [204, corollary 4.8]. The nets of conics are in turn given by

$$(6.121)\quad \begin{aligned} N^{\mathrm{B}} &= \langle x^2 + yz, y^2 + xz, z^2 \rangle, \\ N^{\mathrm{D}} &= \langle x^2 + yz, y^2, z^2 \rangle, \\ N^{\mathrm{E}} &= \langle x^2, y^2, z^2 \rangle. \end{aligned}$$

**Remark 6.56.** It is also possible to study the derived category of the Clifford algebra on $\mathbb{P}^2$ using the derived category of the associated standard conic bundle, as in [37, §5.3].

### 6.4.3 Blowing up Clifford algebras

Using proposition 6.50 we get that the Artin–Schelter regular algebras we need to consider in the construction can be written as graded Clifford algebras, as our construction is insensitive to Zhang twist. We now show using a chain of easy lemmas how it is possible to write the maximal order $p^*\mathcal{S}$ on $\mathrm{Bl}_x\,\mathbb{P}^2$ as a Clifford algebra in the sense of section 6.2.4.

**Lemma 6.57.** There exists an isomorphism

$$\mathcal{S} \cong \mathcal{C}\ell_{\mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0. \tag{6.122}$$

In particular we also have that

$$p^*\mathcal{S} \cong p^*\,\mathcal{C}\ell_{\mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0. \tag{6.123}$$

By functoriality of the Clifford algebra construction we then get the following description.

**Lemma 6.58.** There exists an isomorphism

$$p^*\mathcal{S} \cong \mathcal{C}\ell_{\mathrm{Bl}_x\,\mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathrm{Bl}_x\,\mathbb{P}^2}, q, \mathcal{O}_{\mathrm{Bl}_x\,\mathbb{P}^2}(1))_0. \tag{6.124}$$

*Proof.* This follows from the isomorphism

$$p^*\,\mathcal{C}\ell_{\mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0 \cong \mathcal{C}\ell_{\mathrm{Bl}_x\,\mathbb{P}^2}(E \otimes_k \mathcal{O}_{\mathrm{Bl}_x\,\mathbb{P}^2}, q, \mathcal{O}_{\mathrm{Bl}_x\,\mathbb{P}^2}(H))_0. \tag{6.125}$$

□

Then by proposition 6.10 we get the following description of the Clifford algebra, which puts the category of coherent sheaves over $p^*\mathcal{S}$ on the same footing as that of the noncommutative $\mathbb{P}^1$-bundle as in corollary 6.44. This will allow us to compare the two constructions in section 6.5.

**Corollary 6.59.** There exists exists an equivalence of categories

$$\operatorname{coh} p^*\mathcal{S} \cong \operatorname{qgr}_{\mathbb{P}^1} \mathcal{C}\ell_{\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(E \otimes_k \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q). \tag{6.126}$$

*Proof.* In (6.1) we have seen the classical isomorphism

$$\mathrm{Bl}_x\,\mathbb{P}^2 \cong \mathbf{Proj}\,\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)). \tag{6.127}$$

We apply proposition 6.10 to the morphism $\pi\colon\ \mathrm{Bl}_x\,\mathbb{P}^2 \cong \mathbb{F}_1 \to \mathbb{P}^1$, where

$$\begin{aligned} \mathcal{E} &:= E \otimes_k \mathcal{O}_{\mathbb{P}^2}, \\ \mathcal{L} &:= p^*(\mathcal{O}_{\mathbb{P}^2}(1)), \end{aligned} \tag{6.128}$$

and therefore

$$\mathcal{A} \cong \mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \tag{6.129}$$

using the identification

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \tag{6.130}$$

In this situation we have that $p^*(\mathcal{O}_{\mathbb{P}^2}(1)$ corresponds to the shift by 1 in the sheaf of graded algebras $\mathrm{Sym}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ on $\mathbb{P}^1$. The result follows. □

We state and prove the following easy observation, and come back to this in remark 6.80.

**Proposition 6.60.** The global dimension of $p^*\mathcal{S}$ is 2.

*Proof.* As we have blown up a point in the Azumaya locus, we have that

$$\mathcal{S}|_{\mathbb{P}^2\setminus\{x\}} \cong (p^*\mathcal{S})|_{\mathrm{Bl}_x\,\mathbb{P}^2\setminus E}. \tag{6.131}$$

Because we can check global dimension in the stalks, we conclude that the global dimension is $\leq 2$ in the points of $\mathrm{Bl}_x\,\mathbb{P}^2 \setminus E$. For points on $E$ we use that the global dimension of an Azumaya algebra is equal to the global dimension of the ring. □

## 6.5 Comparing the two constructions

By sections 6.3 and 6.4 the categories $\mathrm{qgr}(\mathbb{S}(_f(\mathcal{O}_{\mathbb{P}^1}(1))_{\mathrm{id}}))$ and $\mathrm{coh}(p^*\mathcal{S})$ are equivalent to a category of the form

$$\mathrm{qgr}\left(\mathcal{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(E\otimes_k \mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)), q\right)\right) \tag{6.132}$$

where $q$ is obtained from a surjective map $\mathrm{Sym}^2(E)\to V$.

Using the isomorphism $\mathrm{Bl}_x\,\mathbb{P}^2 \cong \mathbb{F}_1$ and the blowup $p\colon\ \mathrm{Bl}_x\,\mathbb{P}^2 \to \mathbb{P}^2$ we also write the associated category using

$$p^*\,\mathcal{C}\ell_{\mathbb{P}^2}(E\otimes_k \mathcal{O}_{\mathbb{P}^2}, q, \mathcal{O}_{\mathbb{P}^2}(1))_0. \tag{6.133}$$

by inverting the construction in section 6.4.3.

Using this description the comparison of the abelian categories boils down to understanding how the geometric data defining both categories can be compared. We do this by explaining how we can go back from the data as in (6.132) to the morphism $f\colon\mathbb{P}^1\to\mathbb{P}^1$, and how this gives an inverse to the results of section 6.3.

In section 6.5.1 we discuss the geometry of the linear systems we are interested in: we will need to understand the classification of pencils of binary quartics [235] and pencils of conics [234], in order to set up a correspondence between pencils of binary quartics on one hand, and nets of conics together with the choice of a smooth conic in the net on the other. To set up the correspondence we need to discuss a branched covering of the dual of the net of conics in some detail, this is done in section 6.5.2. This allows us to obtain the comparison between the geometric data in section 6.5.3. Finally we discuss the comparison of the categories in section 6.5.4.

In section 6.5.5 we discuss how the constructions which are compared in this chapter relate to other notions of noncommutative blowing up and noncommutative $\mathbb{P}^1$-bundles. In particular, there is a conjectural relationship between the constructions in this chapter and these other notions, obtained by suitably degenerating the construction. And we discuss how the Hochschild cohomology of these categories is expected to behave, based on the automorphisms of the geometric data.

### 6.5.1 Nets of conics and pencils of binary quartics

In this section we will recall some results on linear systems which we will need to relate the two constructions. The two types which we want to compare are nets of conics (which are used to define graded Clifford algebras) and pencils of binary quartics (which are used to define noncommutative $\mathbb{P}^1$-bundles). In section 6.5.2 we will construct a natural $4:1$-cover of $\mathbb{P}^2$ by $\mathbb{P}^2$ associated to a net of conics, which will allow us to compare pencils of binary quartics to nets of conics together with the choice of a smooth conic in the net, as in section 6.5.3.

**Pencils of conics**

As a tool in setting up the correspondence between nets of conics together with a point and pencils of binary quartics, we need to say a few words on pencils of conics. These are 2-dimensional subspaces of $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Their classification is completely classical. It is also the first (non-trivial) case of Segre's classification of pencils of quadrics using Segre symbols. It is given in table 6.2, and the main observation is that there is a correspondence between the base locus and the types of singular conics which appear in the pencil.

| Segre symbol | base locus | # singular fibres | # double lines |
|---|---|---|---|
| $[1,1,1]$ | $(1,1,1,1)$ | 3 | 0 |
| $[2,1]$ | $(2,1,1)$ | 2 | 0 |
| $[3]$ | $(3,1)$ | 2 | 0 |
| $[(1,1),1]$ | $(2,2)$ | 2 | 1 |
| $[(2,1)]$ | $(4)$ | 1 | 1 |

Table 6.2: Pencils of conics

The number of singular fibres and the number of double lines will allow us to describe the ramification of $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ as in table 6.4.

**Nets of conics**

In section 6.2.3 $n-1$-dimensional linear systems of $n-1$-dimensional quadrics were used to define graded Clifford algebras. We are interested in the special case of nets of conics, i.e. of 3-dimensional subspaces of $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

We will consider a net of conics as a surjective morphism $\phi\colon \mathrm{Sym}^2 E \to V$, where $E$ and $V$ are 3-dimensional vector spaces. Then the net itself is $\mathbb{P}(V^\vee)$, whilst the conics live in $\mathbb{P}(E)$. Associated to this there is the *discriminant* of the net, which is the locus in $\mathbb{P}(V^\vee)$ of the singular conics in the net. It is also known as the *Hessian curve.*

The *Jacobian* is the union of the singular points of each conic in $\mathbb{P}(E)$. Both the discriminant and Jacobian are cubics, but they live in different spaces and they are not necessarily of the same type.

In section 6.5.2 we will construct the morphism $\Theta\colon \mathbb{P}(E) \to \mathbb{P}(V)$, sending a point to the subpencil which has this point in its baselocus. This is a $4:1$-cover. The *branch*

| | curve | ambient space | degree |
|---|---|---|---|
| $\Delta$ | discriminant | $\mathbb{P}(V^\vee)$ | 3 |
| $J$ | Jacobian | $\mathbb{P}(E)$ | 3 |
| $R$ | ramification curve | $\mathbb{P}(E)$ | 3 |
| $B$ | branch curve | $\mathbb{P}(V)$ | $\leq 6$ |
| $C$ | point scheme | $\mathbb{P}(E^\vee)$ | 3 |
| $\widetilde{\Delta}$ | double cover of discriminant | $\mathbb{P}(E^\vee)$ | 3 |
| | conic in the net | $\mathbb{P}(E)$ | 2 |

Table 6.3: Curves associated to a net of conics

*curve* inside $\mathbb{P}(V)$ is the locus of points whose fibres do not have 4 distinct points, Generically the branch curve is the dual of the discriminant, hence it is of degree 6, but if the discriminant is zero the degree is lower.

The *ramification curve* inside $\mathbb{P}(E)$ are those points which are multiple points of a fiber. Using table 6.2 this also means that the associated subpencil of conics has at most two singular points, hence the ramification curve is equal to the Jacobian. In table 6.3 we give an overview of all the curves associated to a net of conics. Observe that the point scheme for an Artin–Schelter regular Clifford algebra is a double cover of the ramification curve, and hence coincides with the double cover of the discriminant constructed by considering the two lines in $\mathbb{P}(E)$ associated to the singular conic parametrised by a point on the discriminant. Hence the curves living in the same ambient $\mathbb{P}^2$ coincide.

The classification of nets of conics is obtained in [234]. We are only interested in basepoint-free nets of conics, otherwise the associated graded Clifford algebra is not Artin–Schelter regular. The classification in this case was summarised in table 6.1, with the labeling from [234]. The double lines in the net of conics necessarily correspond to the singularities in the discriminant for a basepoint-free net of conics.

We now give an example of a net of conics. It will be important in understanding a special element in the classification of noncommutative $\mathbb{P}^1$-bundles, see also example 6.62 and section 6.5.5.

**Example 6.61.** Consider the net of conics given by $x^2, y^2, z^2 + 2xy$. It is (up to base change) the only net of conics of type D. Its discriminant is a conic and line in general position, and the two singularities of the discriminant correspond to the double lines $x^2$ and $y^2$. The Jacobian turns out to be a triangle of lines, but we will not use this.

### Pencils of binary quartics

The only input needed for the construction of a noncommutative $\mathbb{P}^1$-bundle in the sense of [214] is a finite morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4. This data is equivalent to the data of a basepoint-free pencil of binary quartics, i.e. a 2-dimensional subspace of $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$. Indeed, if $\langle f_1, f_2\rangle$ is such a pencil, then $[x : y] \mapsto [f_1(x, y) : f_2(x, y)]$ is a well-defined finite morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$. These pencils have been studied in detail in [235], and we will quickly recall the relevant results here.

As for nets of conics, we can associate the *Jacobian* and the *discriminant* to a pencil of binary quartics. In this case the Jacobian represents the branch points of the associated morphism, and the discriminant are the images of these. We can moreover describe the branching behaviour of the associated morphism using its *symbol*: algebraically it is obtained by considering the roots of the discriminant and listing the multiplicities of the remaining factors. Geometrically it describes the cycle type of the monodromy around each branch point. Its notation is not to be confused with that of Segre symbols as in table 6.2.

The classification is given in table 6.4, where we have already included the comparison to the net of conics, from theorem 6.77.

**Example 6.62.** The "most degenerate" morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is given by $[x^4 : y^4]$. It is ramified in $[0 : 1]$ and $[1 : 0]$ only, where 4 branches come together, hence its symbol is $[(4)(4)]$. It corresponds to type A in the classification of table 6.4.

**Remark 6.63.** In [235, §2] it is remarked that there is no pencil of binary quartics whose symbol is $[(2,2)(2,2)(3,1)]$, which is excluded by considering the monodromy of the would-be corresponding morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4. Using the correspondence we have set up between nets of conics and pencils of binary quartics we can also argue that this case cannot occur using the geometry of cubics: there is no cubic curve with 2 singularities and 1 inflection point.

### 6.5.2 Branched coverings and nets of conics

Let $V$ and $E$ be 3-dimensional vector spaces, and let

$$\phi\colon \operatorname{Sym}^2 E \to V \tag{6.134}$$

be a surjection. This corresponds to a net of conics inside $\mathbb{P}(E)$ parametrised by $\mathbb{P}(V^\vee)$. We will assume that the net is basepoint-free, such that it defines a graded Clifford algebra which is Artin–Schelter regular, as in section 6.2.3.

Define

$$\begin{aligned} S &:= \operatorname{Sym}(E^\vee), \\ R &:= \operatorname{Sym}(V^\vee) \end{aligned} \tag{6.135}$$

and consider $S$ as a graded $R$-algebra via $\phi^\vee\colon V^\vee \to \operatorname{Sym}^2 E^\vee$. Moreover we define

$$A := S/SR_{\geq 1}. \tag{6.136}$$

For the notion of a (graded) Frobenius algebra we refer to [196, §3].

**Lemma 6.64.** The algebra $A$ is a graded Frobenius algebra with Hilbert series $1, 3, 3, 1$.

*Proof.* Because the net of conics is basepoint-free, any choice of basis for the net gives a regular sequence in $S$. The Hilbert series of the associated complete intersection is $(1+t)^3$, e.g. using [203]. Likewise by basepoint-freeness we have that $A$ is Frobenius because it defines a zero-dimensional complete intersection (hence it is Gorenstein). □

If $x \in S$, then we will denote $\overline{x}$ for the induced element in $A$. If $\overline{x} \in A_0$ or $A_1$, then we will also use the notation $\overline{x}$ to indicate the corresponding element in $S_0$ or $S_1$ depending on the context.

There exists a morphism

$$\text{(6.137)}\quad \operatorname{tr}\colon S \to S/(RS_0 \oplus RS_1 \oplus RS_2)$$

which defines a non-degenerate pairing

$$\text{(6.138)}\quad S \otimes_R S \to R(-3) \otimes A_3 : p \otimes q \mapsto \operatorname{tr}(pq),$$

where we have used the isomorphism

$$\text{(6.139)}\quad S/(RS_0 \oplus RS_1 \oplus RS_2) \cong R(-3) \otimes A_3.$$

We will use dual $R$-bases $\{e_0, \dots, e_7\}$ and $\{f_0, \dots, f_7\}$ for $S$, where the degrees of the elements $e_i, f_j$ are

$$\text{(6.140)}\quad |e_i| = |f_{7-i}| = \begin{cases} 0 & i = 0 \\ 1 & i = 1, 2, 3 \\ 2 & i = 4, 5, 6 \\ 3 & i = 7. \end{cases}$$

Moreover we can and will assume that $e_0 = f_7 = 1$ (and therefore $\overline{e}_7 = \overline{f}_0$), $f_4 = e_1$, $f_5 = e_2$, $f_6 = e_3$, and $\overline{f}_1 = \overline{e}_4$, $\overline{f}_2 = \overline{e}_5$, $\overline{f}_3 = \overline{e}_6$. If we introduce the structure constants

$$\text{(6.141)}\quad \overline{e}_i\overline{e}_j = \sum_{l=1}^{3} c_{i,j,l}\overline{e}_{l+3}$$

for the multiplication in $A$, where $i, j = 1, 2, 3$, then we have the following lemma.

**Lemma 6.65.** The structure constants $c_{i,j,l}$ are invariant under permutation of $i, j$ and $l$.

*Proof.* We evaluate $\operatorname{tr}(\overline{e_a e_b e_d})$ for $a, b, d = 1, 2, 3$:

$$\text{(6.142)}\quad \begin{aligned} \operatorname{tr}(\overline{e_a e_b e_d}) &= \sum_{i=1}^{3} c_{a,b,i} \operatorname{tr}(\overline{e_{i+3} e_d}) \\ &= \sum_{i=1}^{3} c_{a,b,i} \operatorname{tr}(\overline{f_i e_d}) \\ &= \sum_{d} c_{a,b,i}\delta_{i,d} \\ &= c_{a,b,d}. \end{aligned}$$

□

By construction we have the identifications

$$\begin{aligned} A_1 &= E^\vee, \\ A_2 &= \operatorname{Sym}^2 E^\vee / V^\vee, \\ A_3 &= \operatorname{Sym}^3 E^\vee / E^\vee V^\vee, \end{aligned} \tag{6.143}$$

and the multiplication

$$\mathrm{mult}\colon A_1 \otimes_k A_1 \to A_2 \tag{6.144}$$

is identified with the quotient map

$$\operatorname{Sym}^2 E^\vee \to \operatorname{Sym}^2 E^\vee / V^\vee, \tag{6.145}$$

where the inclusion of $V^\vee$ into $\operatorname{Sym}^2 E^\vee$ is given by $\phi^\vee$. Because $A$ is graded Frobenius by lemma 6.64 the multiplication in the algebra provides us with a duality in the form of a perfect pairing

$$A_1 \otimes A_2 \to A_3 \tag{6.146}$$

which yields the identifications

$$\begin{aligned} A_1^\vee &= A_2 \otimes A_3^{-1}, \\ A_2^\vee &= A_1 \otimes A_3^{-1}. \end{aligned} \tag{6.147}$$

If we define

$$\alpha\colon A_1 \otimes A_3^{-1} \to \operatorname{Sym}^2 A_2 \otimes A_3^{-2} : e_i \otimes \overline{e}_7^{-1} \mapsto \sum_{j=1}^{3} \overline{e}_i \overline{e}_j \cdot \overline{f}_j \otimes \overline{e}_7^{-2} \tag{6.148}$$

and

$$\beta\colon \operatorname{Sym}^2 A_2 \otimes A_3^{-2} \to V : \overline{e}_{i+3} \cdot \overline{e}_{j+3} \otimes \overline{e}_7^{-2} \mapsto \phi(e_i^\vee \cdot e_j^\vee) \tag{6.149}$$

then these morphisms are compatible with the algebra structure on $A$ in the following way using lemma 6.65.

**Lemma 6.66.** The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 \otimes A_3^{\otimes -1} & \xrightarrow{\alpha} & \operatorname{Sym}^2 A_2 \otimes A_3^{\otimes -2} & \xrightarrow{\beta} & V & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & A_2^\vee & \xrightarrow{\mathrm{mult}^\vee} & \operatorname{Sym}^2 A_1^\vee & \xrightarrow{\phi} & V & \longrightarrow & 0 \end{array} \tag{6.150}$$

where the vertical arrows are the identifications obtained from the graded Frobenius structure, is commutative.

*Proof.* We first discuss the square involving $\alpha$. Note that

$$
\begin{aligned}
\alpha(e_i \otimes \overline{e}_7^{-1}) &= \sum_{j=1}^{3} \overline{e_i e_j} \cdot \overline{e}_{j+3} \otimes \overline{e}_7^{-2} \\
&= \sum_{l,j=1}^{3} c_{i,j,l} \overline{e}_{l+3} \cdot \overline{e}_{j+3} \otimes \overline{e}_7^{-2}
\end{aligned} \tag{6.151}
$$

whereas the lower composition in this square is given by

$$
\begin{aligned}
e_i \otimes \overline{e}_7^{-1} &\to \overline{e}_{i+3}^{\vee} \\
&\to \sum_{p,q=1}^{3} c_{p,q,i} \overline{e}_p^{\vee} \cdot \overline{e}_q^{\vee} \\
&\to \sum_{p,q=1}^{3} c_{p,q,i} \overline{e}_{p+3} \cdot \overline{e}_{q+3} \otimes \overline{e}_7^{-2}.
\end{aligned} \tag{6.152}
$$

It now suffices to use the symmetry of the structure constants as in lemma 6.65.

The fact that the square involving $\beta$ commutes follows immediately from the definition of $\beta$ and the fact that the middle vertical identification is given by

$$
\overline{e_{i+3}} \cdot \overline{e_{j+3}} \otimes \overline{e_7}^{-2} \mapsto \overline{e_i}^{\vee} \overline{e_j}^{\vee}. \tag{6.153}
$$

□

The trace pairing from (6.138) gives an identification

$$
S^{\vee} := \mathrm{Hom}_R(S, R) \cong S(3) \otimes A_3^{\otimes -1} : e_i^{\vee} \mapsto f_i \otimes e_7^{-1} \tag{6.154}
$$

of graded $S$-modules. Using this identification, together with the $R$-dualised multiplication $S \otimes_R S \to S$ we get a copairing

$$
\delta \colon S(-3) \otimes A_3 \to S \otimes_R S \tag{6.155}
$$

such that

$$
\delta(u \otimes \overline{e}_7) = \sum_{i=0}^{7} u e_i \otimes f_i. \tag{6.156}
$$

Similar to (6.35) and [214, lemma 3.24] we see that $\sum_{i=0}^{7} e_i \otimes f_i = \sum_{i=0}^{7} f_i \otimes e_i$ is a central element in $S \otimes_R S$.

We will now do the preliminary algebraic constructions for the results of section 6.5.3, by suitably describing everything in terms of graded modules. Consider the decomposition $S = S_{\mathrm{even}} \oplus S_{\mathrm{odd}}$ of graded $R$-modules into the part of even and odd grading. Then there exists a canonical identification

$$
S_{\mathrm{even}}/R = (R \otimes A_2)(-2). \tag{6.157}
$$

We moreover define a graded $R$-module $\Omega$ as the cokernel in the leftmost morphism of the Koszul sequence, i.e.

$$0 \to R(-6) \xrightarrow{\kappa} V \otimes R(-4) \to \Omega \to 0. \tag{6.158}$$

We have that $\Omega$ induces $\mathrm{T}_{\mathbb{P}(V)}(-3)$ after sheafification, which in turn is isomorphic to $\Omega^1_{\mathbb{P}(V)}$.

Finally consider the morphism

$$\Phi\colon S_{\mathrm{odd}}(-3) \otimes A_3 \to \mathrm{Sym}^2 A_2 \otimes_k R(-4) \tag{6.159}$$

which is the composition of the inclusion into $S(-3) \otimes A_3$, the morphism $\delta$ from (6.155), the projection $S \twoheadrightarrow S_{\mathrm{even}}$ and the quotient $S_{\mathrm{even}} \twoheadrightarrow S_{\mathrm{even}}/R \cong A_2 \otimes_k R(-2)$ in both factors of the tensor product, and the quotient $A_2 \otimes_k A_2 \to \mathrm{Sym}^2 A_2$.

**Proposition 6.67.** There exists an isomorphism

$$\operatorname{coker}\Phi \cong \Omega \otimes A_3^{\otimes 2}. \tag{6.160}$$

More precisely, the induced morphism

$$\mathrm{Sym}^2 A_2 \otimes_k R(-4) \to \operatorname{coker}\Phi \cong \Omega \otimes A_3^{\otimes 2} \tag{6.161}$$

is the composition of $\beta \otimes \mathrm{id}_{R(-4)}$ and the quotient map in (6.158).

*Proof.* The module $S_{\mathrm{odd}}$ contains canonically a graded $R$-submodule $A_1 \otimes R(-1)$. We first describe how $\Phi$ acts on $A_1 \otimes A_3 \otimes R(-4)$. Using the definition this map is given by

$$\overline{e}_i \otimes \overline{e}_7 \overset{\delta}{\mapsto} \sum_{j=0}^{7} e_i e_j \cdot f_j \mapsto \sum_{j=0}^{7} \overline{e_i e_j} \cdot \overline{f}_j = \sum_{j=1}^{3} \overline{e_i e_j} \cdot \overline{f}_j \tag{6.162}$$

for $i = 1, 2, 3$ (where we used the fact that $e_i e_j \notin S_{\mathrm{even}}$ for $j = 0, 4, 5, 6$ and $e_i e_7 \in R$).

In other words it is, up to tensoring with $A_3^{-2}$, precisely the map $\alpha$ in lemma 6.66. More precisely, it is given by the unique map $\alpha'$ making the following diagram commute

$$\begin{array}{ccc} A_1 \otimes A_3^{\otimes -1} \otimes R(-4) & \xrightarrow{\alpha \otimes \mathrm{id}_{R(-4)}} & \mathrm{Sym}^2 A_2 \otimes A_3^{\otimes -2} \otimes R(-4) \\ \downarrow\cong & & \downarrow\cong \\ A_1 \otimes A_3 \otimes R(-4) & \xrightarrow{\alpha'} & \mathrm{Sym}^2 A_2 \otimes R(-4) \end{array} \tag{6.163}$$

Next consider the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & R(-6) \otimes A_3 \otimes A_3 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & S_{\mathrm{odd}}(-3) \otimes A_3 & \xrightarrow{\Phi} & \mathrm{Sym}^2 A_2 \otimes R(-4) & \longrightarrow & \operatorname{coker}\Phi & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \longrightarrow & A_1 \otimes A_3 \otimes R(-4) & \xrightarrow{\alpha'} & \mathrm{Sym}^2 A_2 \otimes R(-4) & \xrightarrow{\beta} & V \otimes R(-4) \otimes A_3^{\otimes 2} & \longrightarrow & 0 \\
 & & \uparrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\tag{6.164}
$$

In particular $\beta \circ \Phi$ induces a map $R(-6) \otimes A_3^2 \to V \otimes R(-4) \otimes A_3^2$.
We claim that this map is trivial on $A_3^{\otimes 2}$:

$$
\begin{aligned}
\overline{e}_7 \otimes \overline{e}_7 \overset{\Phi}{\mapsto} & \sum_{i=4}^{6} \overline{e_7 f_i} \otimes \overline{e_i} \\
&= \sum_{i,j=4}^{6} \mathrm{tr}(e_7 f_i f_j) \overline{e_j} \otimes \overline{e_i} \\
&\overset{\beta}{\mapsto} \sum_{i,j=4}^{6} \mathrm{tr}(e_7 f_i f_j) \phi(\overline{e}_{j-3}^{\vee} \cdot e_{i-3}^{\vee}) \otimes e_7^2 \\
&= \sum_{i,j=4}^{6} \mathrm{tr}(e_7 f_i f_j) \phi(f_j^{\vee} \cdot f_i^{\vee}) \otimes \overline{e}_7^2.
\end{aligned}
\tag{6.165}
$$

The right hand side should be considered as an element of $R_2 \otimes V \otimes A_3^{\otimes 2}$.

Now with the notation of lemma 6.68 (which is an easy result from linear algebra) applied to

$$
\begin{aligned}
\alpha &= \mathrm{tr}(e_7 -) : S^2 E^{\vee} \to R_2 = V^{\vee}, \\
\beta &= \phi^{\vee} : V^{\vee} \to S^2 E^{\vee}, \\
(a_l)_l &= (f_i \cdot f_j)_{i,j=4,5,6}
\end{aligned}
\tag{6.166}
$$

we find

$$
\sum_{i,j=4}^{6} \mathrm{tr}(e_7 f_i f_j) \otimes \phi(f_j^{\vee} \cdot f_i^{\vee}) \otimes \overline{e}_7^2 = \mathrm{id}_V \otimes \overline{e}_7^2
\tag{6.167}
$$

with $\mathrm{id}_V$ considered as an element of $V^{\vee} \otimes V$. As such, by the construction of the Koszul sequence we have that $\beta \circ \Phi \colon R(-6) \otimes A_3^2 \to V \otimes R(-4) \otimes A_3^2$ can be decomposed as $\kappa\otimes\mathrm{id}_{A_3^{\otimes 2}}$ with $\kappa$ as in (6.158). In particular $\operatorname{coker}(\beta\circ\Phi)\cong\operatorname{coker}(\kappa)=\Omega$. Conversely the image of $\beta\circ\Phi$ is given by $\ker\gamma$, such that $\operatorname{coker}(\beta\circ\Phi)\cong\operatorname{im}(\gamma)=\operatorname{coker}(\Phi)$, proving the lemma. ◻

**Lemma 6.68.** Let $V_1, V_2, V_3$ be finite-dimensional vector spaces. Let $\alpha\colon V_1\to V_2$ and $\beta\colon V_3\to V_1$ be linear maps. Let $v_1,\dots,v_n$ be a basis for $V_1$. Then

$$\sum_{i=1}^{n}\alpha(v_i)\otimes\beta^\vee(v_i^\vee) \tag{6.168}$$

is equal to

$$\alpha\circ\beta\colon V_3\to V_2 \tag{6.169}$$

considered as elements of $V_3^\vee\otimes_k V_2$.

We will now use these algebraic results, to describe explicitly a morphism

$$g\colon \mathbb{P}(E)\to\mathbb{P}(V) \tag{6.170}$$

which is a 4 : 1-cover. This is a morphism which exists more generally for nets of curves on smooth projective surfaces, as explained in [77, §11.4.4], and it will be the crucial ingredient in comparing the two constructions. As it turns out there is a purely geometric result comparing morphisms $f\colon\mathbb{P}^1\to\mathbb{P}^1$ of degree 4 to nets of conics and the choice of a smooth conic in the net, which is proven in section 6.5.3, whilst the categorical incarnation comparing the two constructions is given in section 6.5.4.

Consider $Z := \operatorname{Proj} S = \mathbb{P}(E)$ and $T := \operatorname{Proj} R = \mathbb{P}(V)$, where we equip $R$ with the doubled grading. From $\phi\colon \operatorname{Sym}^2 E\twoheadrightarrow V$ we get a corresponding morphism $g\colon Z\to T$.

**Lemma 6.69.** The morphism $g\colon Z\to T$ sends a point to the pencil of conics for which it is a basepoint.

*Proof.* Note that every element in $\operatorname{Sym}^2 E^\vee$ (or rather $(\operatorname{Sym}^2 E^\vee)/k^\times$) defines a conic in $\mathbb{P}(E)$. In particular $\phi^\vee\colon V^\vee\hookrightarrow\operatorname{Sym}^2 E^\vee$ identifies $V^\vee$ (or rather $V^\vee/k^\times$) as a net of conics. More concretely, fix bases $x_1,x_2,x_3$ and $v_1,v_2,v_3$ for $E$ and $V$ respectively and write $\phi(x_ix_j)=\alpha_{1,i,j}v_1+\alpha_{2,i,j}v_2+\alpha_{3,i,j}v_3$. Then $V^\vee$ is the net of conics spanned by

$$\begin{aligned} v_1^\vee &= \sum_{1=i\le j}^{3}\alpha_{1,i,j}x_i^\vee x_j^\vee,\\ v_2^\vee &= \sum_{1=i\le j}^{3}\alpha_{2,i,j}x_i^\vee x_j^\vee,\\ v_3^\vee &= \sum_{1=i\le j}^{3}\alpha_{3,i,j}x_i^\vee x_j^\vee. \end{aligned} \tag{6.171}$$

An element $[t_1:t_2:t_3]\in T=\mathbb{P}(V)$ corresponds to the following one dimensional subspace of $V^\vee/k^\times$ (i.e. the pencil of conics)

$$\{a_1v_1^\vee+a_2v_2^\vee+a_3v_3^\vee \mid a_1t_1+a_2t_2+a_3t_3=0\}/k^\times \tag{6.172}$$

Now $\phi^\vee$ induces a map $g\colon Z \to T$ as follows:

$$Z = \mathbb{P}(E) \xrightarrow{\text{Veronese embedding}} \mathbb{P}(\operatorname{Sym}^2 E) \xrightarrow{\operatorname{Proj}(\operatorname{Sym}(\phi^\vee))} \mathbb{P}(V) \tag{6.173}$$

In particular $g$ acts as follows

$$\begin{aligned}[z_1 : z_2 : z_3] &\mapsto [z_1^2 : z_2^2 : z_3^2 : z_1 z_2 : z_1 z_3 : z_2 z_3] \\ &\mapsto [v_1^\vee(z_1, z_2, z_3) : v_2^\vee(z_1, z_2, z_3) : v_3^\vee(z_1, z_2, z_3)].\end{aligned} \tag{6.174}$$

The lemma follows by combining (6.172) and (6.174). □

**Remark 6.70.** Note that $g([z_1 : z_2 : z_3])$ is not well defined when $[z_1 : z_2 : z_3]$ is a basepoint of the net of conics induced by $\varphi$. We will show in proposition 6.75 below that this situation cannot occur.

From lemma 6.69 we get the following corollary, which is also contained in [77, §11.4.4].

**Corollary 6.71.** With the notation as above as above, let $\xi \in V^\vee$, let $t_\xi \in \mathbb{P}(V^\vee)$ be the associated point and $X_\xi \in \mathbb{P}(V)$ the associated line. Conversely let $Y_\xi \in \mathbb{P}(E)$ be the conic defined by $\xi$. Then $g^{-1}(X_\xi) = Y_\xi$.

*Proof.* Write $\xi = a_1 v_1^\vee + a_2 v_2^\vee + a_3 v_3^\vee$, then $X_w$ is the line in the coordinates $[v_1 : v_2 : v_3]$ defined by $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$. In particular

$$g^{-1}(X_\xi) = \{[x_1 : x_2 : x_3] \mid \sum_{i=1}^{3} a_i v_i^\vee([x_1 : x_2 : x_3]) = 0\}. \tag{6.175}$$

This is equal to $Y_\xi$. □

Now if $g\colon Z \to T$ is an arbitrary finite morphism of smooth projective varieties, then by dualising the multiplication

$$g_*\mathcal{O}_Z \otimes_{\mathcal{O}_T} g_*\mathcal{O}_Z \to g_*\mathcal{O}_Z \tag{6.176}$$

we obtain a copairing

$$g_*\omega_{Z/T} \to g_*\omega_{Z/T} \otimes_{\mathcal{O}_T} g_*\omega_{Z/T}. \tag{6.177}$$

By tensoring it over $g_*\mathcal{O}_Z$ on the left and right with $g_*\omega_{Z/T}^{-1}$ we get an equivalent copairing

$$\delta_{Z/T}\colon g_*\omega_{Z/T}^{-1} \to g_*\mathcal{O}_Z \otimes_{\mathcal{O}_Z} g_*\mathcal{O}_Z. \tag{6.178}$$

This will be the geometric incarnation of (6.155) in the special case we are interested in. This, and other identifications between sheaves and the corresponding graded modules are given in the following lemma.

**Lemma 6.72.** There exist isomorphisms

$$g_*\mathcal{O}_Z \cong \widetilde{S_{\mathrm{even}}}, \tag{6.179}$$

and

$$g_*\omega_{Z/T}^{-1} \cong \widetilde{S_{\mathrm{odd}}(-3)} \otimes_k A_3. \tag{6.180}$$

Using these identifications the morphism (6.178) is the sheafification of the composition

$$S_{\mathrm{odd}}(-3) \otimes A_3 \hookrightarrow S(-3) \otimes A_3 \xrightarrow{\delta} S \otimes_R S \twoheadrightarrow S_{\mathrm{even}} \otimes_R S_{\mathrm{even}}. \tag{6.181}$$

*Proof.* The identification in (6.179) follows from the fact that $R$ is concentrated in even degrees.

The identification in (6.180) is obtained by restricting (6.154) to the even part, which gives

$$g_*\omega_{Z/T} = \widetilde{S_{\mathrm{odd}}(3)} \otimes A_3^{\otimes -1}, \tag{6.182}$$

and using

$$(S_{\mathrm{odd}}(-3) \otimes_k A_3) \otimes_{S_{\mathrm{even}}} (S_{\mathrm{odd}}(3) \otimes_k A_3^{-1}) \cong S_{\mathrm{even}} \tag{6.183}$$

we get the desired identification.

Finally the identification in (6.181) follows from the explicit form of duality for a finite flat morphism. □

We will now define sheaves on $T$, which will induce the sheaves $\mathcal{F}$ and $\mathcal{Q}$ on $X$, as in section 6.3.5. These are

$$\begin{aligned} \overline{\mathcal{F}} &\coloneqq g_*\mathcal{O}_Z/\mathcal{O}_T, \\ \overline{\mathcal{Q}} &\coloneqq \operatorname{coker}(\delta' \colon \omega_{Z/T}^{-1} \to \operatorname{Sym}^2 \overline{\mathcal{F}}) \end{aligned} \tag{6.184}$$

where $\delta'$ is the composition of $\delta_{Z/T}$ with the projection to $\operatorname{Sym}^2 \overline{\mathcal{F}}$. Then we have the following result.

**Proposition 6.73.** There exist isomorphisms

$$\begin{aligned} \overline{\mathcal{F}} &\cong A_2 \otimes_k \mathcal{O}_T(-1), \\ \overline{\mathcal{Q}} &\cong \Omega^1_{T/k} \otimes_k A_3^{\otimes 2}. \end{aligned} \tag{6.185}$$

Moreover we have that the quotient map

$$\operatorname{Sym}^2 \overline{\mathcal{F}} \to \overline{\mathcal{Q}} \tag{6.186}$$

is the composition of the sheafification of the morphism $\beta$ from (6.149)

$$\operatorname{Sym}^2 A_2 \otimes_k \mathcal{O}_T(-2) \to V \otimes_k \mathcal{O}_T(-2) \otimes A_3^{\otimes 2} \tag{6.187}$$

with the quotient map

$$V \otimes_k \mathcal{O}_T(-2) \otimes A_3^{\otimes 2} \to \Omega^1_{T/k} \otimes A_3^{\otimes 2}. \tag{6.188}$$

*Proof.* This follows immediately from the construction together with proposition 6.67 and lemma 6.72. □

### 6.5.3 The geometric comparison

We now consider the situation of section 6.3, i.e. we start with $f\colon Y \to X$ a finite morphism, where $X, Y \cong \mathbb{P}^1$. As in section 6.3.5 we define

$$\begin{aligned} \mathcal{F}_{Y/X} &\coloneqq f_*\mathcal{O}_Y/\mathcal{O}_X \\ \mathcal{Q}_{Y/X} &\coloneqq \operatorname{coker}(\omega_{Y/X}^{-1} \to \operatorname{Sym}^2 \mathcal{F}_{Y/X}). \end{aligned} \tag{6.189}$$

Then we define

$$\begin{aligned} E_{Y/X} &\coloneqq \mathrm{H}^0(X, \mathcal{F}_{Y/X}(1)) \\ V_{Y/X} &\coloneqq \mathrm{H}^0(X, \mathcal{Q}_{Y/X}(2)). \end{aligned} \tag{6.190}$$

Using lemma 6.41 and proposition 6.43 we see that $\dim_k E_{Y/X} = \dim_k V_{Y/X} = 3$, and the quotient morphism $\operatorname{Sym}^2 \mathcal{F}_{Y/X} \to \mathcal{Q}_{Y/X}$ defines a net of conics

$$\phi_{Y/X}\colon \operatorname{Sym}^2 E_{Y/X} \twoheadrightarrow V_{Y/X} \tag{6.191}$$

because the twist in the definition of $E_{Y/X}$ and $V_{Y/X}$ makes the higher sheaf cohomology vanish in the definition of $\mathcal{Q}_{Y/X}$.

We now prove the following lemma, which will allow us to prove that the net of conics $\phi_{Y/X}$ is basepoint-free. Recall from section 6.3.5 that $\mathcal{F}_{Y/X}(1) = \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$ and $f_*\omega_{Y/X}^{-1}(2) = \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. In particular there is a morphism

$$\theta\colon f_*\omega_{Y/X}^{-1}(2) \to \mathcal{O}_{\mathbb{P}^1}(-1) \tag{6.192}$$

which is unique up to scalar multiple.

If $x$ is not a branch point of $f$, then $f^{-1}(x) = \{y_1, y_2, y_3, y_4\}$ and

$$(f_*\omega_{Y/X}^{-1}(2))_x \otimes k(x) \cong (f_*\mathcal{O}_Y)_x \otimes k(x) \cong k(y_1) \oplus k(y_2) \oplus k(y_3) \oplus k(y_4) \tag{6.193}$$

**Lemma 6.74.** For a generic $x \in X$ the map $\theta_x \otimes k(x)$ does not vanish on any of the $k(y_i)$ in (6.193).

*Proof.* As in [214, §3.1] we can restrict the situation to an affine open $\operatorname{Spec} C \subset X$ such that

- $f^{-1}(\operatorname{Spec} C) = \operatorname{Spec} D$,
- $D/C$ is relative Frobenius of rank 4,
- $f_*\omega_{Y/X}^{-1}(2)|_{\operatorname{Spec} C} \cong f_*(\mathcal{O}_D)$.

In particular $\theta$ is given by a morphism $D \to C$. Moreover as $f$ has only finitely many branch points, we can choose $C$ such that $f^{-1}(x)$ consists of 4 points $y_1, y_2, y_3, y_4$ for all $x \in \operatorname{Spec} C$. We will now prove that the lemma holds for all but a finite number of points of $\operatorname{Spec} C$.

Note that if $x$ is any point in $\operatorname{Spec} C$, we can use [213, lemma 3.1] to conclude that

$$(f_*\mathcal{O}_Y)_x \otimes k(x) \cong D \otimes_C k(x) \cong k(y_1) \oplus k(y_2) \oplus k(y_3) \oplus k(y_4) \tag{6.194}$$

is relative Frobenius of rank 4 over $k(x)$. This implies that $k(y_i) \cong k(x)$ for $i = 1, 2, 3, 4$. Let $e_{x,1}, e_{x,2}, e_{x,3}, e_{x,4}$ denote the idempotents in

(6.195) $k(y_1) \oplus k(y_2) \oplus k(y_3) \oplus k(y_4) \cong k(x)^{\oplus 4}$

We now need to prove (for generic $x$) that $(\theta_x \otimes k(x))(e_{x,i}) \neq 0$. For this consider the following commutative diagram

(6.196)

$$\begin{array}{ccccccc} \mathrm{Hom}_C(D,C) & \longrightarrow & \mathrm{Hom}_{k(x)}(D \otimes_C k(x), k(x)) & & \xrightarrow{\mathrm{eval}(e_{x,i})} & & k(x) \\ \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong \\ D & \xrightarrow{-\otimes_C k(x)} & D \otimes_C k(x) & \xrightarrow{\cong} & k(x)^{\oplus 4} & \xrightarrow{\pi_i} & k(x) \cong D_{y_i} \otimes k(y_i). \end{array}$$

Commutativity of the left square follows from the fact that the construction of relative Frobenius pairs is compatible with base change. Commutativity of the right square follows from the fact that the middle vertical isomorphism (or rather its inverse) is given by

(6.197)

$$k(x)^{\oplus 4} \xrightarrow{\cong} \mathrm{Hom}_{k(x)}(k(x)^{\oplus 4}, k(x)) : (a_1, a_2, a_3, a_4) \mapsto ((b_1, b_2, b_3, b_4) \mapsto \sum_{i=1}^{4} a_i b_i)$$

Now recall that every element in $d \in D$ defines a function on $\operatorname{Spec} D$ by sending $y_i \in \operatorname{Spec} D$ to the image of $d$ under the bottom horizontal composition in (6.196). In particular $\theta \in \mathrm{Hom}_C(D, C)$ defines an element in $D$ through the left vertical isomorphism in (6.196) and hence a function $\tilde{\theta}$ on $\operatorname{Spec} D$. By construction we have

(6.198) $\tilde{\theta}(y_i) \neq 0 \;\Leftrightarrow\; (\theta_x \otimes k(x))(e_{x,i}) \neq 0$

The lemma now follows by noticing that the function $\tilde{\theta}$ can only have finitely many zeroes on $\operatorname{Spec} D$. □

**Proposition 6.75.** The net of conics $\phi_{Y/X}$ is basepoint-free.

*Proof.* Recall that $\phi_{Y/X}$ was constructed via the following short exact sequence

(6.199)

$$0 \to \mathrm{H}^0(X, f_* \omega_{Y/X}^{-1}(2)) \xrightarrow{i_{Y/X}} \mathrm{Sym}^2(\mathrm{H}^0(X, \mathcal{F}_{Y/X}(1))) \xrightarrow{\phi_{Y/X}} \mathrm{H}^0(X, \mathcal{Q}_{Y/X}(2)) \to 0$$

In particular it suffices to prove that

(6.200) $\mathrm{im}(i_{Y/X}) \cap \{e \cdot e \mid e \in \mathrm{H}^0(X, \mathcal{F}_{Y/X}(1))\} = \{0\}$.

To see this, note that $\mathrm{im}(i_{Y/X}) = \ker(\phi_{Y/X})$ and

$$\begin{aligned}\ker(\phi_{Y/X}) &= \left\{ \sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j \mid \phi\left(\sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j\right) = 0 \right\} \\ &= \left\{ \sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j \mid \sum_{1=i\leq j}^{3} \alpha_{n,i,j}\beta_{i,j} = 0 \text{ for } n = 1,2,3 \right\} \\ &= \left\{ \sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j \mid v_n^\vee((\beta_{i,j})_{i,j}) = 0 \text{ for } n = 1,2,3 \right\}.\end{aligned} \tag{6.201}$$

where the $x_i$ and $\alpha_{i,j}$ are as in lemma 6.69.

Moreover as in (6.173), we will interpret the functions $v_1^\vee, v_2^\vee, v_3^\vee$ as functions on $\mathbb{P}(\mathrm{Sym}^2 E) \cong \mathbb{P}^5$.

The condition that the net spanned by $v_1^\vee, v_2^\vee, v_3^\vee$ is basepoint-free is equivalent to requiring that the points $[\beta_{1,1} : \beta_{2,2} : \beta_{3,3} : \beta_{1,2} : \beta_{1,3} : \beta_{2,3}]$ for which

$$\sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j \in \ker(\phi_{Y/X})) \tag{6.202}$$

do not lie in the image of the Veronese embedding $\mathbb{P}(E) \cong \mathbb{P}^2 \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 E)$. And finally the latter is equivalent to $\sum_{1=i\leq j}^{3} \beta_{i,j}x_ix_j$ not lying in the image of

$$E \to \mathrm{Sym}^2 E : \sum_{i=1}^{3} \gamma_i x_i \mapsto (\gamma_i x_i)^2 = \sum_{1=i\leq j}^{3} \gamma_i\gamma_j x_i x_j. \tag{6.203}$$

In particular in order to prove the result, it suffices to prove (6.200).

Let $j_{Y/X}\colon \ker(\theta) \to f_*\omega_{Y/X}^{-1}(2) \to \mathrm{Sym}^2(\mathcal{F}_{Y/X}(1))$ be the induced composition. Then, using the fact that $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no global sections, we see that $\mathrm{H}^0(X, j_{Y/X}) = i_{Y/X}$. Moreover as both $\ker(\theta)$ and $\mathrm{Sym}^2(\mathcal{F}_{Y/X}(1))$ are given by direct sums of copies of the structure sheaf we have for each point $x \in X$ that

$$j_{Y/X,x} \otimes_{\mathcal{O}_{X,x}} k(x) = \mathrm{H}^0(X, j_{Y/X}) \otimes_k k(x) = i_{Y/X} \otimes_k k(x) \tag{6.204}$$

and

$$\{(e \cdot e) \otimes_k 1_{k(x)} \mid e \in \mathrm{H}^0(X, \mathcal{F}_{Y/X}(1))\} = \{e_x \cdot e_x \mid e_x \in \mathcal{F}_x \otimes k(x)\}. \tag{6.205}$$

As such it suffices to prove that

$$\mathrm{im}\left(j_{Y/X,x} \otimes_{\mathcal{O}_{X,x}} k(x)\right) \cap \{(e_x \cdot e_x) \otimes_{\mathcal{O}_{X,p}} 1_{k(x)} \mid e_x \in \mathcal{F}_x\} = \{0\} \tag{6.206}$$

holds for some point $x \in X$. For this let $x$ be a generic point as in lemma 6.74

Then as in the proof of lemma 6.37 (using the fact that the $e_i$ form a self-dual basis (remark 6.19)) we see that $j_{Y/X,x} \otimes_{\mathcal{O}_{X,x}} k(x)$ is given by composing the inclusion $\ker(\theta_x) \otimes_{\mathcal{O}_{X,x}} k(x) \subset k(x)^{\oplus 4}$ with

$$k(x)^{\oplus 4} \to \mathrm{Sym}^2(k(x)^{\oplus 4}) \to \mathrm{Sym}^2((k(x)^{\oplus 4})/k(x)) : e_i \mapsto e_i \cdot e_i$$

Note that $\sum_{i,j=1, i\leq j}^{4} a_{i,j}\frac{e_i\cdot e_j+e_j\cdot e_i}{2}$ is of the form $e\cdot e$ if and only if $\alpha_{i,i}\alpha_{j,j} = 4\alpha_{i,j}^2$ holds for all $i<j$. Such an element lies in the image of $j_{Y/X,x}\otimes_{\mathcal{O}_{X,x}} k(x)$ if and only if there is 1 $i$ such that $\alpha_{j,k}=0$ whenever $(j,k)\neq(i,i)$. Hence it suffices to show that $\ker(\theta_x)\otimes_{\mathcal{O}_{X,x}} k(x)$ does not contain an element of the form $\lambda e_i$. This is however guaranteed by the fact that $\theta_x\otimes_{\mathcal{O}_{X,x}} k(x)$ does not vanish on any of the $k(y_i)$ in (6.193). ◻

One final piece of information obtained from $f\colon Y\to X$ in this setting is a character $\xi_{Y/X}\colon V_{Y/X}\to k$, up to scalars, by taking the global sections of the unique non-zero projection

(6.207) $\mathcal{Q}_{Y/X}(2)\cong\mathcal{O}_X\oplus\mathcal{O}_X(1)\to\mathcal{O}_X.$

**Definition 6.76.** An *algebraic quadruple* is a tuple $(E,V,\phi,\xi)$, where $E$ and $V$ are vector spaces of dimension 3, $\phi\colon \operatorname{Sym}^2 E\to V$ is a surjective morphism defining a basepoint-free net of conics, and $\xi\colon V\to k$ is a non-zero morphism.

The procedure described above is a way of getting an algebraic quadruple from a morphism $f\colon\mathbb{P}^1\to\mathbb{P}^1$ of degree 4, by taking $(E_{Y/X},V_{Y/X},\phi_{Y/X},\xi_{Y/X})$. The main result of this section is that there exists a correspondence between such morphisms $f$ and these algebraic quadruples.

From an algebraic quadruple we can define a morphism $f\colon\mathbb{P}^1\to\mathbb{P}^1$ of degree 4. For this we consider the morphism $g\colon Z\to T$ defined in (6.170). From $\xi\in\mathbb{P}(V^\vee)$ we obtain a line $X$ inside $\mathbb{P}(V)$. Then define $f\colon Y\to X$ using the fibre product

(6.208) $$\begin{array}{ccc} Y & \hookrightarrow & Z \\ \downarrow f & & \downarrow g \\ X & \xhookrightarrow{i} & T. \end{array}$$

As explained in corollary 6.71, we can identify $Y$ with the conic parametrised by the point $\xi\in\mathbb{P}(V^\vee)$ in the net of conics. Hence if $\xi$ is taken outside the discriminant locus in $\mathbb{P}(V^\vee)$, then $Y$ is a smooth conic, hence we get a morphism $f\colon\mathbb{P}^1\to\mathbb{P}^1$ of degree 4. The following theorem shows that the algebraic quadruple associated to this morphism is isomorphic to the original algebraic quadruple.

**Theorem 6.77.** Let $(E,V,\phi,\xi)$ be an algebraic quadruple such that $\xi$ defines a point outside the discriminant locus of the net of conics $\phi$. Then

(6.209) $(E,V,\phi,\xi)\cong(E_{Y/X},V_{Y/X},\phi_{Y/X},\xi_{Y/X}).$

*Proof.* Using lemma 6.41 and propositions 6.43 and 6.73 and the fact that $X$ is a curve of degree 1 inside $T$ we have a commutative diagram

(6.210) $$\begin{array}{ccccc} \operatorname{Sym}^2\mathcal{F}_{Y/X} & \twoheadrightarrow & \mathcal{Q}_{Y/X} & \xrightarrow{\cong} & \mathcal{O}_X(-2)\oplus\mathcal{O}_X(-1) \\ \| & & \| & & \uparrow \\ \operatorname{Sym}^2(i^*\overline{\mathcal{F}}) & \twoheadrightarrow & i^*\overline{\mathcal{Q}} & \xrightarrow{\cong} & i^*\Omega^1_{T/k} \end{array}$$

Because $\Omega^1_{T/k}(2)$ sits inside the twist of the Euler exact sequence

$$0 \to \mathcal{O}_T(-1) \xrightarrow{\gamma} V \otimes_k \mathcal{O}_T \to \Omega^1_{T/k}(2) \to 0 \tag{6.211}$$

we get that $\mathrm{H}^0(\Omega^1_{T/k}(2)) = V$. From this we get isomorphisms

$$V \cong V_{Y/X} \tag{6.212}$$

and

$$\begin{aligned} E_{Y/X} &= \mathrm{H}^0(\mathcal{F}_{Y/X}(1)) && \\ &\cong \mathrm{H}^0(i^*(\overline{\mathcal{F}}(1))) && (6.210) \\ &\cong \mathrm{H}^0(i^*(A_2 \otimes_k \mathcal{O}_T)) && \text{proposition 6.73} \\ &\cong \mathrm{H}^0(A_2 \otimes_k \mathcal{O}_X) && \\ &\cong A_2 && \\ &\cong E && (6.143) \text{ and } (6.147) \end{aligned} \tag{6.213}$$

We also get that $\phi = \phi_{Y/X}$ by (6.210).

Finally to show that $\xi_{Y/X} = \xi$ we need to show that $\xi$ equals the composition

$$\begin{array}{ccc} V = \mathrm{H}^0(T, \Omega^1_{T/k}(2)) & \longrightarrow & \mathrm{H}^0(X, i^*\Omega^1_{T/k}(2)) \\ & & \| \\ & & \mathrm{H}^0(X, \mathcal{O}_X \oplus \mathcal{O}_X(1)) \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X) = k. \end{array} \tag{6.214}$$

The fiber in $p \in \mathbb{P}(V)$ of the inclusion is (up to scalar) the morphism $p\colon k \to V$. If we pull back the short exact sequence along the closed immersion $i$ we see that the image of the fibers of all pullbacks are contained in $\ker \xi$, hence we have a complex

$$\mathcal{O}_X(-1) \xrightarrow{i^*\gamma} V \otimes_k \mathcal{O}_X \xrightarrow{\xi \otimes \mathrm{id}_{\mathcal{O}_X}} \mathcal{O}_X. \tag{6.215}$$

Up to scalars this must be the unique non-zero morphism

$$i^*\Omega^1_{T/k} \to \mathcal{O}_X. \tag{6.216}$$

By taking global sections we see that $\xi_{Y/X} = \xi$. □

Unfortunately, we do not have a complete correspondence between the two pieces of geometric data. It is not clear that we can start from $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, produce an algebraic quadruple using geometric arguments, such that the associated morphism is the original $f$ (up to automorphisms of $\mathbb{P}^1$). We phrase this as a conjecture.

**Conjecture 6.78.** Let $f = (f_1, f_2)$ be a basepoint-free pencil of binary quartics. Then there exists an algebraic quadruple $(E, V, \phi, \xi)$ such that $f$ is the associated morphism $\mathbb{P}^1 \to \mathbb{P}^1$.

Luckily, this does not prevent us from completing the comparison between the associated categories, which we perform in section 6.5.4. The geometric correspondence would also follow from a good understanding of the Morita theory of the noncommutative $\mathbb{P}^1$-bundles, but this does not seem within reach.

Moreover, in the classification of the pencils of binary quartics there are 13 types, and for each of these types we can construct an algebraic quadruple giving rise to this pencil of binary quartics. And whenever there are degrees of freedom within a certain class, the degrees of freedom agree with the degrees of freedom in algebraic quadruples of the corresponding type.

**Explicit correspondence** We will now relate the classification of pencils of binary quartics from [235] to algebraic quadruples, i.e. to the classification of nets of conics from [234] and the choice of a smooth conic inside the net. Because these classifications are only available in characteristic 0 we assume this for now (although the arguments for the correspondence work in characteristic not equal to 2, 3). We do this by giving for every element in the classification of pencils of binary quartics the discriminant of the net of conics and the position of the point which is blown up, corresponding to the choice of the smooth conic inside the net.

To understand which type of $f$ we obtain in from an algebraic quadruple, we only need to describe what the inverse image of each point looks like, i.e. what the ramification behaviour is. For this, it suffices to understand what the possible intersections of lines from the pencil of lines with the discriminant cubic are, because there is a correspondence between base loci of pencils of conics and the structure of their singular fibres.

1. The Segre symbol $[1, 1, 1]$ is the generic case, and is not relevant in the construction. It corresponds to a generic line intersecting the discriminant cubic in 3 points, i.e. a pencil of conics with 3 singular fibres.

2. The Segre symbol $[2, 1]$ corresponds to ramification of type $(2, 1, 1)$, and to realise it we must have a tangent line to the discriminant cubic intersecting the discriminant cubic in another (smooth) point: we have two singular fibres of rank 2, one of which is counted twice.

3. The Segre symbol $[3]$ corresponds to ramification of type $(3, 1)$, and to realise it we must have a tangent at an inflection point: we have one singular fibre of rank 2 which is counted thrice.

4. The Segre symbol $[(1, 1), 1]$ corresponds to ramification of type $(2, 2)$, and to realise it we must have a line through a node intersecting the discriminant cubic in another point.

5. The Segre symbol $[(2, 1)]$ corresponds to ramification of type $(4)$, and to realise it we must have a line through a node intersecting the discriminant only in this node.

Using this it is straightforward to set up the correspondence given in table 6.4. It only remains to observe that on a nodal cubic there are 3 inflections points whose tangent lines are not concurrent, and that on an elliptic curve in $\mathbb{P}^2$ with $j = 0$ there exist sets of 3 inflection points whose tangent lines *are* concurrent. If $j \neq 0$ then this is not possible (and recall that there are 9 inflection points in total). This follows immediately from the proof of [7, lemma 1], where the inflection tangents are concurrent if $c = 0$, which gives the elliptic curve defined by $x^3 + y^3 + z^3$, which indeed has $j$-invariant 0.

### 6.5.4 The categorical comparison

The categorical comparison would follow from theorem 6.77 and a positive answer to conjecture 6.78. Lacking this, we give an algebraic proof for the fact that a noncommutative $\mathbb{P}^1$-bundle gives rise to an algebraic quadruple. The only thing left to show for this is that the center of the blowup is outside the discriminant. From the description in section 6.3 and the computation in remark 5.32 it suffices to show that the global dimension of the associated abelian category is finite.

**Proposition 6.79.** The global dimension of $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ is finite.

*Proof.* By [214, lemma 4.6] we have that $\Pi_m^*$ is an exact functor, whilst [214, lemma 5.3] shows that $\Pi_{m,*}$ has cohomological dimension 1. The Grothendieck spectral sequence for Hom in $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ and $\Pi_{m,*}$ then shows that the cohomological dimension of $\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(i))$ is 2, i.e. that $\mathrm{Ext}^n(\Pi_m^*(\mathcal{O}_{\mathbb{P}^1}(i)), \mathcal{M}) = 0$ for $n \geq 3$, using [214, lemma 4.7].

From [214, theorem 5.1, theorem 5.5] we have that there exists a full and strong exceptional collection in $\mathbf{D}^{\mathrm{b}}(\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ consisting of objects of cohomological dimension 2. Because the global dimension of this endomorphism algebra is finite (as it is the path algebra of an acyclic quiver modulo an ideal of relations), and the cohomological dimension of the functor realising the equivalence is 2, we can compute $\mathrm{Ext}^n$ in qgr using uniformly bounded complexes in $\mathbf{D}^{\mathrm{b}}(kQ/I)$, and we can conclude that the global dimension is indeed finite. □

**Remark 6.80.** It would be interesting to give a direct proof that the global dimension of this surface-like category is precisely 2. We expect that this can be deduced from [213, theorem 6.1] using an appropriate analogue of the local-to-global spectral sequence for Ext.

Observe that from theorem 6.82 we can a posteriori conclude that the global dimension is indeed 2, as the global dimension of the equivalent category constructed as a blowup is 2 by proposition 6.60.

**Corollary 6.81.** The center of the blowup in (6.133) is contained in the Azumaya locus of the maximal order on $\mathbb{P}^2$.

*Proof.* Assume on the contrary that $x$ is contained in the discriminant. Then blowing up at $x$ yields a sheaf of algebras on $\mathrm{Bl}_x\,\mathbb{P}^2$ which is not of finite global dimension (and which is not a maximal order), using the local computation from remark 5.32. But this contradicts proposition 6.79, as these two categories are equivalent. □

This does *not* imply conjecture 6.78: there could be non-equivalent $f$ giving rise to an equivalent noncommutative $\mathbb{P}^1$-bundle, and only one of these corresponds to

| label | normal form<br>location of the point | discriminant cubic | moduli |
|---|---|---|---|
| A | [(4)(4)]<br>intersection of tangent lines at conic in singularities | conic and line | 0 |
| B | [(4)(2, 2)(2, 1, 1)]<br>intersection of tangent line at conic in singularity and generic line | conic and line | 0 |
| C | [(4)(3, 1)(2, 1, 1)]<br>intersection of tangent line at singularity and tangent line of inflection point | nodal cubic | 0 |
| D | [(4)(2, 1, 1)(2, 1, 1)(2, 1, 1)]<br>intersection of tangent line at singularity and tangent line at non-inflection point | nodal cubic | 1 |
| E | [(2, 2)(2, 2)(2, 2)]<br>any point | triangle of lines | 0 |
| F | [(2, 2)(2, 2)(2, 1, 1)(2, 1, 1)<br>intersection of tangent lines at two smooth points on conic | conic and line | 1 |
| G | [(2, 2)(3, 1)(3, 1)]<br>intersection of tangent lines at inflection points | nodal cubic | 0 |
| H | [(2, 2)(3, 1)(2, 1, 1)(2, 1, 1)]<br>intersection of non-tangent line through singularity and tangent line at inflection point | nodal cubic | 1 |
| I | [(3, 1)(3, 1)(3, 1)]<br>intersection of three concurrent tangent lines at inflection points | elliptic curve, $j = 0$ | 0 |
| J | [(3, 1)(3, 1)(2, 1, 1)(2, 1, 1)]<br>generic intersection of two tangent lines at inflection points | elliptic curve | 1 |
| K | [(2, 2)(2, 1, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)]<br>intersection of two generic tangent lines | nodal cubic | 2 |
| L | [(3, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)]<br>generic intersection of tangent line at inflection point and another tangent line | elliptic curve | 2 |
| M | [(2, 1, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)(2, 1, 1)]<br>generic point | elliptic curve | 3 |

Table 6.4: Correspondence between pencils of binary quartics and algebraic quadruples

an algebraic quadruple. In other words: it is not clear that there exists a surjection from pencils of binary quartics to algebraic quadruples, but if we only care about these objects up to equivalence of the associated abelian categories, the comparison is complete.

We are now in the position to state and prove the final comparison theorem, giving a correspondence between noncommutative $\mathbb{P}^1$-bundles of rank $(4,1)$ and blowups of quaternionic noncommutative planes outside the discriminant. In this way, we obtain a purely noncommutative incarnation of the isomorphism $\mathrm{Bl}_x \mathbb{P}^2 \cong \mathbb{F}_1$.

**Theorem 6.82.** Every noncommutative $\mathbb{P}^1$-bundle $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ is equivalent to a category of the form $\mathrm{coh}\, p^*\mathcal{S}$ for a quaternionic order $p^*\mathcal{S}$ on $\mathrm{Bl}_x \mathbb{P}^2$ and vice versa.

*Proof.* First let $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ be a noncommutative $\mathbb{P}^1$-bundle. Using the techniques in section 6.3 we know that $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ is equivalent to a category of the form

$$\mathrm{qgr}\, \mathcal{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(E \otimes_k \mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q\right) \tag{6.217}$$

This category is completely determined by a quadruple $(E, V, \phi, \xi)$. Using proposition 6.75, we know that this quadruple is algebraic. In particular the category is equivalent to one of the form $\mathrm{coh}\, p^*\mathcal{S}$. By corollary 6.81 the point (determined by $\xi$) in $\mathbb{P}(V^\vee) \cong \mathbb{P}^2$ blown up by $p\colon \mathbb{F}_1 \to \mathbb{P}^2$ lies in the Azumaya locus of $\mathcal{S}$.

Conversely, let $\mathrm{coh}\, p^*\mathcal{S}$ be the category associated to a quaternionic order on $\mathbb{F}_{\not{k}}$. The techniques in section 6.4 tell us that this category is equivalent to a category of the form

$$\mathrm{qgr}\, \mathcal{C}\ell_{\mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))}\left(E \otimes_k \mathrm{Sym}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)), q\right), \tag{6.218}$$

which again is determined by a quadruple $(E, V, \phi, \xi)$. Using [149, proposition 7], we know that this quadruple is algebraic and by construction $\xi$ determines a point in the Azumaya locus of $\mathcal{S}$. Using [149, proposition 8], we get that this point determines a nonsingular conic in the net $V^\vee$. We can then use theorem 6.77, to find a noncommutative $\mathbb{P}^1$-bundle $\mathrm{qgr}(\mathbb{S}({}_f(\mathcal{O}_{\mathbb{P}^1})_{\mathrm{id}}))$ equivalent to $\mathrm{coh}\, p^*\mathcal{S}$.

□

### 6.5.5 Final remarks

**Blowing up a point on the discriminant**

There is an important difference between the construction of chapter 5 and the construction of a blowup in [222]. In the latter a point on the point scheme on an arbitrary noncommutative surface is constructed resulting in a noncommutative $\mathrm{Bl}_x \mathbb{P}^2$, whereas the former only works for sheaves of orders where a point outside (an isogeny of) the point scheme is blown up. Hence these situations have no overlap. In other terms, the latter construction gives a deformation of the commutative $\mathrm{Bl}_x \mathbb{P}^2$, whereas there is no commutative counterpart for the former.

Likewise there is an important difference between the notion of a noncommutative $\mathbb{P}^1$-bundle of type $(1,4)$ [214] and that of a noncommutative $\mathbb{P}^1$-bundle of type $(1,4)$ [225]. The latter can be used to construct the noncommutative Hirzebruch surface $\mathbb{F}_1$. It is expected that this construction gives the same surfaces as the blowup from [222], but no proof has been found so far.

We can sum up the situation in the following table, where the rows give the same noncommutative surfaces (albeit conjecturally for the first row), whilst the columns are constructions of an analogous nature. This chapter shows that the bottom row gives isomorphic noncommutative surfaces.

| noncommutative $\mathbb{P}^1$-bundle of type $(2,2)$ | abstract blowup on point scheme |
|---|---|
| noncommutative $\mathbb{P}^1$-bundle of type $(1,4)$ | blowup outside ramification |

Table 6.5: Four constructions of a noncommutative $\mathbb{F}_1 = \mathrm{Bl}_x \mathbb{P}^2$

It is possible to degenerate the situation in the bottom row. If one were to blow up a point on the ramification of the maximal order on $\mathbb{P}^2$ and pull back the sheaf of algebras along this morphism in the setting of section 6.4, the resulting sheaf of orders is no longer a maximal order, nor does it have finite global dimension. But there exist two maximal orders in which the pullback can be embedded[2], and these sheaves of algebras describe the construction of [222] in the special case where everything is finite over the center. These two maximal orders correspond to the two inverse images under the isogeny $C \to C/\tau$, where $C$ is the point scheme and $C/\tau$ the ramification.

The method of comparison suggests that there exists a construction of a noncommutative $\mathbb{P}^1$-bundle which does not use a finite morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4, but rather a finite morphism $f\colon C \to \mathbb{P}^1$ of degree 4 where $C$ is the *singular* conic in the net of conics associated to the point which is being blown up. Then the formalism from [213] yields an abelian category of infinite global dimension, which should be the same as the category associated to the non-maximal order of infinite global dimension by methods similar to the ones used in this chapter.

Now the choice of a maximal order (of finite global dimension) containing the pullback gives the construction from the top row in table 6.5. In the construction of the noncommutative $\mathbb{P}^1$-bundle this seems to correspond to a resolution of singularities of the singular conic[3] $C$ degenerating the degree 4 morphism $f\colon C \to \mathbb{P}^1$ to two degree 2 morphisms $f_i\colon \mathbb{P}^1 \to \mathbb{P}^1$. These then degenerate the noncommutative $\mathbb{P}^1$-bundle of type $(1,4)$ to two different noncommutative $\mathbb{P}^1$-bundles of type $(2,2)$. We leave the details for this comparison for future work.

**Hochschild cohomology**

In chapter 4 we computed the Hochschild cohomology of noncommutative planes and quadrics. In [155, 156] a deformation theory of abelian ategories was introduced, which is controlled by their Hochschild cohomology $\mathrm{HH}^\bullet_{\mathrm{ab}}$. This turns out to be a

[2] At least when the point which is blown up is a smooth point on the ramification. If one blows up a singular point, there is only 1 preimage under the isogeny and the maximal order is unique.

[3] If $C$ is a double line there will be only one morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2, which should correspond to there only being one choice of maximal order.

derived invariant, so we can use the finite-dimensional algebra obtained through the full and strong exceptional collection to compute it.

Computer algebra calculations suggest that the result for the noncommutative surfaces considered in this chapter is less varied than it is in the situation of chapter 4, where a wide variety of situations appears. Recall that the method to compute Hochschild cohomology for noncommutative planes and quadrics consists of having an explicit computation for $\mathrm{HH}^1_{\mathrm{ab}}$, a vanishing result for $\mathrm{HH}^i_{\mathrm{ab}}$ with $i \geq 3$ and the Lefschetz trace formula for Hochschild cohomology which then determines the dimension of $\mathrm{HH}^2_{\mathrm{ab}}$ in terms of the number of exceptional objects in the derived category.

For the noncommutative del Pezzo surfaces studied in this chapter, only the noncommutative $\mathbb{P}^1$-bundle associated to the pencil of type A (see also example 6.62) has $\dim_k \mathrm{HH}^1_{\mathrm{ab}}(\mathrm{qgr}\,\mathbb{S}({}_f\mathcal{L}_{\mathrm{id}})) = 1$, all other cases have $\mathrm{HH}^1 = 0$. Because there is a full and strong exceptional collection of length 4 in the derived category this means that the noncommutative $\mathbb{P}^1$-bundle associated to the pencil of type A has $\dim_k \mathrm{HH}^2_{\mathrm{ab}} = 4$, whilst all other cases have $\dim_k \mathrm{HH}^2_{\mathrm{ab}} = 3$.

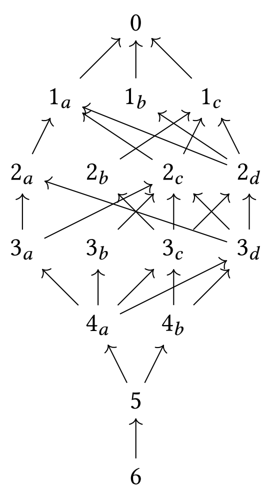
As it turns out, type A is also the only pencil of binary quartics which has a 1-dimensional symmetry group, all other types have a finite (and often trivial) automorphism group [235, §4]. One way of seeing this is that the morphism $f$ of type A is only ramified over two points, and the subgroup of $\mathrm{PGL}_2$ fixing these two points is isomorphic to $\mathbb{G}_{\mathrm{m}}$. The only ingredient missing to mimick the method of chapter 4 is a good understanding of the relationship between automorphisms of the pencil and outer automorphisms of the finite-dimensional algebra obtained from the full and strong exceptional collection to confirm these numerical results.

We can also (heuristically) understand the 1-dimensional first Hochschild cohomology space for type A pencils of binary quartics from the blowup model. Using chapter 4 we have a description of the automorphisms of the elliptic triples. The only elliptic triples we need to consider are those whose automorphism group is positive-dimensional, hence the point scheme is either a conic and a line, or a triangle of lines. Now we are interested in those automorphisms of $C$ (or rather[4] the isogeneous $C/\sigma$) which preserve the point we wish to blow up. In type A of table 6.4 we see that the automorphisms of $C$ preserve the point, hence we get a 1-dimensional automorphism group. For the other relevant types, namely those for which the first Hochschild cohomology of the associated noncommutative $\mathbb{P}^2$ is nonzero (i.e. types B, E and F) we get that at most finitely many automorphisms of the curve also preserve the point we wish to blow up. This heuristic works more generally for all cases in the construction of chapter 5, not just when $n = 2$.

[4]Recall that the discriminant is an isogeny of the point scheme, and is a cubic curve of the same type.

*Chapter 7*

# The point variety of skew polynomial algebras



## 7.1 Introduction

A *skew polynomial algebra* on $n+1$ variables has a presentation

(7.1) $A = \mathbb{C}\langle x_0, x_1, \dots, x_n\rangle/(x_i x_j - q_{i,j} x_j x_i,\ 0 \leq i, j \leq n)$

where all entries of the $(n+1) \times (n+1)$ matrix $Q = (q_{i,j})$ are non-zero and satisfy the relations $q_{i,i} = 1$ and $q_{j,i} = q_{i,j}^{-1}$ for all $i, j$.

If all the variables $x_i$ are given degree one, $A$ is a positively graded algebra with excellent algebraic and homological conditions: it is an iterated Ore extension and an Auslander-regular algebra of dimension $n+1$. They are also examples of (quadratic) Artin–Schelter regular algebras of dimension $n+1$, and as such are a family of Artin–Schelter regular algebras which exist in arbitrary dimension. Other families of such algebras which exist in arbitrary dimension are Sklyanin algebras, or the algebras constructed in [151].

One associates to such an algebra $A$ the *quantum projective space* defined by

(7.2) $\operatorname{qgr} A := \operatorname{gr} A/\operatorname{tors} A$

where $\operatorname{qgr} A$ is the quotient category of the category $\operatorname{gr} A$ of all graded left $A$-modules by the Serre subcategory $\operatorname{tors} A$ of all graded torsion left $A$-modules. This category is well-defined for every graded algebra, and is inspired by Serre's description of the category of coherent sheaves on a projective variety.

An interesting class of objects in $\operatorname{qgr} A$ are the *point modules* of $A$, which are determined by graded left $A$-modules $P = P_0 \oplus P_1 \oplus \dots$ which are *cyclic* (that is, are generated by one element in degree zero), *critical* (implying that all normalising elements of $A$ act on it either as zero or as a non-zero divisor) and have Hilbert series $(1-t)^{-1}$ (that is all graded components $P_i$ have dimension one). These also appear in chapters 3, 5 and 6 and they are a very important tool in noncommutative algebraic geometry.

As such a point module can be written as a quotient $P \simeq A/(Al_1 + \dots + Al_n)$ with linearly independent $l_i \in A_1$, we can associate to it a unique point $x_P = \mathbb{V}(l_1, \dots, l_n)$ defined by the vanishing of the linear forms $l_i$ in $\mathbb{P}^n = \mathbb{P}(A_1^*)$, having as its projective coordinates $u_0, u_1, \dots, u_n$. The *point variety* of $A$ is then the reduced closed subvariety of $\mathbb{P}^n$ defined as

(7.3) $\operatorname{pts}(A) := \{x_P \in \mathbb{P}^n \mid P \text{ a point module of } A\}$

The study of point modules and the point variety was very important in the classification of 3-dimensional Artin–Schelter regular algebras [12, 13]. The first aim of this chapter is to describe the possible subvarieties that can arise as point varieties of skew polynomial algebras in arbitrary dimension, and by doing so provide alternate proofs of some results in [233] which were obtained using methods of graded ring theory.

Coming back to the other examples of Artin–Schelter regular algebras in arbitrary dimension: for Sklyanin algebras one shows that (at least when $n \neq 4$) the point scheme is the elliptic curve used in the definition of the algebra [195], whilst the example of [151] has a bouquet of rational normal curves in $\mathbb{P}^n$ as its point scheme. Both these point schemes are 1-dimensional, an observation we come back to in remark 7.3.

For skew polynomial algebras we will prove the following result in section 7.2.

**Theorem 7.1.** With notation as above we have for all $n \geq 2$

1. the point variety is

$$\text{pts}(A) = \mathbb{V}((q_{i,j}q_{j,k} - q_{i,k})u_iu_ju_k,\ 0 \leq i < j < k \leq n) \tag{7.4}$$

   and hence is the union of a collection of linear subspaces of the form $\mathbb{P}(i_0, \ldots, i_k)$ which is the subspace of $\mathbb{P}^n$ spanned by the points $e_{i_0}, \ldots, e_{i_k}$, where we denote $e_j = [\delta_{0j} : \ldots : \delta_{nj}] \in \mathbb{P}^n$, with $\delta_{ij}$ being the Kronecker delta;

2. $\mathbb{P}(i_0, \ldots, i_k)$ is an irreducible component of $\text{pts}(A)$ if and only if the principal $(k+1) \times (k+1)$ submatrix

$$Q(i_0, \ldots, i_k) = \begin{bmatrix} 1 & q_{i_0,i_1} & \cdots & q_{i_0,i_k} \\ q_{i_1,i_0} & 1 & \cdots & q_{i_1,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ q_{i_k,i_0} & q_{i_k,i_1} & \cdots & 1 \end{bmatrix} \tag{7.5}$$

   of $Q$ is maximal among principal $Q$-submatrices such that $\operatorname{rk} Q(i_0, \ldots, i_k) = 1$;

3. moreover we have that

$$\text{pts}(A) = \mathbb{V}(u_iu_ju_k;\ 0 \leq i < j < k \leq n, \mathbb{P}(i,j,k) \not\subset \text{pts}(A)), \tag{7.6}$$

   and in particular, the point variety of $A$ is determined by all the projective planes $\mathbb{P}^2 = \mathbb{P}(u, v, w)$ it contains.

In section 7.3 we will give a necessary condition for a union of linear subspaces in $\mathbb{P}^n$ to be the point variety of a skew polynomial algebra. Theorem 7.7 implies that this condition is also sufficient for $n \leq 5$.

In section 7.4 we list all possible configurations, and the corresponding degeneration graph, when $n \leq 4$. Moreover we show that in dimension 5 the degeneration graph no longer has a unique endpoint, indicating an interesting feature in the classification of 6-dimensional Artin–Schelter regular algebras.

## 7.2 Description of the point varieties

Because each variable $x_i$ is a normalising element in $A$ we can consider the graded localization at the homogeneous Ore set $\{1, x_i, x_i^2, \ldots\}$. As this localization has an invertible element of degree one it is a strongly graded ring, see [167, §1.4], and therefore is a skew Laurent extension

$$A[x_i^{-1}] = B_i[x_i, x_i^{-1}, \sigma] \tag{7.7}$$

where $B_i$ is the degree zero part of $A[x_i^{-1}]$ and where $\sigma$ is the automorphism on $B_i$ given by conjugation with $x_i$.

The algebra $B_i$ is generated by the $n$ elements $v_j = x_j x_i^{-1}$ and as we have the commutation relations $x_j x_i^{-1} = q_{i,j} x_i^{-1} x_j$ we get the commutation relations

$$\begin{aligned} v_j v_k &= q_{i,j} x_i^{-1} x_j x_k x_i^{-1} \\ &= q_{i,j} q_{j,k} x_i^{-1} x_k x_j x_i^{-1} \\ &= q_{i,j} q_{j,k} q_{i,k}^{-1} x_k x_i^{-1} x_j x_i^{-1} \\ &= q_{i,j} q_{j,k} q_{i,k}^{-1} v_k v_j \end{aligned} \tag{7.8}$$

That is, $B_i$ is again a skew polynomial algebra, this time on $n$ variables $v_j$ with corresponding $n \times n$ matrix $R = (r_{jk})$ with entries

$$r_{j,k} = q_{i,j} q_{j,k} q_{i,k}^{-1} \tag{7.9}$$

One-dimensional representations of $B_i$ correspond to points $(a_j) \in \mathbb{A}^n$ with coordinate functions $y_j, j = 0, \ldots, n, j \neq i$ (via the morphism $v_j \mapsto a_j$) if they satisfy all the defining relations $v_j v_k = r_{jk} v_k v_j$ of $B_i$, that is,

$$(a_j) \in \bigcap_{j \neq i \neq k} \mathbb{V}((1 - r_{jk}) y_j y_k) \tag{7.10}$$

Observe that we can identify this affine space $\mathbb{A}^n$ with $\mathbb{X}(u_i)$ in $\mathbb{P}^n$ with affine coordinates $y_j = u_j u_i^{-1}$. That is, we can identify the projective closure of $\text{rep}_1(B_i)$, the affine variety of all one-dimensional representations of $B_i$, with the following subvariety of $\mathbb{P}^n$

$$\overline{\text{rep}_1(B_i)} = \bigcap_{j \neq i \neq k} \mathbb{V}((q_{i,k} - q_{i,j} q_{j,k}) u_j u_k). \tag{7.11}$$

*Proof of theorem 7.1(1).* Let $A = \mathbb{C}\langle x_0, x_1, x_2 \rangle / (x_i x_j - q_{i,j} x_j x_i, 0 \leq i, j \leq 2)$ be a skew polynomial algebra in 3 variables. Then $\text{pts}(A)$ is determined (see [13]) by the determinant of the following matrix

$$\begin{bmatrix} -q_{0,1} u_1 & u_0 & 0 \\ 0 & -q_{1,2} u_2 & u_1 \\ -q_{0,2} u_2 & 0 & u_0 \end{bmatrix} \tag{7.12}$$

which is equal to $(q_{0,1} q_{1,2} - q_{0,2}) u_0 u_1 u_2$, hence

$$\text{pts}(A) = \mathbb{P}(0,1) \cup \mathbb{P}(0,2) \cup \mathbb{P}(1,2) \text{ or } \text{pts}(A) = \mathbb{P}(0,1,2). \tag{7.13}$$

This proves the claim for $n = 2$.

Let $A$ now be a skew polynomial algebra in $n+1$ variables. If $P$ is a point module of $A$, then each of the variables $x_i$ (being normalising elements) either acts as zero on $P$ or as a non-zero divisor. At least one of the $x_i$ must act as a non-zero divisor, because otherwise $P \simeq \mathbb{C} = A/(x_0, \ldots, x_n)$. But then the localization $P[x_i^{-1}]$ is a graded module over the strongly graded ring $B_i[x_i, x_i^{-1}, \sigma]$ and hence is fully determined by its part of degree zero $(P[x_i^{-1}])_0$, see [12, proposition 7.5], which is a one-dimensional representation of $B_i$ and so $P$ determines a unique point of $\text{rep}_1(B_i)$ described above. Hence, we have the decomposition

$$\text{pts}(A) = \text{rep}_1(B_i) \sqcup \text{pts}(A/(x_i))). \tag{7.14}$$

The quotient $A/(x_i)$ is a skew polynomial algebra in $n$ variables. Hence by induction, we have

$$\text{pts}(A/(x_i)) = \bigcap_{j\neq i, k\neq i, l\neq i} \mathbb{V}((q_{j,l} - q_{j,k}q_{k,l})u_j u_k u_l) \cap \mathbb{V}(u_i). \tag{7.15}$$

But then we have

(7.16)
$$\begin{aligned}\text{pts}(A) &= \overline{\text{rep}_1(B_i)} \cup \text{pts}(A/(x_i)) \\ &= \bigcap_{j\neq i\neq k} \mathbb{V}((q_{i,k} - q_{i,j}q_{j,k})u_j u_k) \cup \bigcap_{j\neq i, k\neq i, l\neq i} \mathbb{V}((q_{j,l} - q_{j,k}q_{k,l})u_j u_k u_l) \cap \mathbb{V}(u_i) \\ &= \bigcap_{0\leq i<j<k\leq n} \mathbb{V}((q_{i,k} - q_{i,j}q_{j,k})u_i u_j u_k)\end{aligned}$$

where the last equality follows from the following lemma.

**Lemma 7.2.** Fix $0 \leq j < k < l \leq n$. If there exists an $i$ such that

$$\begin{cases} q_{i,k} - q_{i,j}q_{j,k} = 0, \\ q_{i,l} - q_{i,j}q_{j,l} = 0, \\ q_{i,l} - q_{i,k}q_{k,l} = 0, \end{cases} \tag{7.17}$$

then $q_{j,l} - q_{j,k}q_{k,l} = 0$.

*Proof.* Easy calculation. □

From the lemma it follows that if $u_j u_k u_l$ belongs to the defining ideal of $\text{pts}(A/(x_i))$, then necessarily for each $i$ either $u_j u_k$, $u_j u_l$ or $u_k u_l$ belongs to the defining ideal of $\overline{\text{rep}_1(B_i)}$. □

In particular, it follows that $\text{pts}(A) = \mathbb{P}^n$ if and only if for all distinct $i, j, k$ we have the relation

$$q_{j,k} = q_{i,k}q_{i,j}^{-1} \tag{7.18}$$

But then, all $2 \times 2$ submatrices of $Q$ have determinant zero as

$$\begin{bmatrix} q_{j,u} & q_{j,v} \\ q_{l,u} & q_{l,v} \end{bmatrix} = \begin{bmatrix} q_{i,u}q_{i,j}^{-1} & q_{i,v}q_{i,j}^{-1} \\ q_{i,u}q_{i,l}^{-1} & q_{i,v}q_{i,l}^{-1} \end{bmatrix} \tag{7.19}$$

and the same applies for $2 \times 2$ submatrices involving the $i$th row or column, so $Q$ must have rank one.

*Proof of theorem 7.1(2).* Observe that

$$(7.20)\quad \mathbb{P}(i_0, \ldots, i_k) = \mathbb{V}(u_{j_1}, \ldots, u_{j_{n-k}})$$

where

$$(7.21)\quad \{0, 1, \ldots, n\} = \{i_0, \ldots, i_k\} \sqcup \{j_1, \ldots, j_{n-k}\}.$$

Therefore, $\mathbb{P}(i_0, \ldots, i_k) \subset \mathrm{pts}(A)$ if and only if

$$(7.22)\quad \mathbb{P}(i_0, \ldots, i_k) = \mathrm{pts}(\overline{A}) \quad \text{with} \quad \overline{A} = \frac{A}{(x_{j_1}, \ldots, x_{j_{n-k}})}$$

and as $\overline{A}$ is again a skew polynomial algebra with corresponding matrix $Q(i_0, \ldots, i_k)$ it follows from the remark above that $\mathrm{rk}\, Q(i_0, \ldots, i_k) = 1$. □

*Proof of theorem 7.1(3).* Recall that $\mathbb{P}(u, v, w) \subset \mathrm{pts}(A)$ if and only if $Q(u, v, w)$ has rank one, which is equivalent to $q_{u,w} = q_{u,v} q_{v,w}$. The statement now follows from theorem 7.1(1). □

**Remark 7.3.** Observe that point varieties of skew polynomial algebras always contain the 1-skeleton of coordinate $\mathbb{P}^1$'s as the principal $2 \times 2$-submatrices always have rank 1. This will also be the generic configuration for skew polynomial algebras. Note that in general a noncommutative $\mathbb{P}^n$ can have no points or only a finite number of point modules; see [230] for examples when $n = 3$. So skew polynomial algebras exhibit very special behaviour among *all* Artin–Schelter regular algebras for $n \geq 3$, just like Sklyanin algebras or the algebras from [151] exhibit this special behaviour.

## 7.3 Possible configurations

Not all configurations of linear subspaces of the above type can occur as point varieties of skew polynomial algebras.

**Example 7.4.** In $\mathbb{P}^3$ only two of the $\mathbb{P}^2$'s (out of four in total) can arise in a proper subvariety $\mathrm{pts}(A) \subsetneq \mathbb{P}^3$. For example, take

$$(7.23)\quad Q = \begin{bmatrix} 1 & a & b & x \\ a^{-1} & 1 & a^{-1}b & c \\ b^{-1} & ab^{-1} & 1 & ab^{-1}c \\ x^{-1} & c^{-1} & a^{-1}bc^{-1} & 1 \end{bmatrix},$$

then, for generic $a, b, c, x$ we have

$$(7.24)\quad \mathrm{pts}(A) = \mathbb{P}(0, 1, 2) \cup \mathbb{P}(1, 2, 3) \cup \mathbb{P}(0, 3).$$

However, if we include another $\mathbb{P}^2$, for example, $\mathbb{P}(0, 1, 3)$ we need the relation $x = ac$ in which case $Q$ becomes of rank one, whence $\mathrm{pts}(A) = \mathbb{P}^3$. This is a consequence of lemma 7.2.

We will present a combinatorial description of all possible configurations in low dimensions. Let $C$ be a collection of $\mathbb{P}^2 = \mathbb{P}(i,j,k)$ contained in $\mathbb{P}^n$. We say that $C$ is *adequate* if the following condition is satisfied

(7.25) $\forall\ 0 \le i \le n, \forall\ \mathbb{P}(j,k,l) \in C, \exists\ \{u,v\} \subset \{j,k,l\}$ such that $\mathbb{P}(i,u,v) \in C$.

Adequacy gives a necessary condition on the collection of $\mathbb{P}^2$'s not contained in the point variety of a skew polynomial algebra.

**Proposition 7.5.** If $A$ is a skew polynomial algebra, then

(7.26) $C_A = \{\mathbb{P}(i,j,k) \mid \mathbb{P}(i,j,k) \not\subset \text{pts}(A)\}$

is an adequate collection.

*Proof.* This is an immediate consequence of lemma 7.2. □

The collection of all coordinates $(q_{i,j})_{i<j}$ in the torus of dimension $\binom{n+1}{2}$ describing skew polynomial algebras with the same reduced point variety is an open subset $T$ of a torus with complement certain subtori describing the coordinates of skew algebras with larger point variety.

In example 7.4 we have $C_A = \{\mathbb{P}(0,1,3), \mathbb{P}(0,2,3)\}$ and $T$ is the complement of $\mathbb{G}_m^4$ (with coordinates $a,b,c,x$) by the sub-torus $\mathbb{G}_m^3$ defined by $x = ac$, describing skew polynomial algebras with point variety $\mathbb{P}^3$. Here, $C_A$ is adequate, but for example $C = \{\mathbb{P}(0,1,3)\}$ is not. In fact, for $n = 3$ it is easy to check that all collections are adequate apart from the singletons, so there are exactly 12 adequate collections.

We say that a collection $C$ of $\mathbb{P}^2$'s in $\mathbb{P}^n$ is *dense* if there exist $0 \le i < j \le n$ such that

(7.27) $\#\,\{\mathbb{P}(i,j,k) \in C\} \ge n-2$

where $k \ne i,j$. For small $n$, adequate collections are always dense.

**Proposition 7.6.** For $n \le 4$ all adequate collections $C$ are dense unless $C = \emptyset$.

*Proof.* For $n = 2$, the proof is trivial. For $n = 3$, it is easily seen that $C = \emptyset$ is the only non-dense collection.

Assume now that $n = 4$ and that $C$ is a non-dense collection. Then we have for all $0 \le i < j \le 4$ that

(7.28) $\#\,\{\mathbb{P}(i,j,k) \in C\} = 0, 1.$

If this quantity is always equal to 0 then $C = \emptyset$, which is adequate. Hence, assume that one of these quantities is equal to 1. Up to permutation by $\text{Sym}_5$, we may assume that $\mathbb{P}(0,1,2) \in C$. Then the only other possible $\mathbb{P}(i,j,k)$ belonging to $C$ is $\mathbb{P}(i,3,4)$ with $i$ either 0, 1 or 2. Again up to permutation, we may assume $i = 0$. But neither the collection $\{\mathbb{P}(0,1,2)\}$ nor $\{\mathbb{P}(0,1,2), \{\mathbb{P}(0,3,4)\}\}$ are adequate (in both cases, take $i = 3$ and $\mathbb{P}(0,1,2)$). □

We can now characterise the possible configurations in small dimensions.

**Theorem 7.7.** Assume $n \leq 5$ and let $C$ be an adequate and dense collection of $\mathbb{P}^2$'s in $\mathbb{P}^n$ with variables $u_i$ for $0 \leq i \leq n$. Then

$$(7.29)\quad \mathbb{V}(u_i u_j u_k \mid \mathbb{P}(i,j,k) \in C)$$

is the point variety $\text{pts}(A)$ of a skew polynomial algebra $A$ with $C = C_A$.

*Proof.* Renumbering the variables if necessary we may assume by density that $\mathbb{P}(0,n)$ is contained in at least $n-2$ of $\mathbb{P}(0,i,n) \in C$. We can write $C$ as a disjoint union

$$(7.30)\quad C = C_1 \sqcup C_2 \sqcup C_3 \sqcup C_4$$

where

$$(7.31)\quad \begin{aligned} C_1 &:= \{\mathbb{P}(p,q,r) \in C \mid p,q,r \notin \{0,n\}\}, \\ C_2 &:= \{\mathbb{P}(0,p,q) \in C \mid p,q \neq 0,n\}, \\ C_3 &:= \{\mathbb{P}(p,q,n) \in C \mid p,q \neq 0,n\}, \\ C_4 &:= \{\mathbb{P}(0,p,n) \in C \mid p \notin \{0,n\}\}. \end{aligned}$$

Note that $\#C_4 \geq n-2$. By adequacy of $C$ we have that $C_1$ is adequate in the variables $u_i$ for $1 \leq i \leq n-1$, $C_1 \sqcup C_2$ is adequate in the variables $u_i$ with $0 \leq i \leq n-1$ and $C_1 \sqcup C_3$ is adequate in the variables $u_i$ with $1 \leq i \leq n$. Therefore, they are dense by proposition 7.6.

Hence, by applying the induction hypothesis twice (which is possible by proposition 7.5), a first time with generic values for $C_1 \sqcup C_2$ and afterwards with specific values for $C_1 \sqcup C_3$, and evaluating the generic values accordingly, we obtain a matrix with non-zero entries

$$(7.32)\quad Q = \begin{bmatrix} 1 & q_{0,1} & \cdots & q_{0,n-1} & x \\ q_{0,1}^{-1} & 1 & \cdots & q_{1,n-1} & q_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{0,n-1}^{-1} & q_{1,n-1}^{-1} & \cdots & 1 & q_{n-1,n} \\ x^{-1} & q_{1,n}^{-1} & \cdots & q_{n-1,n}^{-1} & 1 \end{bmatrix}$$

such that for all principal $3 \times 3$ submatrices $Q(i,j,k)$ with $\{0,n\} \not\subset \{i,j,k\}$ we have

$$(7.33)\quad \operatorname{rk} Q(i,j,k) = 1 \quad \text{if and only if} \quad \mathbb{P}(i,j,k) \notin C_1 \sqcup C_2 \sqcup C_3.$$

But then, the same condition is satisfied for all the matrices

$$(7.34)\quad Q_\lambda = \begin{bmatrix} 1 & q_{0,1} & \cdots & q_{0,n-1} & x \\ q_{0,1}^{-1} & 1 & \cdots & q_{1,n-1} & \lambda q_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{0,n-1}^{-1} & q_{1,n-1}^{-1} & \cdots & 1 & \lambda q_{n-1,n} \\ x^{-1} & \lambda^{-1} q_{1,n}^{-1} & \cdots & \lambda^{-1} q_{n-1,n}^{-1} & 1 \end{bmatrix}$$

with $\lambda \in \mathbb{G}_\text{m}$.

If $\#C_4 = n-1$, a generic value of $x$ will ensure that $\operatorname{rk} Q(0,j,n) > 1$ for $1 \le j \le n-1$.

If $\#C_4 = n-2$ let $i$ be the unique entry $1 \le i \le n-1$ such that $\mathbb{P}(0,i,n) \notin C$, then the rank-one condition on

$$Q(0,i,n) = \begin{bmatrix} 1 & q_{0,i} & x \\ q_{0,i}^{-1} & 1 & \lambda q_{i,n} \\ x^{-1} & \lambda^{-1} q_{i,n}^{-1} & 1 \end{bmatrix} \quad \text{implies} \quad \lambda = q_{0,i}^{-1} q_{i,n}^{-1} x \tag{7.35}$$

and for generic $x$ we can ensure that for all other $1 \le j \ne i \le n-1$ we have that $\operatorname{rk} Q(0,j,n) > 1$. □

One can verify that, up to the $\mathrm{Sym}_6$-action on the variables $u_i$, there are exactly two nonempty adequate collections for $n = 5$ that are *not* dense, which are:

$$\begin{aligned} \mathcal{A} = \{&\mathbb{P}(0,2,4), \mathbb{P}(0,2,5), \mathbb{P}(0,3,4), \mathbb{P}(0,3,5), \mathbb{P}(1,2,4), \mathbb{P}(1,2,5), \\ &\mathbb{P}(1,3,4), \mathbb{P}(1,3,5)\} \end{aligned} \tag{7.36}$$

and

$$\begin{aligned} \mathcal{B} = \{&\mathbb{P}(0,1,3), \mathbb{P}(0,1,5), \mathbb{P}(0,2,4), \mathbb{P}(0,4,5), \mathbb{P}(0,2,3), \mathbb{P}(1,2,4), \\ &\mathbb{P}(1,2,5), \mathbb{P}(1,3,4), \mathbb{P}(2,3,5), \mathbb{P}(3,4,5)\}. \end{aligned} \tag{7.37}$$

The configuration $\mathcal{A}$ is realisable as $C_A$ for a skew polynomial algebra $A$ with matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & x & x \\ 1 & 1 & 1 & 1 & x & x \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ x^{-1} & x^{-1} & 1 & 1 & 1 & 1 \\ x^{-1} & x^{-1} & 1 & 1 & 1 & 1 \end{bmatrix} \tag{7.38}$$

and has as point variety

$$\operatorname{pts}(A) = \mathbb{P}(0,1,2,3) \cup \mathbb{P}(0,1,4,5) \cup \mathbb{P}(2,3,4,5) \tag{7.39}$$

for generic values of $x$.

The configuration $\mathcal{B}$ is realisable as $C_{A'}$ for the skew polynomial algebra $A'$ with defining matrix

$$\begin{bmatrix} 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{7.40}$$

The point variety of this algebra is

$$\begin{aligned} \operatorname{pts}(A') = &\mathbb{P}(0,1,2) \cup \mathbb{P}(1,2,3) \cup \mathbb{P}(2,3,4) \cup \mathbb{P}(0,3,4) \cup \mathbb{P}(0,1,4) \cup \\ &\mathbb{P}(0,2,5) \cup \mathbb{P}(1,3,5) \cup \mathbb{P}(2,4,5) \cup \mathbb{P}(0,3,5) \cup \mathbb{P}(1,4,5) \end{aligned} \tag{7.41}$$

This shows that density is too strong a condition for $C$ to be realised as $C_A$ for some skew polynomial algebra $A$. However, these results may imply that adequacy is a sufficient condition. In particular, all 175 $\mathrm{Sym}_6$-equivalence classes of adequate collections in dimension 5 can be realised as the collection of $\mathbb{P}^2$'s not contained in the point variety of a skew polynomial algebra on 6 variables.

## 7.4 Degeneration graphs

Let $\mathbb{T}_{2,n}$ be the $\binom{n+1}{2}$-dimensional torus parametrising skew polynomial algebras as before with coordinate functions $(q_{i,j})_{i<j}$.

Put $b_{i,j,k} = q_{i,j}q_{j,k}q_{i,k}^{-1}$ for $0 \leq i < j < k \leq n$ and let

(7.42) $I = \{b_{i,j,k} - 1 \mid 0 \leq i < j < k \leq n\}$.

For each $J \subset I$, we obtain a subtorus of $\mathbb{T}_{2,n}$ by taking $\mathbb{V}(J)$. Note however that $\mathbb{V}(J)$ can be equal to $\mathbb{V}(K)$ although $J \neq K$.

We obtain this way a degeneration graph by letting the nodes correspond to possible $\mathbb{V}(J)$, $J \subset I$ and an arrow $\mathbb{V}(J) \to \mathbb{V}(K)$ if $\mathbb{V}(K) \subset \mathbb{V}(J)$.

From the above description of point varieties of skew polynomial algebras, we see that the inclusion graph corresponds to degenerations of skew polynomial algebras to other skew polynomial algebras with a larger point variety.

Some considerations must be made in the calculations of these graphs:

- Let $\mathbb{T}_{3,n}$ be the $\binom{n+1}{3}$-dimensional torus with coordinate functions $(b_{i,j,k})_{i<j<k}$. Then the map $\mathbb{T}_{2,n} \to \mathbb{T}_{3,n}$ defined by $b_{i,j,k} = q_{i,j}q_{j,k}q_{i,k}^{-1}$ is a map of algebraic groups. The kernel $\mathbb{K}$ of this map is a $n$-dimensional torus (as $\mathrm{rk}(Q) = 1$ for all $Q$ associated to elements of $\mathbb{K}$) which acts freely on each $\mathbb{V}(J)$ in the obvious way by

  (7.43) $\mathbb{K} \times \mathbb{V}(J) \to \mathbb{V}(J) : (x, g) \mapsto xg.$

  Therefore, each $\mathbb{V}(J)$ is at least $n$-dimensional, as each subvariety $\mathbb{V}(J)$ contains at least 1 $\mathbb{K}$-orbit.

- The nodes in our graphs are possible subtori up to $\mathrm{Sym}_{n+1}$-action on the variables of the skew polynomial algebras.

- An endpoint corresponds to a unique $\mathbb{K}$-orbit or equivalently to a $n$-dimensional subvariety. For if $\mathbb{V}(J)$ is not $n$-dimensional, then its image in $\mathbb{T}_{3,n}$ is not constant. This implies that there exists a 3-tuple $(i, j, k)$ such that taking $b_{i,j,k} = 1$ gives a subvariety of 1 dimension lower, which corresponds to a larger point variety than the one determined by $\mathbb{V}(J)$.

For $n = 2, 3, 4$, we have calculated the complete degeneration graphs using these methods.

### 7.4.1 Quantum $\mathbb{P}^2$'s

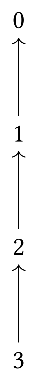
This case is classical: the point variety is either $\mathbb{P}^2$ or the union of the 3 coordinate $\mathbb{P}^1$'s [13].

### 7.4.2 Quantum $\mathbb{P}^3$'s

The degeneration graph is given in figure 7.1. One can easily check by hand that there are 12 adequate collections, and they fall into 4 $\mathrm{Sym}_4$-orbits.

The *label* for a configuration corresponds to the dimension of the loci (in $\mathbb{T}_{2,n}$) parametrising these configurations. The *type* of a configuration describes how many copies of $\mathbb{P}^k$'s there are as irreducible components in the point variety. The commutative situation where the point variety is the whole of $\mathbb{P}^3$ therefore is labeled by 0 and has type $(1, 0, 0)$, whereas the most generic situation (labeled by 3) corresponds to 6 $\mathbb{P}^1$'s whose type we denote by $(0, 0, 6)$.

In this case the degeneration graph is totally ordered, with example 7.4 corresponding to the configuration of type $(0, 2, 1)$.




| label | type |
|---|---|
| 0 | $(1, 0, 0)$ |
| 1 | $(0, 2, 1)$ |
| 2 | $(0, 1, 3)$ |
| 3 | $(0, 0, 6)$ |

Figure 7.1: Degeneration graph for skew $\mathbb{P}^3$'s

### 7.4.3 Quantum $\mathbb{P}^4$'s

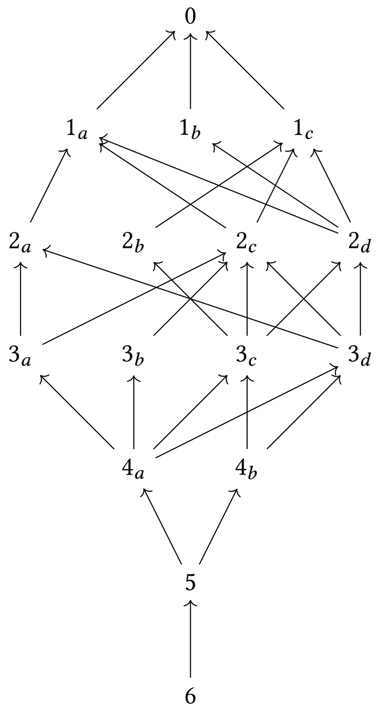
The degeneration graph is given in figure 7.2. There are in total 314 adequate collections, falling into 16 $\mathrm{Sym}_5$-orbits.

This time, the degeneration graph no longer is totally ordered. For example, take the configurations $4_a$ and $4_b$. These differ by *how* the two $\mathbb{P}^2$'s intersect: as we are working in an ambient $\mathbb{P}^4$ this happens in either a point or a line. Via similar arguments it is possible to describe each of these configurations.

Observe that $3_a$ and $3_c$ have the same type, but they are not the same configuration:

- $3_c$ corresponds to three $\mathbb{P}^2$'s intersecting in a common $\mathbb{P}^1$;

- $3_a$ has two $\mathbb{P}^2$'s intersecting only in a point and a third $\mathbb{P}^2$ intersecting the first in two different $\mathbb{P}^1$'s.




| label | type |
|---|---|
| 0 | $(1, 0, 0, 0)$ |
| $1_a$ | $(0, 1, 2, 0)$ |
| $1_b$ | $(0, 0, 5, 0)$ |
| $1_c$ | $(0, 2, 0, 1)$ |
| $2_a$ | $(0, 0, 4, 0)$ |
| $2_b$ | $(0, 0, 4, 2)$ |
| $2_c$ | $(0, 1, 1, 2)$ |
| $2_d$ | $(0, 0, 4, 1)$ |
| $3_a$ | $(0, 0, 3, 3)$ |
| $3_b$ | $(0, 1, 0, 4)$ |
| $3_c$ | $(0, 0, 3, 3)$ |
| $3_d$ | $(0, 0, 3, 2)$ |
| $4_a$ | $(0, 0, 2, 5)$ |
| $4_b$ | $(0, 0, 2, 4)$ |
| 5 | $(0, 0, 1, 7)$ |
| 6 | $(0, 0, 0, 10)$ |

Figure 7.2: Degeneration graph for skew $\mathbb{P}^4$'s

### 7.4.4 Quantum $\mathbb{P}^5$'s

For $n = 5$, we observe the following phenomenon.

**Theorem 7.8.** There are at least 2 endpoints in the degeneration graph for skew polynomial algebras in 6 variables.

*Proof.* An endpoint in the graph corresponds to a $n$-dimensional family of skew polynomial algebras. Let $C$ be the collection

$$\begin{aligned}\{&\mathbb{P}(0,1,2), \mathbb{P}(1,2,3), \mathbb{P}(2,3,4), \mathbb{P}(0,3,4), \mathbb{P}(0,1,4), \\ &\mathbb{P}(0,2,5), \mathbb{P}(1,3,5), \mathbb{P}(2,4,5), \mathbb{P}(0,3,5), \mathbb{P}(1,4,5)\}.\end{aligned} \tag{7.44}$$

Then the complement of $C$ is adequate. We have already constructed an algebra $A'$ with exactly the union of these $\mathbb{P}^2$'s in its point variety. We will show that the family of skew polynomial algebras with these $\mathbb{P}^2$ in its point variety is 5-dimensional. Using the action of $\mathbb{K}$, we may assume that for all $0 \le i \le 4$ we have $q_{i,5} = 1$. If we can now show that there are a finite number of solutions, we are done as we have used up all degrees of freedom. It follows from the second row of $\mathbb{P}^2$'s in the point variety that

$$q_{0,2} = q_{1,3} = q_{2,4} = q_{0,3} = q_{1,4} = 1. \tag{7.45}$$

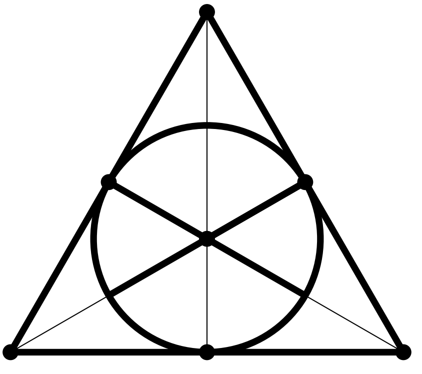
Using the first four $\mathbb{P}^2$'s, we get the conditions

$$q_{0,1} = q_{2,3} = a, \quad q_{1,2} = q_{3,4} = q_{0,4} = a^{-1}. \tag{7.46}$$

Now, the subspace $\mathbb{P}(0, 1, 4)$ belongs to the point variety if and only if $a = a^{-1}$ or equivalently, $a = \pm 1$. The case $a = 1$ leads to the commutative polynomial algebra, while $a = -1$ gives a skew polynomial algebra with exactly these 10 $\mathbb{P}^2$'s in its point variety. □

*Chapter 8*

# Relative tensor triangular Chow groups for coherent algebras



## 8.1 Introduction

In [126], Klein defined and began the study of *relative tensor triangular Chow groups*, a family of K-theoretic invariants attached to a compactly generated triangulated category $\mathcal{K}$ with an action of a rigidly-compactly generated tensor triangulated category $\mathcal{T}$ in the sense of [207]. While in [126], they were used to improve upon and extend results of [125], the initial observation of the present work is that they allow us to enter the realm of *noncommutative* algebraic geometry: if $X$ is a noetherian scheme and $\mathcal{A}$ a (possibly noncommutative) coherent $\mathcal{O}_X$-algebra, then the derived category $\mathcal{K} := \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ admits an action by $\mathcal{T} := \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ which is obtained by deriving the tensor product functor

$$\begin{aligned} \mathrm{Qcoh}(\mathcal{A}) \times \mathrm{Qcoh}(\mathcal{O}_X) &\to \mathrm{Qcoh}(\mathcal{A}) \\ (M, F) &\mapsto M \otimes_{\mathcal{O}_X} F. \end{aligned} \tag{8.1}$$

In this situation, the general machinery of [126] gives us abelian groups $\mathrm{Z}_i^{\Delta}(X, \mathcal{A})$ and $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$, the degree $i$ tensor triangular cycle and Chow groups of $\mathcal{K}$ relative to the action of $\mathcal{T}$. In the test case where $\mathcal{A}$ is coherent and commutative, and hence $\mathcal{A}$ corresponds to a scheme $\mathbf{Spec}\,\mathcal{A}$ and a finite morphism $\mathbf{Spec}\,\mathcal{A} \to X$, we show that $\mathrm{Z}_i^{\Delta}(X, \mathcal{A})$ and $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$ agree with the degree $i$ tensor triangular cycle and Chow groups of $\mathbf{D}^{\mathrm{b}}(\mathbf{Spec}\,\mathcal{A})$ as defined in [125], and hence with the usual dimension $i$ cycle and Chow groups of $\mathrm{Z}_i(\mathbf{Spec}\,\mathcal{A}), \mathrm{CH}_i(\mathbf{Spec}\,\mathcal{A})$ when $\mathbf{Spec}\,\mathcal{A}$ is a regular algebraic variety (see theorem 8.63). This computation serves as a motivation to study the groups $\mathrm{Z}_i^{\Delta}(X, \mathcal{A})$ and $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$ for noncommutative coherent $\mathcal{A}$.

We obtain computations of both invariants when $\mathcal{A}$ is a sheaf of hereditary orders on a curve in section 8.7, and in particular $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$ recovers the classical *stable class group* in this case. We also briefly touch upon the subjects of maximal orders on a surface and orders over a singular base, in the context of noncommutative resolutions of singularities. The case of a finite group algebra over $\mathrm{Spec}\,\mathbb{Z}$ is discussed as a final example.

Let us highlight that the main ingredient for the calculations carried out in this chapter is a new exact sequence which is established in section 8.3 for a general rigidly-compactly generated tensor triangulated category $\mathcal{T}$ acting on a compactly generated triangulated category $\mathcal{K}$.

For the case $\mathcal{K} := \mathbf{D}(\mathrm{Qcoh}(\mathcal{A})), \mathcal{T} := \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ this sequence is

$$0 \to \mathrm{CH}_p^{\Delta}(X, \mathcal{A}) \to \mathrm{K}_0\left(\left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p-1)}\right)^{\mathrm{c}}\right) \to \mathrm{Z}_{p+1}^{\Delta}(X, \mathcal{A}). \tag{8.2}$$

The middle term of the sequence is the Grothendieck group of the subcategory of compact objects of a subquotient of the filtration of $\mathcal{K}$ by dimension of support in $\mathrm{Spc}(\mathcal{T}^{\mathrm{c}})$.

The chapter is structured as follows: in section 8.2 we recall all relevant notions from tensor triangular geometry and the definition of relative tensor triangular cycle and Chow groups. We then establish the exact sequence mentioned above in section 8.3. In section 8.4 we prove some auxilliary results concerning the categories $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ and $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A}))$, most of which should be known to the experts. In section 8.5 we discuss the action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ and contemplate the definition of tensor triangular cycle and Chow groups in this more specific context, including a map $\mathrm{CH}_i^{\Delta}(X, \mathcal{A}) \to \mathrm{CH}_i(X)$ for regular $X$, induced by the forgetful functor $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \to \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$. We then have a look at commutative coherent $\mathcal{O}_X$-algebras in section 8.6 and carry out our computations for orders in section 8.7. The results in section 8.7 motivate the study of relative Chow groups for coherent $\mathcal{O}_X$-algebras in general, by showing that they agree with various invariants in the literature which were defined in an ad hoc way.

## 8.2 Tensor triangular preliminaries

In this section we recall the categorical notions we need. None of the following material is new, our main sources are [23, 24, 126, 207].

### 8.2.1 Tensor triangular geometry

Let us quickly recall the basics of Balmer's tensor triangular geometry. See e.g. [23] for a reference that covers all the material we need (and much more).

**Definition 8.1.** A *tensor triangulated category* is an essentially small triangulated category $\mathcal{C}$ equipped with a symmetric monoidal structure $\otimes$ with unit $\mathbb{I}$ such that the functors $a \otimes -$ are exact for all objects $a \in \mathcal{C}$.

To every tensor triangulated category $\mathcal{C}$, we associate its *Balmer spectrum* $\mathrm{Spc}(\mathcal{C})$, a topological space that is constructed in analogy with the prime ideal spectrum of a commutative ring. By construction of $\mathrm{Spc}(\mathcal{C})$, every object $a \in \mathcal{C}$ has a closed *support* $\mathrm{supp}(a) \subset \mathrm{Spc}(\mathcal{C})$, which satisfies the identities

- $\mathrm{supp}(0) = \emptyset$ and $\mathrm{supp}(\mathbb{I}) = \mathrm{Spc}(\mathcal{C})$,
- $\mathrm{supp}(\Sigma a) = \mathrm{supp}(a)$,
- $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$,
- $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$,
- $\mathrm{supp}(b) \subset \mathrm{supp}(a) \cup \mathrm{supp}(c)$ whenever there is a distinguished triangle

  (8.3) $a \to b \to c \to \Sigma a.$

for all objects $a, b, c \in \mathcal{C}$. One can show that, in a precise sense, the space $\mathrm{Spc}(\mathcal{C})$ and the support function supp are optimal among all pairs of spaces and support functions satisfying the above criteria.

**Example 8.2.** If $X$ is a quasi-compact, quasi-separated scheme, then $\mathcal{C} = \mathbf{D}^{\mathrm{perf}}(X)$, the derived category of perfect complexes on $X$, is a tensor triangulated category with tensor product $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$. We have $\mathrm{Spc}(\mathcal{C}) \cong X$ and under this identification the support $\mathrm{supp}(C^\bullet)$ of some complex $C^\bullet$ is identified with the complement of the set of points $x \in X$ such that $C^\bullet_x$ is acyclic, or equivalently with the support of the total cohomology sheaf $\mathrm{H}^\bullet(C^\bullet) := \bigoplus_i \mathrm{H}^i(C^\bullet)$.

The spectrum $\mathrm{Spc}(\mathcal{C})$ is always a *spectral* topological space, i.e. it is homeomorphic to the prime ideal spectrum of some (usually unknown) commutative ring. Hence, it makes sense to talk about the Krull (co)-dimension of points in $\mathrm{Spc}(\mathcal{C})$. For a subset $S \subset \mathrm{Spc}(\mathcal{C})$, we define

$$\dim(S) := \max_{P \in S} \dim(P) \quad \text{and} \quad \operatorname{codim}(S) := \min_{P \in S} \operatorname{codim}(P), \tag{8.4}$$

where we set $\dim(\emptyset) = -\infty$, $\operatorname{codim}(\emptyset) = \infty$.

### 8.2.2 Supports in large categories

Let $\mathcal{T}$ be a triangulated category.

**Definition 8.3.** The category $\mathcal{T}$ is called a *rigidly-compactly generated tensor triangulated category* if

(i) $\mathcal{T}$ *is compactly generated.* We implicitly assume here that $\mathcal{T}$ has set-indexed coproducts. Note that this implies that $\mathcal{T}$ is not essentially small.

(ii) $\mathcal{T}$ *is equipped with a* compatible closed symmetric monoidal structure

$$\otimes \colon \mathcal{T} \times \mathcal{T} \to \mathcal{T} \tag{8.5}$$

*with unit object* $\mathbb{I}$. Here, a symmetric monoidal structure on $\mathcal{T}$ is *closed* if for all objects $A \in \mathcal{T}$ the functor $A \otimes -$ has a right adjoint $\underline{\mathrm{hom}}(A, -)$. Note that this condition implies that $\otimes$ preserves coproducts in both variables. A *compatible* closed symmetric monoidal structure on $\mathcal{T}$ is one such that the functor $\otimes$ is exact in both variable and such that the two ways of identifying $\Sigma(x) \otimes \Sigma(y)$ with $\Sigma^2(x \otimes y)$ are the same up to a sign. Since adjoints of exact functors are exact (see [169, lemma 5.3.6]) we automatically have that the functor $\underline{\mathrm{hom}}(A, -)$ is exact for all objects $A \in \mathcal{T}$.

(iii) $\mathbb{I}$ *is compact and all compact objects of* $\mathcal{T}$ *are rigid.* Let $\mathcal{T}^c \subset \mathcal{T}$ denote the full subcategory of compact objects of $\mathcal{T}$. Then we require that $\mathbb{I} \in \mathcal{T}^c$ and that all objects $A$ of $\mathcal{T}^c$ are rigid, i.e. for every object $B \in \mathcal{T}$ the natural map

$$\underline{\circ} \colon \underline{\mathrm{hom}}(A, \mathbb{I}) \otimes B \cong \underline{\mathrm{hom}}(A, \mathbb{I}) \otimes \underline{\mathrm{hom}}(\mathbb{I}, B) \to \underline{\mathrm{hom}}(A, B), \tag{8.6}$$

is an isomorphism.

The subcategory $\mathcal{T}^c$ of a rigidly-compactly generated tensor triangulated category $\mathcal{T}$ is a tensor triangulated category in the sense of definition 8.1. Hence, it makes sense to talk about the spectrum $\mathrm{Spc}(\mathcal{T}^c)$.

**Convention 8.4.** Throughout this section we assume that $\mathcal{T}$ is a compactly-rigidly generated tensor triangulated category. *We also assume that* $\mathrm{Spc}(\mathcal{T}^c)$ *is a noetherian topological space.*

**Example 8.5.** If $X$ is a quasi-compact, quasi-separated scheme, then $\mathcal{T} = \mathbf{D}_{Qcoh}(X)$, the derived category of complexes of $\mathcal{O}_X$-modules with quasicoherent cohomology is a rigidly-compactly generated tensor triangulated category with tensor product $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$.

The rigid-compact objects are the perfect complexes in $\mathcal{T}$. By example 8.2, $\mathrm{Spc}(\mathcal{T}^c) = X$ and the condition of convention 8.4 hence holds whenever the space $|X|$ is noetherian, e.g. when $X$ is noetherian. If $X$ is noetherian and separated, $\mathbf{D}_{\mathrm{Qcoh}}(X)$ is equivalent to $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$.

Rigidly-compactly generated tensor triangulated categories come with an associated support theory that extends the notion of support in an essentially small tensor triangulated category. Let us briefly review the theory as introduced in [24]. First recall the concepts of Bousfield and smashing subcategories:

**Definition 8.6.** A thick triangulated subcategory $\mathcal{J} \subset \mathcal{T}$ is *Bousfield* if the Verdier quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{J}$ exists and has a right adjoint. A Bousfield subcategory $\mathcal{J} \subset \mathcal{T}$ is called *smashing* if the right adjoint of the Verdier quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{J}$ preserves coproducts.

If $\mathcal{J}$ is a Bousfield subcategory, there exists a localization functor $\mathrm{L}_{\mathcal{J}}\colon \mathcal{T} \to \mathcal{T}$ (given as the composition of the Verdier quotient $\mathcal{T} \to \mathcal{T}/\mathcal{J}$ and its right adjoint) such that $\mathcal{J} = \ker(\mathrm{L}_{\mathcal{J}})$ and the composition of functors

$$\mathcal{J}^{\perp} \to \mathcal{T} \to \mathcal{T}/\mathcal{J} \tag{8.7}$$

is an exact equivalence, where $\mathcal{J}^{\perp}$ is the full subcategory consisting of those $t \in \mathrm{Ob}(\mathcal{T})$ such that $\mathrm{Hom}(c, t) = 0$ for all $c \in \mathrm{Ob}(\mathcal{J})$. A quasi-inverse of the equivalence is given by the right adjoint of the Verdier quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{J}$. This says that we can actually realise the Verdier quotient $\mathcal{T}/\mathcal{J}$ inside of $\mathcal{T}$ and we will freely (and slightly abusively) confuse $\mathcal{T}/\mathcal{J}$ with $\mathcal{J}^{\perp}$. Also recall, that for every object $a \in \mathcal{T}$ there is a distinguished *localization triangle*

$$\Gamma_{\mathcal{J}}(a) \to a \to \mathrm{L}_{\mathcal{J}}(a) \to \Sigma(\Gamma_{\mathcal{J}}(a)) \tag{8.8}$$

which is unique among triangles $x \to a \to y \to \Sigma(x)$ with $x \in \mathcal{J}$ and $y \in \mathcal{J}^{\perp}$, up to unique isomorphism of triangles that restrict to the identity on $a$. This defines a functor $\Gamma_{\mathcal{J}}(-)$ on $\mathcal{T}$ with essential image $\mathcal{J}$. The functor $\Gamma_{\mathcal{J}}$ is a *colocalization functor*, i.e. $\Gamma_{\mathcal{J}}^{\mathrm{op}}$ is a localization functor on $\mathcal{T}^{\mathrm{op}}$.

**Definition 8.7.** A triangulated subcategory $\mathcal{J} \subset \mathcal{T}$ is called

- $\otimes$*-ideal* if $\mathcal{T} \otimes \mathcal{J} \subseteq \mathcal{J}$.
- *smashing ideal* if it is a $\otimes$-ideal, a Bousfield subcategory and $\mathcal{J}^{\perp} \subset \mathcal{T}$ is also a $\otimes$-ideal.

Smashing ideals are well-behaved: as they are Bousfield subcategories there exists a unique triangle

$$\Gamma_{\mathcal{J}}(\mathbb{I}) \to \mathbb{I} \to \mathrm{L}_{\mathcal{J}}(\mathbb{I}) \to \Sigma(\Gamma_{\mathcal{J}}(\mathbb{I})), \tag{8.9}$$

and by tensoring this triangle with $a \in \mathcal{T}$, we see that we must have $\mathrm{L}_{\mathcal{J}}(a) = \mathrm{L}_{\mathcal{J}}(\mathbb{I}) \otimes a$ and $\Gamma_{\mathcal{J}}(a) = \Gamma_{\mathcal{J}}(\mathbb{I}) \otimes a$.

**Remark 8.8.** Smashing ideals are smashing subcategories: $\mathrm{L}_{\mathcal{J}} = \Gamma_{\mathcal{J}}(\mathbb{I}) \otimes -$ preserves coproducts since it has a right adjoint by definition of a rigidly-compactly generated tensor triangulated category. It follows that the Verdier quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{J}$ must preserve coproducts as well.

An important tool for extending the notion of support from $\mathcal{T}^c$ to $\mathcal{T}$ is the following theorem [107, theorem 3.3.3].

**Theorem 8.9.** Let $\mathcal{S} \subset \mathcal{T}^c$ be a thick $\otimes$-ideal in $\mathcal{T}^c$ (i.e. $\mathcal{T}^c \otimes \mathcal{S} \cong \mathcal{S}$). Let $\langle \mathcal{S} \rangle$ denote the smallest triangulated subcategory of $\mathcal{T}$ that is closed under taking arbitrary coproducts (in $\mathcal{T}$). Then $\langle \mathcal{S} \rangle$ is a smashing ideal in $\mathcal{T}$ and $\langle \mathcal{S} \rangle^c \cong \mathcal{S}$.

**Definition 8.10.** Let $V \subset \mathrm{Spc}(\mathcal{T}^c)$ be a specialization-closed subset. We denote by $\mathcal{T}_V$ the smashing ideal $\langle \mathcal{T}^c_V \rangle$, where $\mathcal{T}^c_V \subset \mathcal{T}^c$ is the thick $\otimes$-ideal $\{a \in \mathcal{T}^c : \mathrm{supp}(a) \subset V\}$. We denote the two associated localization and acyclization functors by $\mathrm{L}_V$ and $\Gamma_V$.

Now let $x \in \mathrm{Spc}(\mathcal{T}^c)$ be a point. The sets $\overline{\{x\}}$ and $Y_x := \{y : x \notin \overline{\{y\}}\}$ are both specialization-closed.

**Definition 8.11** (see [24])**.** Let $x \in \mathrm{Spc}(\mathcal{T}^c)$ and let $\Gamma_x$ denote the functor given as the composition $\mathrm{L}_{Y_x}\Gamma_{\overline{\{x\}}}$. Then, for an object $a \in \mathcal{T}$, we define its *support* as

(8.10) $\mathrm{supp}(a) := \{x \in \mathrm{Spc}(\mathcal{T}^c) : \Gamma_x(a) \neq 0\}$.

**Example 8.12** (see [206])**.** Suppose $X = \mathrm{Spec}\, A$ is an affine scheme with $A$ a noetherian ring. Then $\mathbf{D}_{\mathrm{Qcoh}}(\mathrm{Spec}\, A) \cong \mathbf{D}(\mathrm{Mod}(A))$ and

(8.11) $\mathrm{Spc}(\mathbf{D}(\mathrm{Mod}(A))^c) = \mathrm{Spc}(\mathbf{D}^{\mathrm{perf}}(A)) = \mathrm{Spec}\, A$.

Let $\mathfrak{p} \in \mathrm{Spec}\, A$ be a prime ideal. Then the functor $\Gamma_{\mathfrak{p}}$ is given as $\mathrm{K}_\infty(\mathfrak{p}) \otimes A_{\mathfrak{p}} \otimes -$, where $\mathrm{K}_\infty(\mathfrak{p})$ is the *stable Koszul complex* of the prime ideal $\mathfrak{p}$.

In particular, if $\mathrm{Supp}(C^\bullet)$ denotes the complement of the set of points where $C^\bullet$ is acyclic, then we see that $\mathrm{supp}(C^\bullet) \subset \mathrm{Supp}(C^\bullet)$. The set $\mathrm{supp}(C^\bullet)$ is sometimes known as the *small support* of $C^\bullet$ and coincides with the set of prime ideals $\mathfrak{p}$ such that $k(\mathfrak{p}) \otimes^{\mathbf{L}} C^\bullet \neq 0$.

**Remark 8.13.** In comparison to the notion of support of an essentially small tensor triangulated category, the support of an object of $\mathcal{T}$ is still a well-behaved construction. For example, we have $\mathrm{supp}(\bigoplus_i a_i) = \bigcup_i \mathrm{supp}(a_i)$, but $\mathrm{supp}(a)$ needs not be closed. If $a \in \mathcal{T}^c$, then $\mathrm{supp}(a)$ coincides with the notion of support from section 8.2.1 and hence it will be closed.

### 8.2.3 Relative supports and tensor triangular Chow groups

We shall now adapt to a situation where we consider triangulated categories $\mathcal{K}$ that don't necessarily have a symmetric monoidal structure themselves, but rather admit an *action* by a tensor triangulated category $\mathcal{T}$. Let us recall from [207] what it means for $\mathcal{T}$ to have an action $*$ on $\mathcal{K}$.

We are given a biexact bifunctor

(8.12) $*\colon \mathcal{T} \times \mathcal{K} \to \mathcal{K}$

that commutes with coproducts in both variables, whenever they exist. Furthermore we are given natural isomorphisms

$$\begin{aligned} \alpha_{x,y,a} &: (X \otimes Y) * a \xrightarrow{\sim} x * (y * a) \\ l_a &: \mathbb{I} * a \xrightarrow{\sim} a \end{aligned} \tag{8.13}$$

for all objects $x, y \in \mathcal{T}, a \in \mathcal{K}$. These natural isomorphisms should satisfy a list of natural coherence relations that we omit here, but rather refer the reader to [207].

**Example 8.14.** Any rigidly-compactly generated tensor triangulated category has an action on itself via its monoidal structure.

Let us now assume that we are given a tensor triangulated category $\mathcal{T}$ with an action $*$ on a triangulated category $\mathcal{K}$, where $\mathcal{K}$ is assumed to be compactly generated as well (and so we implicitly mean that it has all coproducts). As in the previous section, we still assume that $\mathrm{Spc}(\mathcal{T}^c)$ is a noetherian topological space. Let us first describe a procedure to construct smashing subcategories of $\mathcal{K}$.

**Lemma 8.15.** Suppose $V \subset \mathrm{Spc}(\mathcal{T})$ is a specialization-closed subset. Then the full subcategory

(8.14) $\Gamma_V(\mathbb{I}) * \mathcal{K} = \{a \in \mathcal{K} : a \cong \Gamma_V(\mathbb{I}) * b \text{ for some } b \in \mathcal{K}\}$

is smashing. The corresponding localization and colocalization functors are given by $\mathrm{L}_V(\mathbb{I}) * -$ and $\Gamma_V(\mathbb{I}) * -$, respectively.

*Proof.* It is shown in [207, lemma 4.4] that the subcategory $\Gamma_V(\mathbb{I}) * \mathcal{K}$ is Bousfield with

(8.15) $(\Gamma_V(\mathbb{I}) * \mathcal{K})^{\perp} = \mathrm{L}_V(\mathbb{I}) * \mathcal{K} := \{a \in \mathcal{K} : a \cong \mathrm{L}_V(\mathbb{I}) * b \text{ for some } b \in \mathcal{K}\}$.

Both $\Gamma_V(\mathbb{I}) * \mathcal{K}$ and $\mathrm{L}_V(\mathbb{I}) * \mathcal{K}$ are $\mathcal{T}$-submodules, and we have a localization triangle

(8.16) $\Gamma_V(\mathbb{I}) \to \mathbb{I} \to \mathrm{L}_V(\mathbb{I}) \to \Sigma(\Gamma_V(\mathbb{I}))$.

Applying the functor $- * a$ to this triangle shows that the localization and colocalization functors associated to the Bousfield subcategory are given by $\mathrm{L}_V(\mathbb{I}) * -$ and $\Gamma_V(\mathbb{I}) * -$, respectively. Since $\mathrm{L}_V(\mathbb{I}) * -$ preserves coproducts by defintion of an action, it follows that $\Gamma_V(\mathbb{I}) * \mathcal{K}$ is indeed smashing. □

Following [207], we can now assign to any object $a \in \mathcal{K}$ a support in $\mathrm{Spc}(\mathcal{T}^c)$ as follows:

**Definition 8.16.** Let $x \in \mathrm{Spc}(\mathcal{T}^c)$. Then, for an object $a \in \mathcal{K}$, we define its *support* as

(8.17) $\mathrm{supp}_{\mathcal{T}}(a) := \{x \in \mathrm{Spc}(\mathcal{T}^c) : \Gamma_x(\mathbb{I}) * a \neq 0\}$.

If there is no risk of confusion, we will usually drop the subscript $\mathcal{T}$ and just write $\mathrm{supp}(a)$ instead. Furthermore, we will abbreviate the expression $\Gamma_x(\mathbb{I}) * a$ by $\Gamma_x a$.

Let us state two important properties of the support.

**Proposition 8.17** (see [207, lemma 5.7])**.** Let $V$ be a specialization-closed subset of $\mathrm{Spc}(\mathcal{T}^c)$ and $a$ an object of $\mathcal{K}$. Then

(8.18) $\mathrm{supp}(\Gamma_V(a)) = \mathrm{supp}(a) \cap V$

and

(8.19) $\mathrm{supp}(\mathrm{L}_V(a)) = \mathrm{supp}(a) \cap (\mathrm{Spc}(\mathcal{T}^c) \setminus V)$.

**Definition 8.18.** For every specialization-closed subset $V \subset \mathrm{Spc}(\mathcal{T}^c)$, the subcategory $\mathcal{K}_V$ is defined as the essential image of the functor $\Gamma_V(\mathbb{I}) * -$. For every integer $p$ the subcategory $\mathcal{K}_{(p)}$ is defined as $\Gamma_{V_{\leq p}}(\mathbb{I}) * \mathcal{K}$, where $V_{\leq p} \subset \mathrm{Spc}(\mathcal{T}^c)$ is the subset of all points $x$ such that $\dim(x) \leq p$.

**Remark 8.19.** In [126], $\mathcal{K}_{(p)}$ is defined differently, namely as the full subcategory of $\mathcal{K}$ on the collection of objects $\{a \in \mathcal{K} : \dim(\mathrm{supp}(a)) \leq p\}$. This coincides with definition 8.18 whenever supp detects vanishing, i.e. whenever $\mathrm{supp}(a) = \emptyset \Leftrightarrow a = 0$ holds. Indeed, if $a \in \Gamma_{V_{\leq p}}(\mathbb{I}) * \mathcal{K}$, then $a \cong \Gamma_{V \leq p}(\mathbb{I}) * b$ for some $b \in \mathcal{K}$ and it follows from proposition 8.17 that $\mathrm{supp}(a) \subset V_{\leq p}$. Conversely, if

$$\dim(\mathrm{supp}(a)) \leq p \Leftrightarrow \mathrm{supp}(a) \subset V_{\leq p}, \tag{8.20}$$

we have a localization triangle

$$\Gamma_{V_{\leq p}}(\mathbb{I}) * a \to a \to \mathrm{L}_{V_{\leq p}}(\mathbb{I}) * a \to \Sigma(\Gamma_{V_{\leq p}}(\mathbb{I})), \tag{8.21}$$

and it follows from proposition 8.17 that $\mathrm{supp}(\mathrm{L}_{V_{\leq p}}(\mathbb{I})) = \emptyset$ and hence $\mathrm{L}_{V_{\leq p}}(\mathbb{I}) \cong 0$. This implies $\Gamma_{V_{\leq p}}(\mathbb{I}) * a \cong a$ and shows that $a \in \Gamma_{V_{\leq p}}(\mathbb{I}) * \mathcal{K}$. By [207, theorem 6.9], supp detects vanishing when the action of $\mathcal{T}$ on $\mathcal{K}$ satisfies the *local-to-global principle*, see remark 8.22.

**Proposition 8.20** (See [207, corollary 4.11])**.** Let $V \subset \mathrm{Spc}(\mathcal{T}^c)$ be specialization-closed. The category $\mathcal{K}_V$ is compactly generated.

We now come to the definition of the central invariant that is studied in this chapter. For a triangulated category $\mathcal{C}$, we shall denote by $\mathcal{C}^\natural$ its *idempotent completion*, a triangulated category with a fully faithful inclusion $\mathcal{C} \to \mathcal{C}^\natural$ which is universal for the property that all idempotents in $\mathcal{C}^\natural$ split (see [25] for a detailed discussion). Let us first write down a diagram of Grothendieck groups:

$$\begin{array}{ccc} \mathrm{K}_0(\mathcal{K}^c_{(p)}) & \xrightarrow{q^\natural} & \mathrm{K}_0((\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)})^\natural) \ (= \mathrm{K}_0((\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c)) \\ \downarrow i & & \\ \mathrm{K}_0(\mathcal{K}^c_{(p+1)}) & & \end{array} \tag{8.22}$$

Here, $q^\natural$ is the map induced by the composition of the Verdier quotient functor

$$\mathcal{K}^c_{(p)} \to \mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)} \tag{8.23}$$

and the inclusion into the idempotent completion of the latter category. The morphism $i$ is induced by the inclusion functor. The identification

$$(\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)})^\natural \cong (\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)})^c \tag{8.24}$$

holds by [131, theorem 5.6.1] since $\mathcal{K}_{(p-1)}$ is compactly generated by proposition 8.20.

**Definition 8.21** (See [126])**.** The *dimension p tensor triangular cycle group* of $\mathcal{K}$ relative to the action $*$ is defined as

$$\mathrm{Cyc}^\Delta_p(\mathcal{T}, \mathcal{K}) := \mathrm{K}_0((\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)})^\natural). \tag{8.25}$$

The *dimension p tensor triangular Chow group* of $\mathcal{K}$ relative to the action $*$ is defined as

$$\mathrm{CH}_p^{\Delta}(\mathcal{K}) := \mathrm{Cyc}_p^{\Delta}(\mathcal{K})/q^{\natural}(\ker(i)). \tag{8.26}$$

**Remark 8.22.** In [126], the definition of relative tensor triangular cycle and Chow groups was given under the assumption that another technical condition, the *local-to-global principle*, is satisified. The principle asserts that for any object $a \in \mathcal{K}$, the smallest localizing subcategory of $\mathcal{K}$ that is closed under the action of $\mathcal{T}$ and contains $a$ equals the smallest localizing subcategory of $\mathcal{K}$ that is closed under the action of $\mathcal{T}$ and contains all the objects $\Gamma_x a$ for all $x \in \mathrm{Spc}(\mathcal{T}^c)$ (see [207, definition 6.1]). While it is not necessary for the statement of definition 8.21 to make sense, the local-to-global principle makes dealing with these invariants easier (see remark 8.19), and it is satisfied very often. In particular, it will be satisfied in our main case of interest by [207, theorem 6.9], when we consider actions of the derived category of quasicoherent sheaves on a noetherian separated scheme. In order to keep the exposition of the chapter a bit lighter, we will not go into further details concerning this topic.

Let us illustrate our definitions with an example that explains the name "tensor triangular Chow group". The following theorem is a slight variation of [126, corollary 3.6], and is based on Quillen's result describing the Chow groups using the coniveau spectral sequence [182, proposition 5.14].

**Theorem 8.23.** Let $X$ be a separated regular scheme of finite type over a field. Consider the action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on itself via $- \otimes^{\mathbf{L}} -$. Then for all $p \geq 0$, we have isomorphisms

$$\begin{aligned} \mathrm{Cyc}_p^{\Delta}(\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)), \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))) &\cong \mathrm{Cyc}_p(X) \\ \mathrm{CH}_p^{\Delta}(\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)), \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))) &\cong \mathrm{CH}_p(X), \end{aligned} \tag{8.27}$$

where $\mathrm{Cyc}_p(X)$ and $\mathrm{CH}_p(X)$ denote the dimension $p$ cycle and Chow groups of $X$.

*Proof.* This is [126, corollary 3.6], with codimension replaced by dimension. The former statement is proved by showing that the analogous cycle and Chow groups $\mathrm{Cyc}^p_{\Delta}(\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)), \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)))$ and $\mathrm{CH}^p_{\Delta}(\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)), \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)))$ which are defined via a filtration by *codimension* of support, are isomorphic to certain terms on the $E^1$ and $E^2$ page of Quillen's coniveau spectral sequence associated to $X$, which happen to be isomorphic to $\mathrm{Cyc}^p(X)$ and $\mathrm{CH}^p(X)$, respectively.

In order to prove the "dimension" version of the statement, we see that the same argument shows that

$$\mathrm{Cyc}_p^{\Delta}(\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)), \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))) \cong E^1_{p,-p} \tag{8.28}$$

and

$$\mathrm{CH}_p^{\Delta}(\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)), \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))) \cong E^2_{p,-p} \tag{8.29}$$

where $E^1_{p,-p}$ and $E^2_{p,-p}$ are obtained from the *niveau* spectral sequence of $X$, which happen to be isomorphic to $\mathrm{Cyc}_p(X)$ and $\mathrm{CH}_p(X)$ (see e.g. [161] for the identification of $E^1_{p,-p}$ and $E^2_{p,-p}$ with $\mathrm{Cyc}_p(X)$ and $\mathrm{CH}_p(X)$). □

**Remark 8.24** (see [126, §4])**.** We can actually do better and also recover $\mathrm{CH}_p(X)$ for singular schemes. In order to do so, one lets $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ act on $\mathbf{K}(\mathrm{Inj}\, X)$, the homotopy category of injective quasicoherent injective sheaves on $X$, instead of considering the action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on itself. Later on, we shall be interested in the action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on the derived category of a coherent $\mathcal{O}_X$-algebra.

## 8.3 An exact sequence

In this section we derive an exact sequence that will give us a new description of $\mathrm{CH}^{\Delta}_p(\mathcal{T},\mathcal{K})$ as an image of a map in a K-theoretic localization sequence. It will be especially useful for computing $\mathrm{CH}^{\Delta}_0(\mathcal{T},\mathcal{K})$ when $\dim(\mathrm{Spc}(\mathcal{T}^c)) = 1$. Let $\mathcal{T}$ be a rigidly-compactly generated triangulated category that has an action $*$ on a compactly generated triangulated category $\mathcal{K}$ and assume that $\mathrm{Spc}(\mathcal{T}^c)$ is a noetherian topological space. Then we know that $\mathcal{K}_{(p)}$ is a compactly generated subcategory of $\mathcal{K}$ for all $p \geq 0$ and we have an exact sequence of triangulated categories

$$(8.30)\quad \mathcal{K}_{(p)}/\mathcal{K}_{(p-1)} \to \mathcal{K}_{(p+1)}/\mathcal{K}_{(p-1)} \to \mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}.$$

Since the inclusion functor $\mathcal{K}_{(p)} \to \mathcal{K}_{(p+1)}$ admits a coproduct-preserving right adjoint $\Gamma_{V_{\leq p}}(\mathbb{I}) * -$, the same is true for both functors in the sequence (8.30). Hence it restricts to a sequence of compact objects

$$(8.31)\quad \left(\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}\right)^{c} \to \left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p-1)}\right)^{c} \to \left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}\right)^{c}$$

which is exact up to factors. Applying $\mathrm{K}_0$ to this diagram yields a sequence of abelian groups

$$(8.32)\quad \mathrm{Z}^{\Delta}_p(\mathcal{T},\mathcal{K}) \xrightarrow{\iota} \mathrm{K}_0\left(\left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p-1)}\right)^{c}\right) \xrightarrow{\pi} \mathrm{Z}^{\Delta}_{p+1}(\mathcal{T},\mathcal{K})$$

which is exact at the middle term.

**Lemma 8.25.** The map $\pi$ is surjective if and only if $\mathcal{K}^c_{(p+1)}/\mathcal{K}^c_{(p)}$ is idempotent complete.

*Proof.* We have $\left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}\right)^{c} \cong \left(\mathcal{K}^c_{(p+1)}/\mathcal{K}^c_{(p)}\right)^{\natural}$ and hence $\mathcal{K}^c_{(p+1)}/\mathcal{K}^c_{(p)}$ is a dense triangulated subcategory of $\left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}\right)^{c}$. Thomason's classification of these subcategories (see [215]) then shows that $\mathrm{im}(\pi)$ is maximal if and only if the inclusion $\mathcal{K}^c_{(p+1)}/\mathcal{K}^c_{(p)} \hookrightarrow \left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p)}\right)^{c}$ is essentially surjective which happens if and only if the former category is idempotent complete. □

We shall now be concerned with the kernel of $\iota$. Our goal is to prove the following statement:

**Proposition 8.26.** In the notation of definition 8.21, we have $\ker(\iota) = q^{\natural}(\ker(i))$. Hence, we obtain an exact sequence

$$(8.33)\quad 0 \to \mathrm{CH}^{\Delta}_p(\mathcal{T},\mathcal{K}) \xrightarrow{\bar{\iota}} \mathrm{K}_0\left(\left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p-1)}\right)^{c}\right) \xrightarrow{\pi} \mathrm{Z}^{\Delta}_{p+1}(\mathcal{T},\mathcal{K})$$

which is exact on the right if and only if $\mathcal{K}^c_{(p+1)}/\mathcal{K}^c_{(p)}$ is idempotent complete.

**Lemma 8.27.** Let $\mathcal{K}$ be a triangulated category and $\mathcal{L} \subset \mathcal{K}$ a triangulated subcategory. Consider the full triangulated subcategories $\mathcal{L}^\natural, \mathcal{K} \subset \mathcal{K}^\natural$. Then $\mathcal{L}^\natural \cap \mathcal{K} \cong \mathcal{L}$ as full subcategories of $\mathcal{K}^\natural$.

*Proof.* It is clear that an object $A \in \mathcal{L}$ is both contained in $\mathcal{L}^\natural$ and $\mathcal{K}$. For the converse inclusion, suppose that $A$ is in $\mathcal{L}^\natural \cap \mathcal{K}$. Any object $A \in \mathcal{L}^\natural$ can be written as a pair $(A', e)$, where $A'$ is an object of $\mathcal{L}$ and $e$ is an idempotent endomorphism $A' \to A'$ in $\mathcal{L}$. Similarly, the objects $B$ of $\mathcal{K}$ in $\mathcal{K}^\natural$ are identified with exactly the pairs $(B', \mathrm{id}_B)$. It follows that $A$ can be written in the form $(A', \mathrm{id}_{A'})$ with $A' \in \mathcal{L}$. Hence, $A$ is in the image of the inclusion functor $\mathcal{L}^\natural \to \mathcal{K}^\natural$. □

**Lemma 8.28.** In the situation of lemma 8.27, assume that $\mathcal{L}, \mathcal{K}$ are essentially small and consider the diagram of Grothendieck groups

$$
\begin{array}{ccc}
\mathrm{K}_0(\mathcal{L}) & \xrightarrow{\alpha} & \mathrm{K}_0(\mathcal{K}) \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\sigma} \\
\mathrm{K}_0(\mathcal{L}^\natural) & \xrightarrow{\beta} & \mathrm{K}_0(\mathcal{K}^\natural)
\end{array}
\tag{8.34}
$$

induced by the inclusion functors. Then $\ker(\beta) = \rho(\ker(\alpha))$.

*Proof.* By the commutativity of the diagram, it is clear that $\ker(\beta) \supseteq \rho(\ker(\alpha))$, so let us prove the converse inclusion. Consider an element $[a] \in \ker(\beta)$, i.e. $[a] = 0$ in $\mathrm{K}_0(\mathcal{K}^\natural)$.

By Thomason's classification of dense triangulated subcategories (see [215]) applied to $\mathcal{K} \subset \mathcal{K}^\natural$, we have

$$\mathcal{K} = \{x \in \mathcal{K}^\natural : [x] \in \mathrm{im}(\sigma)\}. \tag{8.35}$$

Since $0 \in \mathrm{im}(\sigma)$, we must have that $a \in \mathcal{K} \subset \mathcal{K}^\natural$, and by lemma 8.27 it follows that $a \in \mathcal{L}$. Thus, $[a] \in \mathrm{im}(\rho)$ and since $\sigma$ is injective (see [215, corollary 2.3]), it follows that $[a] \in \ker(\alpha)$. □

*Proof of proposition 8.26.* Consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{K}_0(\mathcal{K}^c_{(p)}) & \xrightarrow{i} & \mathrm{K}_0(\mathcal{K}^c_{(p+1)}) \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle h} \\
\mathrm{K}_0\left(\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)}\right) & \xrightarrow{k} & \mathrm{K}_0\left(\mathcal{K}^c_{(p+1)}/\mathcal{K}^c_{(p-1)}\right) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle l} \\
\underbrace{\mathrm{K}_0\left(\left(\mathcal{K}^c_{(p)}/\mathcal{K}^c_{(p-1)}\right)^\natural\right)}_{=\mathrm{Z}^\Delta_p(\mathcal{T},\mathcal{K})} & \xrightarrow{\iota} & \underbrace{\mathrm{K}_0\left(\left(\mathcal{K}^c_{(p+1)}/\mathcal{K}^c_{(p-1)}\right)^\natural\right)}_{=\mathrm{K}_0\left(\left(\mathcal{K}_{(p+1)}/\mathcal{K}_{(p-1)}\right)^c\right)}
\end{array}
\tag{8.36}
$$

where all maps are induced by inclusions of subcategories or Verdier quotient functors and in particular, we have $q^\natural = j \circ q$. Since $\ker(h) = i(\ker(q))$, we obtain that $\ker(k) = q(\ker(i))$. Therefore, it suffices to show that $\ker(\iota) = j(\ker(k))$, which follows from lemma 8.28. The last statement of the proposition is lemma 8.25. □

**Remark 8.29.** When $\dim(\operatorname{Spc}(\mathcal{T}^c)) = 1$, proposition 8.26 exhibits $\mathrm{CH}_0^\Delta(\mathcal{T}, \mathcal{K})$ as a subgroup of $\mathrm{K}_0(\mathcal{K}^c)$. If $X$ is a regular algebraic curve, $\mathcal{T} = \mathcal{K} = \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$, then we recover the well-known isomorphism

$$\mathrm{K}_0(X) \cong \mathrm{CH}_0(X) \oplus \mathrm{Z}_1(X) \tag{8.37}$$

using theorem 8.23: the map $\pi$ is surjective by lemma 8.25, since

$$\begin{aligned} \mathbf{D}^{\mathrm{perf}}(\mathrm{coh}(X))_{(1)}/\,\mathbf{D}^{\mathrm{perf}}(\mathrm{coh}(X))_{(0)} &\cong \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(X))/\,\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(X))_{(0)} \\ &\cong \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(X)/\,\mathrm{coh}(X)_{\leq 0}), \end{aligned} \tag{8.38}$$

(see [125, §3.2], compare corollary 8.43) and the latter category is idempotent complete since it is the bounded derived category of an abelian category (see [25]). Furthermore, $\mathrm{Z}_1(X)$ is free abelian and hence the exact sequence splits. Again, as in remark 8.24, we can drop the regularity assumption and consider the action of the category $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ with the derived tensor product on $\mathbf{K}(\mathrm{Inj}\,X)$ instead. Then we obtain

$$\mathrm{G}_0(X) \cong \mathrm{CH}_0(X) \oplus \mathrm{Z}_1(X). \tag{8.39}$$

## 8.4 Derived categories of quasicoherent $\mathcal{O}_X$-algebras

In this section, we first recall some well-known facts about the category of quasicoherent right $\mathcal{A}$-modules $\mathrm{Qcoh}(\mathcal{A})$, and its derived category $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$. We show how to realise the functor $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \to \mathbf{D}(\mathrm{Mod}(\mathcal{A}_x))$ that takes stalks at $x \in X$ as a localization of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ and prove a technical result about the filtration of $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A}))$ by dimension of support. At this point we will not need to assume that $\mathcal{A}$ is coherent, quasi-coherence is enough. Starting from section 8.5 we will impose the coherence condition to make the action well-behaved on the level of compact objects, and to make the different notions of support agree.

### 8.4.1 Quasicoherent modules over quasicoherent $\mathcal{O}_X$-algebras

Let $X$ be a scheme. In this section we recall some basic facts about modules over an $\mathcal{O}_X$-algebra $\mathcal{A}$. The material we present here should be well-known (or at least hardly surprising) to most experts.

An $\mathcal{O}_X$-algebra $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-modules $\mathcal{A}$ together with a multiplication map $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ that is associative and has unit, and is $\mathcal{O}_X$-bilinear[1]. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is *quasicoherent*, if it is so as an $\mathcal{O}_X$-module. The pair $(X, \mathcal{A})$ is a ringed space, and hence it makes sense to talk about quasicoherent right $\mathcal{A}$-modules. It is not hard to show that if $\mathcal{A}$ is a quasicoherent $\mathcal{O}_X$-algebra, then a right $\mathcal{A}$-module is quasicoherent if and only if it is quasicoherent as an $\mathcal{O}_X$-module. Furthermore, quasicoherent

[1]This last condition implies that $\mathcal{O}_X$ acts centrally on $\mathcal{A}$.

right $\mathcal{A}$-modules over a quasicoherent $\mathcal{O}_X$-algebra $\mathcal{A}$ have a local description analogous to quasicoherent $\mathcal{O}_X$-modules [240, proposition 1.15].

**Proposition 8.30.** Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra and $U \subset X$ an affine open. Define $A := \Gamma(U, \mathcal{A})$. Then the functor $\Gamma(U, -)$ induces an equivalence of categories

(8.40) $\{\text{quasicoherent right } \mathcal{A}|_U\text{-modules}\} \xrightarrow{\sim} \{\text{right } A\text{-modules}\}.$

Since the notion of coherence is general as well, it applies to right $\mathcal{A}$-modules. We shall primarily be interested in the case where $X$ is noetherian and $\mathcal{A}$ is a *coherent* $\mathcal{O}_X$-*algebra*, i.e. one that is coherent as an $\mathcal{O}_X$-module.

**Lemma 8.31.** Suppose $X$ is noetherian and $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra. Then a right $\mathcal{A}$-module $M$ is coherent if and only if it is coherent as an $\mathcal{O}_X$-module.

*Sketch of the proof.* Let us first notice that under the given conditions, $\mathcal{A}$ is a sheaf of right-noetherian rings. A right $\mathcal{A}$-module is hence coherent if and only if it is locally of finite type. Therefore, it suffices to show that a right $\mathcal{A}$-module is locally of finite type over $\mathcal{A}$ if and only if it is so over $\mathcal{O}_X$, which is straightforward. □

**Proposition 8.32.** The category $\mathrm{Qcoh}(\mathcal{A})$ is Grothendieck abelian.

*Proof.* The category $\mathrm{Qcoh}(\mathcal{A})$ is exactly the category of modules over the right-exact monad corresponding to the adjunction $\mathcal{A} \otimes_{\mathcal{O}_X} - \dashv U$. Then [26, lemma A.3] applies and shows that $\mathrm{Qcoh}(\mathcal{A})$ is Grothendieck abelian, since $\mathrm{Qcoh}(\mathcal{O}_X)$ is so. □

The following notion is central for our further considerations:

**Definition 8.33.** Let $M \in \mathrm{Qcoh}(\mathcal{O}_X)$. The *support* $\mathrm{Supp}(M)$ is the set of points $x \in X$ such that $M_x \neq 0$. If $N \in \mathrm{Qcoh}(\mathcal{A})$, then $\mathrm{Supp}(N) := \mathrm{Supp}(U(N)) \subset X$.

### 8.4.2 The derived category of a quasicoherent $\mathcal{O}_X$-algebra

In the following, *we shall always assume that $X$ is a noetherian separated scheme* and that $\mathcal{A}$ is a quasicoherent $\mathcal{O}_X$-algebra. These are not the strongest possible assumptions for (most of) the results in this section, but in section 8.5 we will need these (and stronger) conditions to develop the machinery of relative tensor triangular Chow groups.

#### Basic properties

In this section we study the category $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$, the derived category of quasicoherent right-$\mathcal{A}$-modules. Let us first note that $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ exists, since $\mathrm{Qcoh}(\mathcal{A})$ is Grothendieck abelian by proposition 8.32. Furthermore, since the forgetful functor $U$ is exact, it directly descends to give a functor $U\colon \mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \to \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$. Its right adjoint $\mathcal{A} \otimes_{\mathcal{O}_X} -$ induces a left-derived functor

(8.41) $\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}_X} -\colon \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)) \to \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$

which is computed by first taking K-flat resolutions in $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ and then applying $\mathcal{A} \otimes_{\mathcal{O}_X} -$.

**Proposition 8.34.** There is an adjunction $(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}_X} -) \dashv U$.

*Proof.* This follows since the derived functors of an adjoint pair, if they exist, are again adjoint [200, tag 09T5]. □

**Theorem 8.35.** The category $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ is compactly generated, and a complex in $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ is compact if and only if it is perfect, i.e. it is locally quasi-isomorphic to a bounded complex of projective modules of finite rank.

*Proof.* This can be shown using Rouquier's cocoverings. See [55, theorem 3.14]. □

**Convention 8.36.** In the following, we shall denote the full subcategory of perfect complexes over $\mathcal{A}$ by $\mathbf{D}^{\mathrm{perf}}(\mathcal{A}) \subset \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$. Whenever $S \subset |X|$ is a subset, we shall denote by $\mathbf{D}_S(\mathcal{A})$ $(\mathbf{D}^{\mathrm{b}}_S(\mathrm{coh}(\mathcal{A})), \mathbf{D}^{\mathrm{perf}}_S(\mathcal{A}))$ the corresponding full subcategories consisting of complexes $C^\bullet$ with $\mathrm{Supp}(\mathrm{H}^\bullet(C^\bullet)) \subset S$. If $S = V_{\leq p}$, the subset of all points of dimension $\leq p$, we shall replace the subscript "$V_{\leq p}$" by "$\leq p$".

### Taking stalks

Let us consider a point $x \in X$ and the inclusion $\mathrm{Spec}\,\mathcal{O}_{X,x} \to X$. If we equip $\mathrm{Spec}\,\mathcal{O}_{X,x}$ with the sheaf of rings $\mathcal{A}_x$, we obtain a morphism of ringed spaces

$$(8.42)\quad i_x \colon (\mathrm{Spec}\,\mathcal{O}_{X,x}, \mathcal{A}_x) \to (X, \mathcal{A})$$

and the general theory of ringed spaces gives us a pair of adjoint functors

$$(8.43)\quad \begin{array}{c} \mathrm{Mod}(\mathcal{A}_x) \\ {\scriptstyle (i_x)_*}\downarrow \;\uparrow{\scriptstyle (i_x)^*} \\ \mathrm{Qcoh}(\mathcal{A}) \end{array}$$

which fits into a commutative diagram

$$(8.44)\quad \begin{array}{ccc} \mathrm{Mod}(\mathcal{O}_{X,x}) & \xleftarrow{U} & \mathrm{Mod}(\mathcal{A}_x) \\ {\scriptstyle (i_x)_*}\downarrow \;\uparrow{\scriptstyle (i_x)^*} & & {\scriptstyle (i_x)_*}\downarrow \;\uparrow{\scriptstyle (i_x)^*} \\ \mathrm{Qcoh}(\mathcal{O}_X) & \xleftarrow[U]{} & \mathrm{Qcoh}(\mathcal{A}) \end{array}$$

and satisfies $(i_x)^* \circ (i_x)_* = \mathrm{id}$. The map $\mathrm{Spec}\,\mathcal{O}_{X,x} \to X$ is quasi-separated and quasi-compact (recall that we assumed that $X$ noetherian). Therefore the functor $(i_x)_*$ indeed produces quasicoherent $\mathcal{O}_X$-modules, and hence also quasicoherent $\mathcal{A}$-modules, since quasi-coherence can be checked after applying $U$.

Since $X$ was separated, the map $i_x$ is affine and thus the functor $(i_x)_*$ is exact on the level of $\mathcal{O}_{X,x}$-modules. Since $U$ preserves and reflects exactness, it follows that $(i_x)_*$ is exact on the level of $\mathcal{A}_x$-modules as well.

Furthermore, the morphism $\mathrm{Spec}\,\mathcal{O}_{X,x} \to X$ is flat and hence $(i_x)^*$ is exact on both levels as well.

Since the derived functors of an adjoint pair are again adjoint [200, tag 09T5], we obtain an adjunction

$$\begin{array}{c} \mathbf{D}(\mathrm{Mod}(\mathcal{A}_x)) \\ (i_x)_* \big\downarrow \big\uparrow (i_x)^* \\ \mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \end{array} \tag{8.45}$$

which still satisfies $(i_x)^* \circ (i_x)_* = \mathrm{id}$ since there was no need to derive any of the two functors.

**Proposition 8.37.** Let $X$ be a noetherian separated scheme and $\mathcal{A}$ a quasicoherent $\mathcal{O}_X$-algebra. Let $x \in X$ and $\mathbf{D}_{Y_x}(\mathcal{A}) \subset \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ be the full subcategory of complexes $C^\bullet$ such that $\mathrm{Supp}(\mathrm{H}^\bullet(C^\bullet)) \subset Y_x = \{y \in X | x \notin \overline{\{y\}}\}$. Then $\mathbf{D}_{Y_x}(\mathcal{A}) \cong \ker(i_x)^*$ and the functor $(i_x)^*$ induces an exact equivalence

$$\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))/\,\mathbf{D}_{Y_x}(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}(\mathrm{Mod}(\mathcal{A}_x)). \tag{8.46}$$

*Proof.* The first part follows from the identity $\mathrm{H}^\bullet((i_x)^*C^\bullet) = (i_x)^*(\mathrm{H}^\bullet(C^\bullet))$.

Since $(i_x)^* \circ (i_x)_* = \mathrm{id}$, we must have that $(i_x)_*$ is fully faithful. It is well-known (see e.g. [188, lemma 3.4]) that we therefore get an exact sequence of triangulated categories

$$\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))/\ker(i_x)^* \xrightarrow{\sim} \mathbf{D}(\mathrm{Mod}(\mathcal{A}_x)), \tag{8.47}$$

which finishes the proof by the first part of the proposition. □

### Filtrations of the bounded derived category of coherent sheaves

Let us now assume that $X$ is a noetherian scheme and that $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra. We record the following, essentially trivial lemma for later use.

**Lemma 8.38.** Let $\mathcal{J} \subset \mathcal{O}_X$ be an ideal sheaf and $M$ an $\mathcal{A}$-module. Then

$$\mathcal{J}M = 0 \Leftrightarrow (\mathcal{A} \cdot \mathcal{J})M = 0. \tag{8.48}$$

*Proof.* Easy local computation. □

**Definition 8.39.** A sheaf of ideals $\mathcal{I} \subset \mathcal{A}$ is called *central*, if for any open $U \subset X$, the ideal $\mathcal{I}(U) \subset \mathcal{A}(U)$ can be generated by central elements.

**Proposition 8.40.** Let

$$0 \to A \to B \to C \to 0 \tag{8.49}$$

be an exact sequence of coherent $\mathcal{A}$-modules with $\mathrm{Supp}(A) = V \subset X$. Then there exists a commutative diagram of $\mathcal{A}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow \mathrm{id} & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array} \tag{8.50}$$

with exact rows and such that $\mathrm{Supp}(B'), \mathrm{Supp}(C') \subset V$.

*Proof.* Let $\mathcal{J} \subset \mathcal{O}_X$ denote the radical ideal corresponding to the closed subset $V$. Then there exists $n_0 \in \mathbb{N}$ such that $\mathcal{J}^n A = 0$ for all $n \geq n_0$, and by lemma 8.38 it follows that $(\mathcal{A} \cdot \mathcal{J}^n)A = (\mathcal{A} \cdot \mathcal{J})^n A = 0$ all $n \geq n_0$. For each $n$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\ & & A & \xrightarrow{\bar{\iota}} & B/(\mathcal{A}\cdot\mathcal{J})^n B & \xrightarrow{\bar{\pi}} & C/(\mathcal{A}\cdot\mathcal{J})^n C & \longrightarrow & 0 \end{array} \tag{8.51}$$

where $\bar{\iota}, \bar{\pi}$ are induced by $\iota, \pi$ respectively and the non-labeled vertical maps are the canoncial projections. We claim that for $n$ large enough, $\bar{\iota}$ is a monomorphism. As we can check injectivity locally, let $X = \bigcup_{i=1}^r U_i$ with $U_i = \mathrm{Spec}\, R_i$ open affine. Then, on each $U_i$, the problem looks as follows: we are given an $R_i$-algebra $S_i$, an ideal $J_i \subset R_i$, an exact of $S_i$-modules

$$0 \to A_i \to B_i \to C_i \to 0 \tag{8.52}$$

and we know that for all $n \geq n_i$, $J^n A = 0$. Diagram (8.51) translates as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_i & \xrightarrow{\iota_i} & B_i & \xrightarrow{\pi_i} & C_i & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ & & A_i & \xrightarrow{\overline{\iota_i}} & B_i/(S_i\cdot J_i)^n B_i & \xrightarrow{\overline{\pi_i}} & C/(S_i\cdot J_i)^n C_i & \longrightarrow & 0 \end{array} \tag{8.53}$$

We will now use the Artin–Rees lemma, which is in general not valid for noncommutative rings, but does hold for central ideals like $S_i \cdot J_i$ (see [171, theorem 7.2.1]): there exists $q_i \in \mathbb{N}$ such that for all $m_i \geq q_i$ we have

$$A_i \cap (S_i \cdot J_i)^n B_i = (S_i \cdot J_i)^{n-q_i}(A_i \cap (S_i \cdot J_i)^{q_i} B_i). \tag{8.54}$$

Now note that $\ker(\overline{\iota_i}) = A_i \cap (S_i \cdot J_i)^n B_i$, and thus the Artin-Rees lemma tells us that if we choose $m_i$ such that $n - q \geq n_i$, then $\ker(\overline{\iota_i}) = 0$, i.e. $\overline{\iota_i}$ is injective. Now, if we choose $n = \max_i m_i$, then $\overline{\iota_i}$ will be injective for all $i$, proving that $\bar{\iota}$ is a monomorphism.

To conclude the proof, note that for any coherent $\mathcal{A}$-module $M$, we have that

$$\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}_{\mathcal{O}_X}(M)) \tag{8.55}$$

since $M$ is also $\mathcal{O}_X$-coherent. But by lemma 8.38, we know that $\mathcal{J}^n$ annihilates

$$M/(\mathcal{A} \cdot \mathcal{J})^n M = M/(\mathcal{A} \cdot \mathcal{J}^n)M \tag{8.56}$$

as $\mathcal{A} \cdot \mathcal{J}^n$ does so. It follows that

$$\mathrm{Supp}(B/(\mathcal{A} \cdot \mathcal{J})^n B), \mathrm{Supp}(C/(\mathcal{A} \cdot \mathcal{J})^n C) \subset \mathrm{V}(\mathcal{J}^n) = \mathrm{V}(\mathcal{J}) = V. \tag{8.57}$$

□

**Definition 8.41.** For $p \in \mathbb{Z}$, denote by $\mathrm{coh}(\mathcal{A})_{\leq p}$ the full subcategory of $\mathrm{coh}(\mathcal{A})$ consisting of those $\mathcal{A}$-modules $M$ with $\dim(\mathrm{Supp}(M)) \leq p$.

**Remark 8.42.** The properties of $\mathrm{Supp}(-)$ easily imply that $\mathrm{coh}(\mathcal{A})_{\leq p}$ is a Serre subcategory of $\mathrm{coh}(\mathcal{A})_{\leq q}$ if $p \leq q$.

**Corollary 8.43.** The natural functors (see definition 8.41 and convention 8.36 for the notation)

$$\begin{gathered} \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A})_{\leq p}) \to \mathbf{D}^{\mathrm{b}}_{\leq p}(\mathrm{coh}(\mathcal{A})) \\ \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A})_{\leq p}) / \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A})_{\leq p-1}) \to \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A})_{\leq p} / \mathrm{coh}(\mathcal{A})_{\leq p-1}) \end{gathered} \tag{8.58}$$

are equivalences of categories.

*Proof.* The conclusion of proposition 8.40 is the condition of [123, §1.15, lemma (c1)] which makes the above functors equivalences. □

## 8.5 Relative tensor triangular Chow groups of a coherent $\mathcal{O}_X$-algebra

In this section, we obtain a definition of the relative tensor triangular cycle and Chow groups of a coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ by means of an action of the derived category of quasicoherent $\mathcal{O}_X$-modules $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on the derived category of quasicoherent right $\mathcal{A}$-modules $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$. We then derive some basic properties of these groups, including a group homomorphism induced by the forgetful functor that relates $\mathrm{CH}^{\Delta}_i(X, \mathcal{A})$ to $\mathrm{CH}_i(X)$ when $X$ is regular.

The general approach we use for the relative tensor triangular Chow groups works for all quasicoherent $\mathcal{O}_X$-algebras $\mathcal{A}$, but as we will see below, the coherent case will turn out to be more manageable, since then two notions of support will agree for bounded complexes of coherent $\mathcal{A}$-modules. Therefore we only develop the theory in this setting, which is sufficient for the examples.

### 8.5.1 The action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$

The bifunctor

$$- \otimes_{\mathcal{O}_X} - \colon \mathrm{Qcoh}(\mathcal{O}_X) \times \mathrm{Qcoh}(\mathcal{A}) \to \mathrm{Qcoh}(\mathcal{A}) \tag{8.59}$$

gives rise to a bifunctor

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_X} - \colon \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X)) \times \mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \to \mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \tag{8.60}$$

by taking K-flat resolution in the first variable and applying $- \otimes_{\mathcal{O}_X} -$. This defines an action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$, where the unitor and associator isomorphisms (8.13) are induced by those on the level of complexes, i.e. the natural isomorphisms

$$\begin{gathered} (A^\bullet \otimes_{\mathcal{O}_X} B^\bullet) \otimes_{\mathcal{O}_X} X^\bullet \xrightarrow{\sim} A^\bullet \otimes_{\mathcal{O}_X} (B^\bullet \otimes_{\mathcal{O}_X} X^\bullet) \\ \mathcal{O}_X \otimes_{\mathcal{O}_X} X^\bullet \xrightarrow{\sim} X^\bullet \end{gathered} \tag{8.61}$$

for $A^\bullet, B^\bullet$ complexes of quasicoherent $\mathcal{O}_X$-modules and $X^\bullet$ a complex of quasicoherent right $\mathcal{A}$-modules.

**Remark 8.44.** The action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ satisfies the local-to-global principle (see remark 8.22) since the action $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on itself does so.

We will now continue to derive some properties of the notion of support that the action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ induces on objects of the latter category.

**Proposition 8.45.** Let $V \subset X$ be a specialization-closed subset. Then $\mathbf{D}_V(\mathrm{Qcoh}(\mathcal{A}))$ coincides with the subcategory $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))_V$ of all complexes $C^\bullet \in \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ such that $\mathrm{supp}(C^\bullet) \subset V$. In particular, the subcategories $\mathbf{D}_V(\mathrm{Qcoh}(\mathcal{A}))$ are smashing.

*Proof.* If $C^\bullet$ is a complex of quasicoherent right $\mathcal{A}$-modules, then we need to show that $\mathrm{supp}(C^\bullet) \subset V \Leftrightarrow \mathrm{Supp}(C^\bullet) \subset V$. If $X = \bigcup_i U_i$ is an open cover, then it suffices to show that $\mathrm{supp}(C^\bullet) \cap U_i \subset V \cap U_i \Leftrightarrow \mathrm{Supp}(C^\bullet) \cap U_i \subset V \cap U_i$ for all $i$. Let $U_i = \mathrm{Spec}\, R_i, i = 1, \ldots, n$ be a cover of $X$ by affine opens with closed complements $Z_i$ and set $V_i := U_i \cap V$. Notice that the sets $V_i$ are still specialization-closed in $U_i$. We have $\mathrm{supp}(C^\bullet|_{U_i}) = \mathrm{supp}(\mathrm{L}_{Z_i}\mathcal{O}_X * C^\bullet) = \mathrm{supp}(C^\bullet) \cap U_i$ by proposition 8.17 and $\mathrm{Supp}(C^\bullet|_{U_i}) = \mathrm{Supp}(\mathrm{L}_{Z_i}\mathcal{O}_X * C^\bullet) = \mathrm{Supp}(C^\bullet) \cap U_i$ since localization is exact. Hence we have reduced to showing that

$$\mathrm{supp}(C^\bullet|_{U_i}) \subset V_i \Leftrightarrow \mathrm{Supp}(C^\bullet|_{U_i}) \subset V_i \text{ for } i = 1, \ldots n. \tag{8.62}$$

But now, we can assume that $\mathcal{A}$ is given as an $R_i$-algebra $A$ and $C^\bullet|_{U_i}$ a complex of right $A$-modules. Since both supp and Supp can be computed by first applying the forgetful functor $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \to \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$, the result follows from [189, proposition 3.14], where it is shown that for the complex of $R_i$-modules $C^\bullet|_{U_i}$, the sets $\mathrm{supp}(C^\bullet|_{U_i})$ and $\mathrm{Supp}(C^\bullet|_{U_i})$ have the same minimal elements.

The last statement follows from the first and lemma 8.15. □

Let us show that supp and Supp coincide for small complexes.

**Proposition 8.46.** Let $C^\bullet \in \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ such that $\mathrm{H}^*(C^\bullet)$ is bounded and coherent. Then $\mathrm{supp}(C^\bullet) = \mathrm{Supp}(C^\bullet)$.

*Proof.* As in the proof of proposition 8.45, we notice that if $X = \bigcup_i U_i$ is a cover by affine opens with complements $Z_i$, then it suffices to show that

$$\underbrace{\mathrm{supp}(C^\bullet) \cap U_i}_{=\mathrm{supp}(C^\bullet|_{U_i})} = \underbrace{\mathrm{Supp}(C^\bullet) \cap U_i}_{=\mathrm{Supp}(C^\bullet|_{U_i})} \tag{8.63}$$

for all $i$. Hence, we have reduced to the affine case, where the result is implied from the corresponding one for complexes in $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$. But the latter is well known. □

**Remark 8.47.** Given a coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ (on which $\mathcal{O}_X$ acts by centrally by assumption) we can consider $\mathrm{Z}(\mathcal{A})$ as a commutative coherent $\mathcal{O}_X$-algebra. Let

$$\pi\colon Z := \mathbf{Spec}_X\, \mathrm{Z}(\mathcal{A}) \to X \tag{8.64}$$

be the relatively affine scheme given by $\mathrm{Z}(\mathcal{A})$. We can consider $\mathcal{A}$ as a coherent $\mathcal{O}_Z$-algebra, which we will denote $\mathcal{B}$, and by [55, proposition 3.5] we have an equivalence $\mathrm{Qcoh}_X \mathcal{A} \cong \mathrm{Qcoh}_Z \mathcal{B}$. The action of $\mathbf{D}(\mathrm{Qcoh}\, X)$ and $\mathbf{D}(\mathrm{Qcoh}\, Z)$ will be different in general.

### 8.5.2 Unwinding the definitions

With all the technical material we have assembled so far, let us look once more at definition 8.21. Let $\mathcal{T} = \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ and $\mathcal{K} = \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$. Recall that $X$ is a noetherian separated scheme and $\mathcal{A}$ is a coherent sheaf of $\mathcal{O}_X$-algebras.

**Convention 8.48.** We will write

$$\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) \quad \text{and} \quad \mathrm{CH}_i^{\Delta}(X,\mathcal{A}) \tag{8.65}$$

for the groups $\mathrm{Z}_i^{\Delta}(\mathcal{T},\mathcal{K})$ and $\mathrm{CH}_i^{\Delta}(\mathcal{T},\mathcal{K})$, respectively.

We have

$$\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) = \mathrm{K}_0\left((\mathcal{K}_{(i)}/\mathcal{K}_{(i-1)})^{\mathrm{c}}\right) \tag{8.66}$$

by definition, and both categories $\mathcal{K}_{(i)}, \mathcal{K}_{(i+1)}$ are compactly generated. Hence, we have that

$$(\mathcal{K}_{(i)}/\mathcal{K}_{(i-1)})^{\mathrm{c}} \cong \left((\mathcal{K}_{(i)})^{\mathrm{c}}/(\mathcal{K}_{(i-1)})^{\mathrm{c}}\right)^{\natural} \tag{8.67}$$

by [131, theorem 5.6.1]. Furthermore, $(\mathcal{K}_{(i)})^{\mathrm{c}}$ coincides with the full subcategory of $\mathcal{K}^{\mathrm{c}}$ consisting of objects with support in codimension $\geq i$ by [126, proposition 2.23]. From theorem 8.35, we have

$$\mathcal{K}^{\mathrm{c}} \cong \mathbf{D}^{\mathrm{perf}}(\mathcal{A}) \subset \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A})). \tag{8.68}$$

Because $\mathcal{A}$ is assumed to be coherent, the supports Supp and supp coincide for objects of $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A}))$ by proposition 8.46. It follows that

$$\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) = \mathrm{K}_0\left(\left(\mathbf{D}^{\mathrm{perf}}_{\leq i}(\mathcal{A})/\,\mathbf{D}^{\mathrm{perf}}_{\leq i-1}(\mathcal{A})\right)^{\natural}\right). \tag{8.69}$$

If $\mathcal{A}$ is additionally of finite global dimension, $\mathbf{D}^{\mathrm{perf}}(\mathcal{A}) \cong \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A}))$ and we get from corollary 8.43 that

$$\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) = \mathrm{K}_0\left(\mathbf{D}^{\mathrm{b}}\left(\mathrm{coh}_{\leq i}(\mathcal{A})/\,\mathrm{coh}_{\leq i-1}(\mathcal{A})\right)\right) = \mathrm{K}_0\left(\mathrm{coh}_{\leq i}(\mathcal{A})/\,\mathrm{coh}_{\leq i-1}(\mathcal{A})\right). \tag{8.70}$$

Similarly, we deduce in this case an isomorphism of sequences of abelian groups

(8.71)

$$\begin{array}{ccccc}
\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) & \longrightarrow & \mathrm{K}_0\left((\mathcal{K}_{(i+1)}/\mathcal{K}_{(i-1)})^{\mathrm{c}}\right) & \longrightarrow & \mathrm{Z}_{i+1}^{\Delta}(X,\mathcal{A}) \\
\| & & \| & & \| \\
\mathrm{K}_0\left(\mathrm{coh}_{\leq i}(\mathcal{A})/\,\mathrm{coh}_{\leq i-1}(\mathcal{A})\right) & \xrightarrow{\iota} & \mathrm{K}_0\left(\mathrm{coh}_{\leq i+1}(\mathcal{A})/\,\mathrm{coh}_{\leq i-1}(\mathcal{A})\right) & \xrightarrow{\pi} & \mathrm{K}_0\left(\mathrm{coh}_{\leq i+1}(\mathcal{A})/\,\mathrm{coh}_{\leq i}(\mathcal{A})\right)
\end{array}$$

which are exact in the middle. Hence, we deduce from proposition 8.26 an isomorphism $\mathrm{CH}_i^{\Delta}(X,\mathcal{A}) \cong \mathrm{im}(\iota) = \ker(\pi)$ for this situation. The lower sequence is the end of the K-theory long exact localization sequence for the Serre localization

$$\mathrm{coh}_{\leq i}(\mathcal{A})/\,\mathrm{coh}_{\leq i-1}(\mathcal{A}) \to \mathrm{coh}_{\leq i+1}(\mathcal{A})/\,\mathrm{coh}_{\leq i-1}(\mathcal{A}) \to \mathrm{coh}_{\leq i+1}(\mathcal{A})/\,\mathrm{coh}_{\leq i}(\mathcal{A}) \tag{8.72}$$

and hence

$$\mathrm{CH}_i^{\Delta}(X,\mathcal{A}) \cong \mathrm{coker}\left(\mathrm{K}_1\left(\mathrm{coh}_{\leq i+1}(\mathcal{A})/\,\mathrm{coh}_{\leq i}(\mathcal{A})\right) \to \mathrm{K}_0\left(\mathrm{coh}_{\leq i}(\mathcal{A})/\,\mathrm{coh}_{\leq i-1}(\mathcal{A})\right)\right). \tag{8.73}$$

There is also a local description of $\mathrm{Z}_i^{\Delta}(X,\mathcal{A})$. Abstractly, it follows from [205] and [126, proposition 2.18, lemma 2.19], that

$$\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) = \coprod_{x\in X_{(i)}} \mathrm{K}_0\left((\Gamma_x\mathcal{K})^{\mathrm{c}}\right), \tag{8.74}$$

where $X_i$ is the set of points $x \in X$ such that $\dim(x) = i$.

**Lemma 8.49.** Suppose $\mathcal{A}$ is coherent. Then

$$(\Gamma_x\mathcal{K})^{\mathrm{c}} \cong \mathbf{D}^{\mathrm{perf}}_{\{x\}}(\mathcal{A}_x). \tag{8.75}$$

*Proof.* Since for any object $A \in \mathcal{K}$ we have, by definition, $\Gamma_x A = \Gamma_{\overline{\{x\}}}\mathrm{L}_{Y_x}\mathcal{O}_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} A$, it follows that

$$\Gamma_x\mathcal{K} = \Gamma_{\overline{\{x\}}}\mathcal{O}_X * \left(\mathrm{L}_{Y_x}\mathcal{O}_X * \mathcal{K}\right). \tag{8.76}$$

The subcategory $\mathbf{D}_{Y_x}(\mathcal{A})$ is smashing by proposition 8.45. It follows from lemma 8.15 and proposition 8.37 that $\mathrm{L}_{Y_x}\mathcal{O}_X * \mathcal{K} \cong \mathbf{D}(\mathrm{Mod}(\mathcal{A}_x))$. The compact objects of $\Gamma_x\mathcal{K}$ are given by the compact objects $a$ of $\mathrm{L}_{Y_x}\mathcal{O}_X * \mathcal{K}$ with $\mathrm{supp}(a) \subset \overline{\{x\}}$: the inclusion functor $I\colon \Gamma_x\mathcal{K} \to \mathrm{L}_{Y_x}\mathcal{O}_X * \mathcal{K}$ has a coproduct-preserving right adjoint $\Gamma_{\overline{\{x\}}}(\mathbb{I}) * -$ and hence preserves compactness. Thus, the compact objects of $\Gamma_x\mathcal{K}$ embed into the compact objects of $\mathrm{L}_{Y_x}\mathcal{O}_X * \mathcal{K}$ with support in $\overline{\{x\}}$. On the other hand, if $a$ is a compact object of $\mathrm{L}_{Y_x}\mathcal{O}_X * \mathcal{K}$ with support in $\overline{\{x\}}$, then the localization triangle

$$\Gamma_{\overline{\{x\}}}(\mathbb{I}) * a \to a \to L\overline{\{x\}}(\mathbb{I}) * a \to \Sigma\left(\Gamma_{\overline{\{x\}}}(\mathbb{I}) * a\right) \tag{8.77}$$

and proposition 8.17 show that $\Gamma_{\overline{\{x\}}}(\mathbb{I}) * a \cong a$, and hence $a$ belongs to the essential image of the embedding $I$.

Since $\mathbf{D}(\mathrm{Mod}(\mathcal{A}_x))^{\mathrm{c}} \cong \mathbf{D}^{\mathrm{perf}}(\mathcal{A}_x)$ and supp = Supp for its objects by proposition 8.46, the desired description follows. □

**Lemma 8.50.** Let $(R, \mathfrak{m})$ be a commutative noetherian local ring and $A$ a (module-)finite $R$-algebra. Then a right $A$-module $M$ has finite length over $A$ if and only if it has finite length over $R$.

*Proof.* Recall that a right module has finite length if and only if it is both artinian and noetherian. Hence, if $M$ has finite length over $R$, it must also have finite length over $A$, since every chain of $A$-submodules of $M$ is also a chain of $R$-submodules.

In order to prove that right $A$-modules of finite $A$-length also have finite $R$-length, it suffices to show that all simple right $A$-modules have finite $R$-length: one can then refine finite composition series over $A$ to finite composition series over $R$. In order to study simple right $A$-modules it suffices to consider simple modules over $A/\mathrm{J}(A)$, since the Jacobson radical annihilates all simple modules, by definition. We have $\mathrm{J}(R) = \mathfrak{m}$ and by [148, corollary 5.9], it follows that $\mathfrak{m}A \subset \mathrm{J}(A)$, and hence we have a surjection $A/\mathfrak{m}A \twoheadrightarrow A/\mathrm{J}(A)$. By assumption, $A/\mathfrak{m}A$ is a finite $R$-module with support contained in $\{\mathfrak{m}\}$ and hence has finite length over $R$. It follows that $A/\mathrm{J}(A)$ has finite $R$-length as well. Hence, the finite length right modules over $A/\mathrm{J}(A)$ have finite length over $R$, which holds in particular for the simple ones. □

**Corollary 8.51.** Suppose $\mathcal{A}$ is coherent. Then

$$\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) = \coprod_{x\in X_{(i)}} \mathrm{K}_0\left(\mathbf{D}^{\mathrm{perf}}_{\mathrm{fl}}(\mathcal{A}_x)\right). \tag{8.78}$$

where $\mathbf{D}^{\mathrm{perf}}_{\mathrm{fl}}(\mathcal{A}_x) \subset \mathbf{D}^{\mathrm{perf}}(\mathcal{A}_x)$ denotes the full subcategory of complexes with finite length cohomology. If furthermore $\mathcal{A}$ has finite global dimension, then

$$\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) = \coprod_{x\in X_{(i)}} \mathrm{K}_0\left(\mathbf{D}^{\mathrm{b}}(\mathrm{fl}\,\mathcal{A}_x)\right), \tag{8.79}$$

where $\mathrm{fl}\,\mathcal{A}_x$ denotes the abelian category of right $\mathcal{A}_x$-modules of finite length.

*Proof.* For the first statement, it suffices to prove that $\mathbf{D}^{\mathrm{perf}}_{\{x\}}(\mathcal{A}_x) \cong \mathbf{D}^{\mathrm{perf}}_{\mathrm{fl}}(\mathcal{A}_x)$ by lemma 8.49. This follows from lemma 8.50 since a complex $C^\bullet \in \mathbf{D}^{\mathrm{perf}}(\mathcal{A}_x)$ has support in $\{x\}$ if and only if $\operatorname{Supp}\mathrm{H}^\bullet(C^\bullet)) \subset \{x\}$ if and only if $\mathrm{H}^\bullet(C^\bullet)$ has finite $\mathcal{O}_{X,x}$-length if and only if $\mathrm{H}^\bullet(C^\bullet)$ has finite $\mathcal{A}_x$-length.

For the second assertion, corollary 8.43 gives

$$\mathbf{D}^{\mathrm{perf}}_{\{x\}}(\mathcal{A}_x) \cong \mathbf{D}^{\mathrm{b}}_{\{x\}}(\mathrm{mod}(\mathcal{A}_x)) \cong \mathbf{D}^{\mathrm{b}}(\mathrm{mod}(\mathcal{A}_x)_{\{x\}}) \tag{8.80}$$

Now a finitely generated right $A_x$-module has support in $\{x\}$ if and only if it has finite length as an $R$-modules if and only if it has finite length as a right $\mathcal{A}_x$-module by lemma 8.50. This shows that $\mathrm{mod}(\mathcal{A}_x)_{\{x\}} \cong \mathrm{fl}\,\mathcal{A}_x$ and finishes the proof. □

Corollary 8.51 makes it possible to give a computation of $\mathrm{Z}_i^{\Delta}(X,\mathcal{A})$ in large generality.

**Theorem 8.52.** Let $X$ be a noetherian scheme and $\mathcal{A}$ a coherent $\mathcal{O}_X$-algebra of finite global dimension. Then

$$\mathrm{Z}_i^{\Delta}(X,\mathcal{A}) = \bigoplus_{x\in X_{(i)}} \mathbb{Z}^{r_x} \tag{8.81}$$

where $r_x < \infty$ is the number of isomorphism classes of simple right modules of $\mathcal{A}_x$.

*Proof.* By corollary 8.51, it suffices to show that $\mathrm{K}_0(\mathbf{D}^{\mathrm{b}}(\mathrm{fl}\,\mathcal{A}_x) = \mathrm{K}_0(\mathrm{fl}\,\mathcal{A}_x) = \mathbb{Z}^{r_x}$ with $r_x < \infty$. From the proof of lemma 8.50 we see that the simple $\mathcal{A}_x$-modules correspond to the simple $\mathcal{A}_x/\mathrm{J}(\mathcal{A}_x)$-modules, and that the latter algebra is of finite length over $\mathcal{O}_{X,x}$. This implies that $\mathcal{A}_x/\mathrm{J}(\mathcal{A}_x)$ is right Artinian and hence has $r_x < \infty$ simple right modules (all of them occur in a composition series of $\mathcal{A}_x$ over itself by the Jordan-Hölder theorem). A standard induction on the composition multiplicities of these simple modules shows that $\mathrm{K}_0(\mathrm{fl}\,\mathcal{A}_x) = \mathbb{Z}^{r_x}$ as desired. □

Let us finish the section with an easy observation concerning the vanishing of $\mathrm{Z}_i^{\Delta}(X, \mathcal{A})$ and $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$.

**Proposition 8.53.** Suppose $\dim(\mathrm{supp}(\mathcal{A})) = n$. Then

$$\mathrm{Z}_i^{\Delta}(X, \mathcal{A}) = \mathrm{CH}_i^{\Delta}(X, \mathcal{A}) = 0 \tag{8.82}$$

for all $i > n$.

*Proof.* If $i > n$, then $\mathcal{K}_i = \mathcal{K}_{i-1} = \mathcal{K}$ and hence

$$\mathrm{Z}_i^{\Delta}(X, \mathcal{A}) = \mathrm{K}_0\left((\mathcal{K}_{(i)}/\mathcal{K}_{(i-1)})^{\mathrm{c}}\right) = 0, \tag{8.83}$$

which also implies $\mathrm{CH}_i^{\Delta}(X, \mathcal{A}) = 0$. □

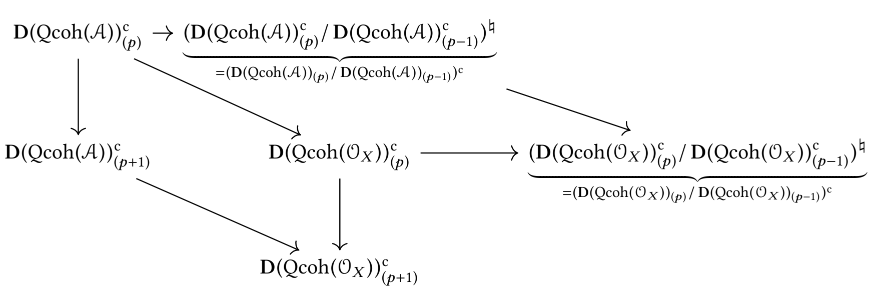
### 8.5.3 Comparison to Chow groups of $X$ for coherent $\mathcal{O}_X$-algebras on regular schemes

Suppose that $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra and that $X$ is regular. By definition of supp, the forgetful functor $U\colon \mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \to \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ induces functors

$$\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))_{(p)} \to \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))_{(p)} \tag{8.84}$$

for all $p \geq 0$. If $C^\bullet$ is a perfect complex in $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$, then $U(C^\bullet)$ will be an object of $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(X)) = \mathbf{D}^{\mathrm{perf}}(X)$ and hence $U$ preserves compactness. Hence, we obtain a commutative diagram

(8.85)




in which the horizontal arrows are given by the Verdier quotient followed by the inclusion into the idempotent completion, the vertical arrows are inclusions and the diagonal ones are induced by $U$.

**Remark 8.54.** It is possible to construct the above diagram without assuming $X$ to be regular: the main obstruction is for $U$ to preserve compactness. This happens for example, when $U$ admits a coproduct-preserving right adjoint. But the functor $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(U(\mathcal{A}), -)$ is always right adjoint to $U$. It will preserve coproducts if $U(\mathcal{A})$ is a perfect complex over $X$ by [168, proof right after example 1.13]. Hence, we see that, instead of assuming that $X$ is regular, it suffices that $U(\mathcal{A})$ is perfect. If $X$ is regular this is, of course, always the case.

**Proposition 8.55.** Suppose that $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra on a noetherian regular scheme $X$. Let $\mathcal{T} = \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ and $\mathcal{K} = \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$. Then the forgetful functor $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))_{(p)} \to \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))_{(p)}$ induces group homomorphisms

(8.86)
$$\mathrm{Z}^{\Delta}_p(X, \mathcal{A}) \to \mathrm{Z}^{\Delta}_p(X, \mathcal{O}_X) = \mathrm{Z}_p(X) \quad \text{and} \quad \mathrm{CH}^{\Delta}_p(X, \mathcal{A}) \to \mathrm{CH}^{\Delta}_p(X, \mathcal{O}_X) = \mathrm{CH}_p(X)$$

for all $p \geq 0$.

*Proof.* This follows immediately from theorem 8.23 and the definitions of $\mathrm{Z}^{\Delta}_p(X, \mathcal{A})$ and $\mathrm{CH}^{\Delta}_p(X, \mathcal{A})$ by applying $\mathrm{K}_0(-)$ to (8.85). □

**Remark 8.56.** If in proposition 8.55 we only assume that $U(\mathcal{A})$ is perfect instead of $X$ being regular (see remark 8.54), then $U$ still gives group homomorphisms

(8.87)
$$\mathrm{Z}^{\Delta}_p(X, \mathcal{A}) \to \mathrm{Z}^{\Delta}_p(X, \mathcal{O}_X) \quad \text{and} \quad \mathrm{CH}^{\Delta}_p(X, \mathcal{A}) \to \mathrm{CH}^{\Delta}_p(X, \mathcal{O}_X)$$

for all $p \geq 0$.

**Remark 8.57.** Extension of scalars $\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ has a a coproduct-preserving right adjoint $U$ and hence preserves compact object. For $C^\bullet \in \mathbf{D}^{\mathrm{perf}}(X)$, we have

(8.88)
$$\mathrm{supp}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}_X} C^\bullet) = \mathrm{supp}(\mathcal{A}) \cap \mathrm{supp}(C^\bullet)$$

from which we deduce that $\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ restricts to

(8.89)
$$\mathbf{D}^{\mathrm{perf}}(\mathcal{O}_X)_{(p)} \to \mathbf{D}^{\mathrm{perf}}(\mathcal{A})_{(p)}$$

for all $p \geq 0$. Hence, by a similar argument as for $U$, we obtain that extension of scalars induces morphisms $\mathrm{CH}^{\Delta}_p(X, \mathcal{O}_X) \to \mathrm{CH}^{\Delta}_p(X, \mathcal{A})$.

Note however, that if $\dim(\mathrm{supp}(\mathcal{A})) = q$, then these morphisms are necessarily trivial for $p > q$ since $\mathrm{Z}^{\Delta}_p(X, \mathcal{A}) = \mathrm{CH}^{\Delta}_p(X, \mathcal{A}) = 0$ in this case by proposition 8.53.

## 8.6 The case of coherent commutative $\mathcal{O}_X$-algebras

In the following, we will show, how the framework we have set up lets us deal with finite morphisms between noetherian schemes. Let $X$ be a noetherian separated scheme and $\mathcal{A}$ a *commutative* $\mathcal{O}_X$-algebra which is *coherent* as an $\mathcal{O}_X$-module. Then $\mathcal{A}$ corresponds to an affine morphism $\varphi\colon Y := \mathbf{Spec}_X \mathcal{A} \to X$ and there is an equivalence of categories $\Theta\colon \mathrm{Qcoh}(\mathcal{A}) \cong \mathrm{Qcoh}(\mathcal{O}_Y)$ that makes the following diagram commute up to natural isomorphism:

$$\begin{array}{ccc} \mathrm{Qcoh}(\mathcal{A}) & \xrightarrow{\Theta} & \mathrm{Qcoh}(\mathcal{O}_Y) \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle \varphi_*} & \\ & \mathrm{Qcoh}(\mathcal{O}_X) & \end{array} \tag{8.90}$$

Let us note that $\Theta$ also restricts to an equivalence between the subcategories of coherent modules and the restriction makes a diagram similar to (8.90) commute, with Qcoh(−) replaced by coh(−). The following three results should be well-known.

**Lemma 8.58.** The morphism $\varphi$ is finite. In particular, $Y$ is noetherian and separated.

*Proof.* This is an immediate consequence of the construction of $\mathbf{Spec}\,\mathcal{A}$: over each open affine $U = \operatorname{Spec} R$ of $X$ lies an open affine $\operatorname{Spec} \mathcal{A}(U)$, and $\mathcal{A}(U)$ is a finite $R$-module since $\mathcal{A}$ was assumed to be a coherent sheaf on $X$. □

**Lemma 8.59.** Let $f\colon Y \to X$ be a morphism of schemes and assume $X$ locally noetherian.

1. For any coherent $\mathcal{O}_X$-module $M$, we have $\mathrm{Supp}(f^*(M)) = f^{-1}(\mathrm{Supp}(M))$.
2. Suppose $f$ is finite. For any closed subset $Z \subset \mathrm{im}(f)$, we have

   $$\dim(f^{-1}(Z)) = \dim(Z) \tag{8.91}$$

   and for any closed set $W \subset Y$, we have

   $$\dim(f(W)) = \dim(W) \tag{8.92}$$

*Sketch of the proof.* For the first assertion we can assume that $X, Y$ are affine, in this case the statement is proved in [15, exercise 3.19(viii)]. For the second statement, we consider the fibre square

$$\begin{array}{ccc} f^{-1}Z & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & \mathrm{im}(f) \end{array} \tag{8.93}$$

and use that for finite and surjective morphisms, domain and codomain have the same Krull dimension. The last assertion follows from the second one by considering the composition $f|_W\colon W \to Y \xrightarrow{f} X$. □

**Proposition 8.60.** Suppose $X$ is a locally noetherian scheme and $f\colon X \to Y$ is an affine closed morphism and $M$ a quasicoherent $\mathcal{O}_X$-module. Then

$$\text{(8.94)}\quad \operatorname{Supp}(f_*M) = f(\operatorname{Supp}(M)).$$

*Sketch of the proof.* We shall compute the stalks of the sheaf $f_*M$ at $y \in Y$. Since $f$ is closed, this can be done using all opens on $X$, i.e. $(f_*M)_y = \varinjlim_{V \supset f^{-1}(y)} M(V)$. The set $f^{-1}(y)$ will be contained in an affine open $\operatorname{Spec} R \subset X$ because $f$ is affine and hence, we can assume that $M$ is an $R$-module and $P := f^{-1}(y)$ is a set of prime ideals of $R$. We rewrite

$$\text{(8.95)}\quad (f_*M)_y = \varinjlim_{V \supset f^{-1}(y)} M(V) = \varinjlim_{D(r) \supset P} M_r,$$

where $D(r)$ runs over the basic opens of $\operatorname{Spec}(R)$ that contain $P$. From this, we see that $(f_*M)_y = S^{-1}M$, where $S := R \setminus \bigcup_{\mathfrak{p} \in P} \mathfrak{p}$. It follows that $(f_*M)_y = 0$ if and only if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in P = f^{-1}(y)$, which proves the claim. □

**Corollary 8.61.** The equivalence $\Theta\colon\ \operatorname{Qcoh}(\mathcal{A}) \to \operatorname{Qcoh}(\mathcal{O}_Y)$ respects dimension of support: if $M \in \operatorname{Qcoh}(\mathcal{A})$, then $\dim(\operatorname{Supp}_X(M)) = \dim(\operatorname{Supp}_Y(\Theta(M)))$ . Hence, $\Theta$ induces exact equivalences

$$\text{(8.96)}\quad \operatorname{Qcoh}(\mathcal{A})_{\leq p} \xrightarrow{\sim} \operatorname{Qcoh}(\mathcal{O}_Y)_{\leq p}$$

for all $p \geq 0$.

*Proof.* By definition and (8.90), we have

$$\text{(8.97)}\quad \dim(\operatorname{Supp}(M)) = \dim(\operatorname{Supp}_X(U(M))) = \dim(\operatorname{Supp}_X(\varphi_*(\Theta(M)))).$$

Since $\mathcal{A}$ was assumed to be coherent, $\varphi$ is finite by lemma 8.58 and it follows from proposition 8.60 that

$$\text{(8.98)}\quad \dim(\operatorname{Supp}_X(\varphi_*(\Theta(M)))) = \dim(\varphi(\operatorname{Supp}_X(\Theta(M))))$$

as finite morphisms are in particular affine and (universally) closed. By lemma 8.59, the latter quantity is equal to $\dim(\operatorname{Supp}_Y(\Theta(M)))$, which proves the claim. □

**Corollary 8.62.** The functor $\Theta$ induces an equivalence

$$\text{(8.99)}\quad \mathbf{D}(\operatorname{Qcoh}(\mathcal{A}))_{(p)} \cong \mathbf{D}(\operatorname{Qcoh}(\mathcal{O}_Y))_{(p)}$$

for all $p \geq 0$.

*Proof.* The equivalence $\Theta$ is exact (as any equivalence of abelian categories) and hence induces and equivalence $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A})) \cong \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_Y))$. Now, it suffices to remark that for $C^\bullet \in \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ we have

$$\begin{aligned} C^\bullet \in \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))_{(p)} &\Leftrightarrow \mathrm{H}^\bullet(C^\bullet) \in \mathrm{Qcoh}(\mathcal{A})_{\leq p} \\ &\Leftrightarrow \mathrm{H}^\bullet(\Theta(C^\bullet)) \in \mathrm{Qcoh}(\mathcal{O}_Y)_{\leq p} \\ &\Leftrightarrow \Theta(C^\bullet) \in \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))_{(p)} \end{aligned} \tag{8.100}$$

where we used proposition 8.45 and corollary 8.61. □

**Theorem 8.63.** Let $X$ be a separated scheme of finite type over a field and $\mathcal{A}$ a coherent sheaf of *commutative* $\mathcal{O}_X$-algebras. Then

$$\mathrm{CH}_p^\Delta(X, \mathcal{A}) \cong \mathrm{CH}_p^\Delta(Y, \mathcal{O}_Y) \tag{8.101}$$

for all $p \geq 0$. In particular if $\mathbf{Spec}\,\mathcal{A}$ is regular ($\Leftrightarrow$ $\mathcal{A}$ has finite global dimension), then

$$\mathrm{CH}_p^\Delta(X, \mathcal{A}) \cong \mathrm{CH}_p(\mathbf{Spec}\,\mathcal{A}). \tag{8.102}$$

*Proof.* There is a diagram

(8.103)

$$\begin{array}{ccccc}
\mathrm{K}_0\left((\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))_{(p)})^{\mathrm{c}}\right) & \to & \mathrm{K}_0\left((\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))_{(p)}/\,\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))_{(p+1)})^{\mathrm{c}}\right) & & \\
\downarrow & \searrow & & \searrow & \\
\mathrm{K}_0\left((\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))_{(p-1)})^{\mathrm{c}}\right) & & \mathrm{K}_0\left((\mathbf{D}(\mathcal{O}_Y)_{(p)})^{\mathrm{c}}\right) & \longrightarrow & \mathrm{K}_0\left((\mathbf{D}(\mathcal{O}_Y)_{(p)}/\,\mathbf{D}(\mathcal{O}_Y)_{(p+1)})^{\mathrm{c}}\right) \\
& \searrow & \downarrow & & \\
& & \mathrm{K}_0\left((\mathbf{D}(\mathcal{O}_Y)_{(p-1)})^{\mathrm{c}}\right) & &
\end{array}$$

where all diagonal arrows are isomorphisms induced by $\Theta$, as follows from corollary 8.62. This immediately gives the desired isomorphisms of Chow groups. □

## 8.7 Relative tensor triangular Chow groups for orders

In this section we study relative tensor triangular Chow groups for a special class of coherent $\mathcal{O}_X$-algebras: orders. These are particularly well-behaved *noncommutative* algebras, whose definition we recall in section 8.7.1. In their modern incarnation they were defined in [19] and the main reference is [184]. The main goal is to show that they coincide with other invariants in the literature, as is the case in the commutative setting where tensor triangular Chow groups agree with the classical Chow groups, see [125, 126].

We give some general results on cycle groups in section 8.7.2, based on theorem 8.52. We get a description of the top degree cycle groups for any order in proposition 8.69.

Finally we will use the structure theory for hereditary orders over discrete valuation rings to describe all cycle groups of hereditary orders and the codimension one cycle groups of tame orders, making the result in theorem 8.52 concrete in a well-known example.

In section 8.7.3 we discuss Chow groups for orders. An easy corollary of the theory is a description of the top degree Chow group in proposition 8.72. More importantly, we recall the definition of various class groups in the theory of orders, and we show that these classical invariants agree with the appropriate tensor triangular Chow groups.

In section 8.7.4 we study the Chow groups of group rings over Dedekind domains, for which it is again possible to relate the tensor triangular Chow groups to classical invariants. We give some explicit examples on how one can compute them for integral group rings, using tools from algebraic number theory and representation theory.

### 8.7.1 Preliminaries on orders

In this section we will introduce some basic results about orders on schemes. There are no new results here, but the literature at this level of generality is somewhat scattered.

Observe that for most of this section we will assume that we are working in a central simple algebra. This corresponds to the more geometric approach to the theory of orders. In section 8.7.4 we will relax this condition, and consider algebras which are only separable over the generic point, as is common in representation theory and algebraic number theory. We will explain how the results of sections 8.7.2 and 8.7.3 change in this more general situation.

**Definition 8.64.** Let $X$ be an integral normal noetherian scheme with function field $K$. Let $A_K$ be a central simple $K$-algebra. An $\mathcal{O}_X$*-order* $\mathcal{A}$ in $A_K$ is a torsion-free coherent $\mathcal{O}_X$-algebra whose generic fibre is $A_K$.

We say that $\mathcal{A}$ is a *maximal order* if it is not properly contained in another order.

In [184] (maximal) orders are studied in both the geometric and arithmetic setting, mostly in the case of dimension 1. The behaviour of orders in higher dimension quickly becomes more and more complicated.

We will need two more classes of orders, besides just the maximal ones. Recall that Auslander–Goldman characterised maximal orders as those orders which are reflexive as $\mathcal{O}_X$-modules, and for which $\mathcal{A}_{\eta_Y}$ is a maximal order over the discrete valuation ring $\mathcal{O}_{X,\eta_Y}$, for all $\eta_Y$ a point of codimension 1. In dimension one there is a larger class of orders whose behaviour is as nice as that of the maximal orders, and of which maximal orders are a special instance.

**Definition 8.65.** Assume that $X$ is regular and of dimension 1. Then we say that $\mathcal{A}$ is an *hereditary order* if $\mathcal{A}(U)$ is of global dimension 1 for every affine open $U \subseteq X$.

For hereditary (and maximal) orders in dimension 1 there exists an extensive structure theory. Inspired by the Auslander–Goldman maximality criterion we can introduce a final class of orders, for which one can bootstrap the structure theory of hereditary orders.

**Definition 8.66.** We say that $\mathcal{A}$ is a *tame order* if it is reflexive as an $\mathcal{O}_X$-module, and $\mathcal{A}_{\eta_Y}$ is an hereditary order over the discrete valuation ring $\mathcal{O}_{X,\eta_Y}$, for all $\eta_Y$ a point of codimension 1.

The notion of tame generalises hereditary orders to higher dimensions.

We now give some examples of orders for which we can describe the tensor triangular cycle and Chow groups.

**Example 8.67.** The easiest examples of maximal orders are matrix algebras and their étale twisted forms: Azumaya algebras.

**Example 8.68.** An example of an hereditary but non-maximal order on $\mathbb{P}^1_k$ is

$$\mathcal{A} := \begin{pmatrix} \mathcal{O}_{\mathbb{P}^1_k} & \mathcal{O}_{\mathbb{P}^1_k} \\ \mathcal{O}_{\mathbb{P}^1_k}(-p) & \mathcal{O}_{\mathbb{P}^1_k} \end{pmatrix} \tag{8.104}$$

where $p \in \mathbb{P}^1_k$ is a closed point. The algebra structure is induced from the embedding in $\mathrm{Mat}_2(\mathcal{O}_{\mathbb{P}^1_k})$.

For each closed point $q \neq p$ we see that $\mathcal{A}_q$ is isomorphic to the matrix ring over $\mathcal{O}_{\mathbb{P}^1_k,q}$, whereas for the point $p$ we get the non-maximal order

$$\mathcal{A}_p \cong \begin{pmatrix} \mathcal{O}_{\mathbb{P}^1_k,p} & \mathcal{O}_{\mathbb{P}^1_k,p} \\ \mathfrak{m} & \mathcal{O}_{\mathbb{P}^1_k,p} \end{pmatrix}. \tag{8.105}$$

It is precisely this non-maximality that will contribute to the structure of the relative Chow group, see corollary 8.82.

### 8.7.2 Cycle groups

Using theorem 8.52 we have a complete description of cycle groups of coherent $\mathcal{O}_X$-algebras. In this section we discuss what happens in the special case of orders. First we observe that the top-dimensional Chow group always is of the same form.

**Proposition 8.69.** Let $X$ be an integral normal noetherian scheme of dimension $n$. Let $\mathcal{A}$ be an order on $X$. Then

$$\mathrm{Z}^{\Delta}_n(X,\mathcal{A}) \cong \mathbb{Z}. \tag{8.106}$$

*Proof.* Let $\eta$ be the unique generic point of $X$. Then $\mathcal{A}_\eta$ is a central simple algebra over the function field $\mathcal{O}_{X,\eta}$ and by Morita theory we can conclude from theorem 8.52, as there is a unique simple for a division algebra. □

There are several issues in computing the cycle and Chow groups for orders in other degrees:

1. there is no general structure theory for (maximal) orders on local rings in arbitrary dimension;
2. even if there is such a description (as will be the case in dimension 1) the non-splitness of the central simple algebra over the generic point will play an important role, because the higher K-theory of central simple algebras (let alone orders) is different in general from the K-theory of the center.

Nevertheless, in the one-dimensional case we can obtain an explicit description.

First we consider the complete local case, for which there exists an explicit description of hereditary orders [184, §39]. In this affine situation we will use ring-theoretical notation from op. cit. In particular, we consider a (complete) discrete valuation ring $(R, \mathfrak{m})$ whose field of fractions is denoted $K$, and an hereditary $R$-order $\Lambda$ in a central simple $K$-algebra $A \cong \mathrm{Mat}_n(D)$, where $D$ is a division algebra over $K$. Then there exists a unique maximal $R$-order $\Delta$ in $D$, and we have a block decomposition

$$(8.107)\quad \Lambda = \begin{pmatrix} \Delta & \mathrm{rad}\,\Delta & \mathrm{rad}\,\Delta & \dots & \mathrm{rad}\,\Delta \\ \Delta & \Delta & \mathrm{rad}\,\Delta & \dots & \mathrm{rad}\,\Delta \\ \Delta & \Delta & \Delta & \dots & \mathrm{rad}\,\Delta \\ \dots & & & & \dots \\ \Delta & \Delta & \Delta & \dots & \Delta \end{pmatrix}^{n_1,\dots,n_r}$$

where the block decomposition is given by putting $\mathrm{Mat}_{n_i \times n_j}(\Delta)$ (resp. $\mathrm{Mat}_{n_i \times n_j}(\mathrm{rad}\,\Delta)$) if $i \geq j$ (resp. $i < j$). In particular, $\sum_{i=1}^r n_i = n$.

**Definition 8.70.** The number of blocks $r$ in the block decomposition is the *type* of $\Lambda$.

The following result can be proved along the same lines as theorem 8.81, but we give an alternative proof here using dévissage in algebraic K-theory [182, §5].

**Proposition 8.71.** Let $R$ be a complete discrete valuation ring, with fraction field $K$ and residue field $k$. Let $\Lambda$ be an hereditary $R$-order in the central simple $K$-algebra $A$. Then

$$(8.108)\quad \mathrm{Z}_0^{\Delta}(R, \Lambda) \cong \mathbb{Z}^r$$

where $r$ is the type of $\Lambda$.

*Proof.* By dévissage for algebraic K-theory and the invariance of K-theory under nilpotent thickenings applied to [184, corollary 39.18(iii)] we have that

$$(8.109)\quad \mathrm{K}_0(\mathrm{fl}\,\Lambda) \cong \mathrm{K}_0(\Lambda/\mathrm{rad}\,\Lambda).$$

By [184, (39.17)] we have

$$(8.110)\quad \mathrm{K}_0(\Lambda/\mathrm{rad}\,\Lambda) \cong \bigoplus_{i=1}^{r} \mathrm{K}_0(\mathrm{Mat}_{n_i}(\Delta/\mathrm{rad}\,\Delta)) \cong \mathbb{Z}^{\oplus r}$$

where $\Delta/\mathrm{rad}\,\Delta$ is a skew field over $k$ by [184, corollary 17.5].

Similarly one can by dévissage appeal to [184, corollary 39.18(v)] for the conclusion. ☐

### 8.7.3 Chow groups in the regular case

In this section we prove the main results for orders: corollary 8.76 shows that for an hereditary order over a Dedekind domain the 0th relative Chow group agrees with the reduced projective class group, and if the order is moreover maximal it agrees with the ideal class group. These are classical invariants that will be introduced below.

In the setting of a quasiprojective curve over a field we get the analogous result in corollary 8.78, from which we obtain theorem 8.81.

As an immediate corollary to proposition 8.69 and the description of the rational equivalence we have the following general result.

**Proposition 8.72.** With notation and assumptions as in proposition 8.69 we have that

$$\mathrm{CH}_n^{\Delta}(X, \mathcal{A}) \cong \mathbb{Z}. \tag{8.111}$$

*Proof.* We have that $q^{\natural}(\ker(i))$ from (8.36) is zero because $i$ is an isomorphism if $p \geq n$. □

A similar proof of course works for every coherent $\mathcal{O}_X$-algebra, where the cycle group is given by the Grothendieck group of a certain finite-dimensional algebra over the function field, in particular it is easy to construct examples for which

$$\mathrm{CH}_n^{\Delta}(X, \mathcal{A}) \neq \mathbb{Z}, \tag{8.112}$$

e.g. by taking $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{O}_X$.

**Classical invariants** In the 1-dimensional case the only other tensor triangular Chow group we need to describe is $\mathrm{CH}_0^{\Delta}$, see proposition 8.53. We will do this using proposition 8.26, which allows us to interpret the tensor triangular Chow groups in terms of classical invariants such as the ideal class group and the reduced projective class group, whose definitions we now recall in the affine setting.

Let $R$ be a Dedekind domain, and denote its quotient field by $K$. Let $\Lambda$ be an $R$-order inside a central simple $K$-algebra $A$. Let $M$, $N$ be left $\Lambda$-modules. We say that they are *stably isomorphic* if there exists an integer $r$ and an isomorphism $M \oplus \Lambda^{\oplus r} \cong N \oplus \Lambda^{\oplus r}$.

**Definition 8.73.** The *ideal class group* (or *stable class group*) $\mathrm{Cl}\,\Lambda$ of $\Lambda$ consists of the stable isomorphism classes of left $\Lambda$-ideals (i.e. those submodules $I$ such that $KI = A$), where the group structure is defined in [184, theorem 35.5].

It is a one-sided generalisation of the usual class group (or Picard group). There also exists a two-sided version, which is different in general, see remark 8.77. Because we are only considering the module structure on one side, it is the former and not the latter that is important to us.

In this case the localisation sequence that is used to define rational equivalence in the zeroth Chow group as in (8.73) is also known as the *Bass–Tate sequence* [104, 150]. We will now recall the description from [183, §2]. In the relevant degrees the localization sequence takes on the form

$$\mathrm{K}_1(\Lambda) \to \mathrm{K}_1(A) \to \mathrm{K}_0(\mathrm{fl}\,\Lambda) \to \mathrm{K}_0(\Lambda) \to \mathrm{K}_0(A) \to 0. \tag{8.113}$$

We can also apply dévissage to the term $\mathrm{K}_0(\mathrm{fl}\,\Lambda)$, and obtain

$$\mathrm{K}_0(\mathrm{fl}\,\Lambda) \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R \setminus \{0\}} \mathrm{K}_0(\mathrm{fl}\,\Lambda_{\mathfrak{p}}). \tag{8.114}$$

**Definition 8.74.** The *reduced projective class group* $\widetilde{K_0}(\Lambda)$ of $\Lambda$ is the kernel of the morphism $K_0(\Lambda) \twoheadrightarrow K_0(A)$ in (8.113).

In some texts the reduced projective class group is also denoted $SK_0$.

Observe that the reduced projective class group is the kernel of a *split* epimorphism, because $K_0(A) \cong \mathbb{Z}$ is projective. So to compute the reduced projective class group it suffices to compute $K_0(\Lambda)$.

The connection between these two types of class groups is given by [184, theorem 36.3] and [183, (2.9)]. The first result says that for a maximal order we have that

$$\text{(8.115)}\quad \mathrm{Cl}\,\Lambda \cong \widetilde{K_0}(\Lambda),$$

whilst the latter describes the ideal class group in general as a *subgroup* of the reduced projective class group via the short exact sequence

$$\text{(8.116)}\quad 0 \to \mathrm{Cl}\,\Lambda \to \widetilde{K_0}(\Lambda) \xrightarrow{\lambda_0} \bigoplus_{\mathfrak{p}\in\operatorname{Spec} R\setminus\{(0)\}} \widetilde{K_0}(\Lambda_{\mathfrak{p}}) \to 0$$

In particular, if $\Lambda$ is maximal, then $\lambda_0$ is the zero map: by [184, theorem 18.7] we have indeed that $\mathrm{Cl}\,\Lambda_{\mathfrak{p}} = \widetilde{K_0}(\Lambda_{\mathfrak{p}})$ is zero.

Moreover, we know by Jacobinski that $\mathrm{Cl}\,\Lambda \cong \mathrm{Cl}\,\Lambda'$, for $\Lambda \subseteq \Lambda'$ an inclusion of *hereditary* orders [184, theorem 40.16]. In particular it suffices to compute the ideal class group of a maximal order containing $\Lambda$, provided one starts with an hereditary order.

**Remark 8.75.** It is possible to reprove Jacobinski's result using (8.116) and the results used in the proof of proposition 8.80: if $\Lambda$ is an hereditary order, then $K_0(\Lambda_{\mathfrak{p}}) \cong \mathbb{Z}^{\oplus r-1}$ for $\mathfrak{p}$ a maximal ideal of $R$, where $r$ is the type of $\Lambda_{\mathfrak{p}}$, because the last terms of (8.113) reduce to the *split* short exact sequence

$$\text{(8.117)}\quad 0 \to \mathbb{Z}^{\oplus r-1} \to \mathbb{Z}^{\oplus r} \to \mathbb{Z} \to 0.$$

As an immediate corollary to proposition 8.26 we have the following main result. In particular, by the above discussion we obtain an explicit description of the relative tensor triangular Chow groups in the case of an order $\Lambda$ over a Dedekind domain $R$.

**Corollary 8.76.** We have that

$$\text{(8.118)}\quad \mathrm{CH}_0^{\Delta}(R,\Lambda) \cong \widetilde{K_0}(\Lambda).$$

If $\Lambda$ is moreover hereditary, then

$$\text{(8.119)}\quad \mathrm{CH}_0^{\Delta}(R,\Lambda) \cong \widetilde{K_0}(\Lambda) \cong \mathrm{Cl}\,\Lambda' \oplus \mathbb{Z}^{r-1}$$

where $\Lambda'$ is a maximal order containing $\Lambda$ and $r$ is the maximal length of a chain of inclusions of orders.

In section 8.7.4 we will encounter another situation in which we can express the relative tensor triangular Chow groups in terms of class groups of orders, but there the behaviour with respect to inclusions in maximal orders is different.

**Remark 8.77.** In [184, theorem 40.9] a description of the (two-sided) Picard group is given. It combines information about the local type (see proposition 8.71) and the ramification. This differs from the tensor triangular Chow groups, for which the local type shows up as copies of $\mathbb{Z}$, not in the form of torsion quotients.

**Hereditary orders on curves** Up to now we only looked at hereditary orders on Dedekind domains. In [135, 185] the case of hereditary orders on smooth (quasi-)projective curves over a field $k$ is studied, mostly from a representation theory point of view.

Let $C$ be an irreducible quasiprojective curve over $\operatorname{Spec} k$. Let $\mathcal{A}$ be an hereditary order in the central simple $k(C)$-algebra $A$. In this situation corollary 8.76 becomes the following statement.

**Corollary 8.78.** We have that

$$\mathrm{CH}_0^{\Delta}(C,\mathcal{A}) \cong \ker\left(\mathrm{K}_0(\mathcal{A}) \twoheadrightarrow \mathrm{K}_0(A) \cong \mathbb{Z}\right). \tag{8.120}$$

One can use the results of [185] to compute Grothendieck groups of hereditary orders in this setting. The results in this paper are stated only for $k$ algebraically closed. In this case we have by Tsen's theorem that $\mathrm{Br}(k) = \mathrm{Br}(k(C)) = 0$, which means that the central simple $k(C)$-algebra $A$ is always of the form $\mathrm{Mat}_n(k(C))$, i.e. it is unramified.

If $k$ is not algebraically closed, then one should change the definition of $r$ in [185, proposition 2.1]: it should only incorporate the local types of the hereditary order, not the ramification of a maximal order containing it. The reason why the definition using ramification works in the algebraically closed case is because every central simple $k(C)$-algebra is automatically unramified, and so is every maximal order. But if $\mathrm{Br}(k(C)) \neq 0$ there are ramified maximal orders.

The correct definition should only account for the length of a chain of orders containing $\mathcal{A}$ and terminating in a maximal order $\overline{\mathcal{A}}$. If $\mathcal{A}$ is itself already maximal we will say that this length is 0.

**Proposition 8.79.** Let $\mathcal{A}$ be a sheaf of hereditary $\mathcal{O}_C$-orders. Let $r_p$ be the type of the hereditary $\mathcal{O}_{C,p}$-order $\mathcal{A}_p$. Then the maximal length of a chain of orders containing $\mathcal{A}$ is independent of the maximal order in which it terminates and is equal to

$$\sum_{p\in C}(r_p - 1). \tag{8.121}$$

*Proof.* This follows from the proof of [184, theorem 40.8]. □

We can now formulate [185, proposition 2.1] in such a way that it is also valid over non-algebraically closed fields. By the discussion above the formulation of loc. cit. can be misinterpreted if one does not assume throughout that $k$ is algebraically closed.

**Proposition 8.80.** Let $\mathcal{A}$ be a sheaf of hereditary $\mathcal{O}_C$-orders. Let $\overline{\mathcal{A}}$ be a maximal order containing $\mathcal{A}$. Then

$$\mathrm{K}_0(\mathcal{A}) \cong \mathrm{K}_0(\overline{\mathcal{A}}) \oplus \mathbb{Z}^{\oplus\rho} \tag{8.122}$$

where $\rho := \sum_{p\in C_{(0)}}(r_p - 1)$.

*Proof.* This follows from proposition 8.79 and [186, theorem 1.14]. □

We are now ready to prove the main result for hereditary orders on quasiprojective curves.

**Theorem 8.81.** Let $\mathcal{A}$ be a sheaf of hereditary $\mathcal{O}_C$-orders. Let $\overline{\mathcal{A}}$ be a maximal order containing $\mathcal{A}$. Then

$$\begin{aligned} \mathrm{CH}_0^\Delta(C,\mathcal{A}) &\cong \mathrm{Cl}(\overline{\mathcal{A}}) \oplus \mathbb{Z}^{\oplus\rho} \\ \mathrm{CH}_1^\Delta(C,\mathcal{A}) &\cong \mathbb{Z} \end{aligned} \tag{8.123}$$

where $\rho \coloneqq \sum_{p\in C_{(0)}}(r_p - 1)$.

*Proof.* By [185, proposition 2.1] we obtain that

$$\mathrm{K}_0(\mathcal{A}) \cong \mathrm{K}_0(\overline{\mathcal{A}}) \oplus \mathbb{Z}^{\oplus\rho}. \tag{8.124}$$

Now we apply corollary 8.78 to conclude. □

We now discuss some situation in which these Chow groups can be described more explicitly, which reduces to having an explicit description of the ideal class group of a maximal order in this geometric setting.

**Corollary 8.82.** Let $k$ be algebraically closed. Then for every $\mathcal{A}$ as in theorem 8.81 we have that

$$\mathrm{CH}_0^\Delta(C,\mathcal{A}) \cong \mathrm{Pic}\, C \oplus \mathbb{Z}^{\oplus\rho}. \tag{8.125}$$

If $k$ is not algebraically closed the same description holds as long as $A \cong \mathrm{Mat}_n(k(C))$.

*Proof.* By Tsen's theorem we know that $\mathrm{Br}(k(C)) = 0$, so $A \cong \mathrm{Mat}_n(k(C))$. The maximal orders in $A$ are all of the form $\mathrm{End}_X(\mathcal{E})$ for $\mathcal{E}$ a vector bundle of rank $n$, and by Morita theory we can conclude because $\mathrm{K}_0(\overline{\mathcal{A}}) \cong \mathrm{K}_0(\mathcal{O}_C) \cong \mathrm{Pic}(C) \oplus \mathbb{Z}$. □

**Remark 8.83.** It would be interesting to develop the notion of functoriality for relative tensor triangular Chow groups, as was done for the non-relative case in [125]. One example would be the observation that the functor

$$- \otimes_R \mathrm{Mat}_n(R)\colon R\text{-mod} \to \mathrm{Mat}_n(R)\text{-mod} \tag{8.126}$$

induces multiplication by $n$ on the level of Grothendieck groups. In more general settings (e.g. inclusions of orders) one expects similar interesting behaviour.

If $k$ is not algebraically closed we have an inclusion

$$\mathrm{Br}\, C \hookrightarrow \mathrm{Br}\, k(C) \tag{8.127}$$

sending an Azumaya algebra to the central simple algebra at the generic point of $C$. In the special case of $C = \mathbb{P}^1_k$ we moreover have that $\mathrm{Br}(\mathbb{P}^1_k) \cong \mathrm{Br}(k)$.

If the class of the central simple $k(C)$-algebra $\mathcal{A}_\eta$ in the Brauer group $\mathrm{Br}(k(C))$ actually comes from $\mathrm{Br}(C)$ in the inclusion (8.127) we say that it is *unramified*. Because $C$ is nonsingular of dimension 1 we have that every maximal order in the unramified central simple algebra $\mathcal{A}_\eta$ is actually an Azumaya algebra [6, 20], and we can describe the Chow groups up to *controlled* torsion. The situation of corollary 8.82 is a special case of this where the Azumaya algebra is split, where $n = 1$.

**Corollary 8.84.** Let $\mathcal{A}$ be an hereditary order as in theorem 8.81 such that $\mathcal{A}_\eta$ is an unramified central simple $k(C)$-algebra, and denote $\rho = \sum(e_i - 1)$. Let $n$ be the degree of $\mathcal{A}_\eta$ over $k(C)$. Then

$$\mathrm{CH}_0^\Delta(C, \mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n] \cong \left(\operatorname{Pic} C \oplus \mathbb{Z}^{\oplus\rho}\right) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]. \tag{8.128}$$

*Proof.* Denote by $\overline{\mathcal{A}}$ any maximal order containing $\mathcal{A}$. By the assumptions it is necessarily an Azumaya algebra.

Using [5, corollary 1.2] we have that there exists an isomorphism

$$\mathrm{K}_0(C) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n] \cong \mathrm{K}_0(\overline{\mathcal{A}}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n], \tag{8.129}$$

and by theorem 8.81 we can conclude. □

**Remark 8.85.** In this case op. cit. gives that the map induced on $\mathrm{K}_0$ by $- \otimes_{\mathcal{O}_C} \mathcal{A}$ has torsion (co)kernel of exponent $n^4$.

**Maximal orders on surfaces** There is another invariant in the literature which is a special case of relative Chow groups for orders [9, §3.7]. In op. cit. these are defined for a (terminal) maximal order $\mathcal{A}$ on a (smooth) projective surface $X$ over an algebraically closed field $k$. Here we don't need a precise definition of a terminal maximal order, only that it has finite global dimension [9, corollary 3.3.5]. Less explicitly so, they have also been defined in a more specific setting in [229]. In both cases this intersection theory for sheaves of orders is used to show that the center of a quadratic Artin–Schelter regular algebra is necessarily $\mathbb{P}^2$.

Comparing definitions, we have that the filtration obtained by the tensor action is the same as the filtration by dimension of support corollary 8.43 on the abelian level, which is precisely the filtration used in op. cit. They define a divisor group for $\mathcal{A}$, and as the filtrations are the same we see that

$$\operatorname{Div}(\mathcal{A}) \cong \operatorname{Cyc}_1^\Delta(X, \mathcal{A}). \tag{8.130}$$

Moreover, they define a group $\mathrm{G}_1(\mathcal{A})$ (not to be confused with higher K-theory of coherent sheaves), using the localization sequence (8.73), as the two-dimensional analogue of the reduced projective class group. In particular, combining (8.73) and [9, proposition 3.7.8] we have that

$$\mathrm{G}_1(\mathcal{A}) \cong \mathrm{CH}_1^\Delta(X, \mathcal{A}). \tag{8.131}$$

Moreover, in [9, proposition 3.7.12] an explicit description of $\mathrm{G}_1(\mathcal{A})$ (and hence the codimension-one Chow group) is given in their situation as

$$0 \to k(X)^\times / \det D^\times \to \mathrm{CH}_1^\Delta(X, \mathcal{A}) \to \operatorname{Pic} X \to 0 \tag{8.132}$$

where $D$ is the division algebra over $k(X)$ Morita equivalent to $\mathcal{A}_\eta$.

**Remark 8.86.** A point not addressed here is the relationship between relative tensor triangular Chow groups for hereditary orders on smooth quasiprojective curves and various Chow groups for "orbifold curves". By [61] there exists a correspondence between these when working over an algebraically closed base field of characteristic zero. Observe that by [232] the Chow groups of the orbifold curve are (up to torsion) the same as the Chow groups of the coarse moduli space. Hence the relative tensor triangular Chow groups of an hereditary order on a smooth quasiprojective curve are different from the Chow groups of its associated orbifold curve, because the stackiness shows up as copies of $\mathbb{Z}$ and not as torsion.

This raises at least two questions:

1. is there a purely commutative (relative) setup that recovers the relative Chow groups of the order from the orbifold curve?

2. is there an analogue of [125] identifying the Chow group defined by Vistoli with the tensor triangular Chow group of its derived category?

### 8.7.4 Chow groups of (integral) group rings

In this section we consider the situation where the scheme $X$ is $\operatorname{Spec} R$ for a Dedekind domain $R$, and the coherent $\mathcal{O}_X$-algebra is given by (the sheafification of) the integral group ring $RG$, for a finite group $G$ of order $n$. Observe that in this situation the global dimension of $RG$ is often infinite. Especially the case where $R$ is the ring of integers in an algebraic number field is interesting, where it combines the representation theory of finite groups and algebraic number theory.

As in section 8.7.3 we obtain that we can express in the relative tensor triangular Chow groups in terms of classical invariants, see theorem 8.90.

If we denote $K$ the field of fractions of $R$, then we will relax definition 8.64 by allowing $KG$ to be a separable $K$-algebra. By Maschke's theorem this will be the case if the characteristic of $K$ does not divide $n$ and $K$ is a perfect field. We will assume this throughout, and it is of course satisfied in the case where $K$ is an algebraic number field.

By the Artin–Wedderburn decomposition theorem we have that $KG$ has a direct product decomposition

$$(8.133)\quad KG \cong \prod_{i=1}^{t} \operatorname{Mat}_{n_i}(D_i)$$

whose factors are matrix rings over division rings over $K$. In particular we allow the conditions in definition 8.64 to be relaxed in two directions: we can have multiple factors, and the division algebras can have centers which are larger than $K$.

This allows us to describe the top degree cycle and Chow groups.

**Theorem 8.87.** Let $R$ be a Dedekind domain such that $RG$ defines an order in $KG$. Then

$$(8.134)\quad \operatorname{Cyc}_1^{\Delta}(R, RG) \cong \operatorname{CH}_1^{\Delta}(R, RG) \cong \mathbb{Z}^{\oplus t}$$

where $t$ is the number of simple factors in the Artin–Wedderburn decomposition of $KG$.

*Proof.* This is a straightforward generalisation of propositions 8.69 and 8.72, taking the more general notion of order into account. □

An easy example of the dependence on the field of fractions is given by considering the group rings $\mathbb{Z}\operatorname{Cyc}_p$ and $\mathbb{Z}[\zeta_p]\operatorname{Cyc}_p$, for a cyclic group of prime order $p \geq 3$, where $\zeta_p$ is a primitive $p$th root of unity.

**Example 8.88.** We have that $\mathbb{Q}\operatorname{Cyc}_p \cong \mathbb{Q}\times\mathbb{Q}(\zeta_p)$, so

$$\mathrm{CH}_1^\Delta(\mathbb{Z},\mathbb{Z}\operatorname{Cyc}_p) \cong \mathbb{Z}^{\oplus 2}. \tag{8.135}$$

On the other hand $\mathbb{Q}(\zeta_p)\operatorname{Cyc}_p \cong \prod_{i=0}^{p-1}\mathbb{Q}(\zeta_p)$, hence

$$\mathrm{CH}_1^\Delta(\mathbb{Z}[\zeta_p],\mathbb{Z}[\zeta_p]\operatorname{Cyc}_p) \cong \mathbb{Z}^{\oplus p}. \tag{8.136}$$

**Remark 8.89.** More generally we have that the integral group ring $\mathbb{Z}G$ considered as a sheaf of algebras over $\operatorname{Spec}\mathbb{Z}$ has highest Chow group

$$\mathrm{CH}_1^\Delta(\mathbb{Z},\mathbb{Z}G) \cong \mathbb{Z}^t \tag{8.137}$$

where $t$ is the number of conjugacy classes of cyclic subgroups of $G$ [191, corollary 13.1.2].

For the zero-dimensional Chow groups we obtain a result similar to corollary 8.76. We will not cover the zero-dimensional cycle groups explicitly: there is no uniform description possible but the techniques of theorem 8.87 go through.

**Theorem 8.90.** Let $R$ be a Dedekind domain such that $RG$ defines an order in $KG$. Then

$$\mathrm{CH}_0^\Delta(R,RG) \cong \widetilde{\mathrm{K}_0}(RG) \cong \operatorname{Cl} RG. \tag{8.138}$$

*Proof.* The first isomorphism follows from proposition 8.26. The second isomorphism is [66, remarks 49.11(iv)]. □

The second isomorphism is indeed somewhat special to the situation of group rings: for an hereditary order $\Lambda$ we had that $\operatorname{Cl}\Lambda \cong \operatorname{Cl}\Lambda'$ if $\Lambda \subseteq \Lambda'$ is an inclusion of orders, reducing the computation of the class group to that of a maximal order. To compute the class group of a group ring, observe that $RG$ is maximal if and only if it is hereditary, which happens if and only if $n \in R^\times$ [184, theorem 41.1].

Moreover, the inclusion of $RG$ into a maximal order $\Lambda'$ usually only induces an epimorphism of class groups. In particular one obtains a short exact sequence

$$0 \to \mathrm{D}(RG) \to \mathrm{Cl}(RG) \cong \widetilde{\mathrm{K}}_0(RG) \to \mathrm{Cl}(\Lambda') \to 0 \tag{8.139}$$

as in [66, (49.33)], independent of the choice of $\Lambda'$.

In the case where $R$ is the ring of integers in an algebraic number field, we get by the Jordan–Zassenhaus theorem that $\operatorname{Cl} RG$ (and therefore $\mathrm{CH}_0^\Delta(R,RG)$) is a finite abelian group, generalising the theory of class groups and class numbers of $R$ to the situation of group rings. This is significantly different from the situation for hereditary orders, where the inclusion in a maximal order was responsible for copies $\mathbb{Z}$ in the Chow groups. More information and some explicit expressions can be found in [100, 183].

To end this discussion we give some examples of explicit computations of $\mathrm{Cl}\,\mathbb{Z}G$.

**Example 8.91.** If one considers the situation of example 8.88, then the (necessarily unique) maximal order in $\mathbb{Q}\times\mathbb{Q}(\zeta_p)$ is $\mathbb{Z}\times\mathbb{Z}[\zeta_p]$, and we [66, theorem 50.2] we obtain the following

$$\mathrm{CH}_0^\Delta(\mathbb{Z},\mathbb{Z}\,\mathrm{Cyc}_p)\cong\mathrm{Cl}(\mathbb{Z}[\zeta_p]). \tag{8.140}$$

The order of this group is the class number of the cyclotomic field $\mathbb{Q}(\zeta_p)$. For example if $p=23$ then $\mathrm{CH}_0^\Delta(\mathbb{Z},\mathbb{Z}\,\mathrm{Cyc}_{23})\cong\mathbb{Z}/3\mathbb{Z}$.

Using the class numbers of cyclotomic fields it is possible to give a complete classification of the finite abelian groups for which $\mathrm{Cl}(\mathbb{Z}G)$ (and therefore $\mathrm{CH}_0^\Delta(\mathbb{Z},\mathbb{Z}G)$) is zero: by [66, corollary 50.17] this is only the case if $G$ is cyclic of order $\leq 11$, cyclic of order 13, 14, 17, 19 or the Klein group of order 4.

### 8.7.5 Chow groups in the singular case

Finally we discuss a single example where the base is singular, but the order is a noncommutative resolution and in particular has finite global dimension. Observe that this case is covered by the general results in section 8.5.2. By no means is this a complete discussion, it is given to suggest possible future research.

We will work in the setting of [54, remark 2.7]. Consider

$$R_1\coloneqq k[[x,y]]/(xy), R_2\coloneqq k[[x,y]]/(y^2-x^3) \tag{8.141}$$

which are the complete local rings for the nodal (resp. cuspidal) curve singularity, with maximal ideals $\mathfrak{m}_i$. Denote their normalizations by $\widetilde{R}_i$. Then the *Auslander order* is introduced in op. cit., and it is given by

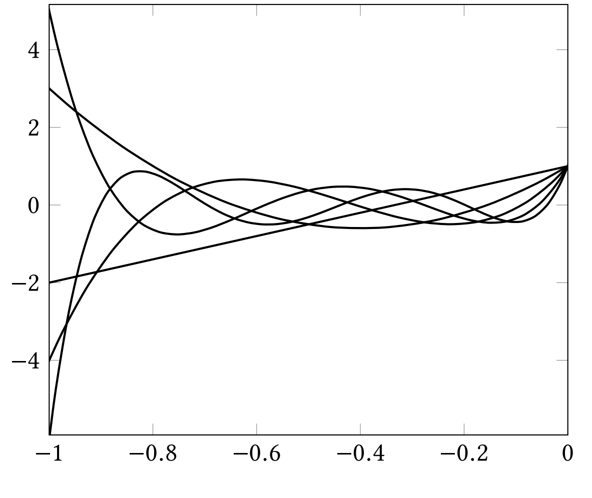
$$A_i\coloneqq\begin{pmatrix}\widetilde{R}_i & \mathfrak{m}_i\\ \widetilde{R}_i & R_i\end{pmatrix} \tag{8.142}$$

It can be seen that these orders have 3 (resp. 2) simple modules, in particular we get the following description of the cycle groups in dimension 0

$$\begin{aligned}\mathrm{K}_0(A_1\text{-}\mathrm{fl})&\cong\mathbb{Z}^{\oplus3},\\ \mathrm{K}_0(A_2\text{-}\mathrm{fl})&\cong\mathbb{Z}^{\oplus2}.\end{aligned} \tag{8.143}$$

*Appendix* A

# The Hochschild cohomology of projective space




($P_n^{(0,1)}(2t+1)$ *for* $n = 1, \ldots, 5$)

## A.1 Introduction

The Hochschild cohomology of an associative algebra is an invariant which is used to control its deformation theory as an associative algebra. Its definition has been generalised to abelian and dg categories [156]. It can be given the structure of a *Gerstenhaber algebra*, which is useful in the context of deformation theory. This structure combines that of a supercommutative algebra and a graded Lie superalgebra.

In this short appendix we will describe the Gerstenhaber algebra structure as detailed as possible, for the Hochschild cohomology of $\mathbb{P}^n$. For the various definitions of Hochschild cohomology for a smooth projective variety one is referred to section 1.3.3. For such a variety we have the Hochschild–Kostant–Rosenberg decomposition expressing the Hochschild cohomology of sheaf cohomology of exterior powers of the tangent bundle. We will use this decomposition, and various refinements that take the Gerstenhaber algebra structure into account. In appendix A.2 we will recall the notion of Jacobi polynomials. These are a standard object in the study of special functions, and form one of the classes of orthogonal polynomials.

In appendix A.3 we discuss the Hochschild–Kostant–Rosenberg theorem in the special case of $\mathbb{P}^n$. In proposition A.2 we use the Borel–Weil–Bott theorem to show that the Hochschild cohomology agrees with the global sections of the sheaves of polyvector fields. This is suggested in [76, §10] as an interesting example to study the Gerstenhaber algebra structure on Hochschild cohomology.

Using the Borel–Weil–Bott theorem we can indeed also describe the structure of $\mathrm{HH}^i(\mathbb{P}^n)$ as a representation of the Lie algebra $\mathrm{HH}^1(\mathbb{P}^n) = \mathfrak{sl}_{n+1}$, and we can express its dimension in a combinatorial way using the hook-length formula. Then we show in corollary A.6 that the Hilbert polynomial of the Hochschild cohomology actually turns out to belong to a well-known family of orthogonal polynomials, namely that it is a Jacobi polynomial.

Finally in appendix A.4 we discuss the algebra structure on $\mathrm{HH}^\bullet(\mathbb{P}^n)$. In particular it is known that as an algebra it is generated by the part in degree 1, i.e. the algebra has many non-zero cup products [238]. We give an alternative and more geometric proof of this result.

Except for the observation that it is possible to interpret the Hilbert polynomial of the Hochschild cohomology as a Jacobi polynomial, the results in this note are easy applications of well-known methods in the study of algebraic groups, Lie algebras and partial flag varieties.

It would be interesting to generalise these results to other homogeneous varieties. The first step would be to consider Grassmannians and quadric hypersurfaces, but computations in low dimensions suggest that the coefficients of the Hilbert polynomial in this case are not known in [OEIS], so determining whether they correspond to a certain class of known polynomials will be harder. In general, [73, proposition 2] shows that for (almost all) partial flag varieties the tangent bundle has no higher cohomology, but I do not know what happens for exterior powers. Another example of interest would be del Pezzo surfaces, or more generally surfaces for which $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, so that $\mathrm{HH}^1(S) = \mathrm{Lie}\,\mathrm{Aut}(S)$.

## A.2 Jacobi polynomials

In appendix A.3.2 we will show that the Hilbert polynomial of the Hochschild cohomology of $\mathbb{P}^n$ is (an affine transformation of) a certain Jacobi polynomial. We will quickly recall their definition, for more information one is referred to [2, §22]. In particular, we will use the explicit description from [2, (22.3.2)].

**Definition A.1.** Let $n, \alpha, \beta \geq 0$ be integers. The *Jacobi polynomial* associated to $n \geq 0$ and $\alpha, \beta > -1$ is defined as

$$\text{(A.1)}\quad \mathrm{P}_n^{(\alpha,\beta)}(t) := \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{2^m\Gamma(\alpha+m+1)} (t-1)^m.$$

These Jacobi polynomials for fixed $\alpha$ and $\beta$ are orthogonal polynomials on the interval $[-1, 1]$, for the weight $(1-x)^\alpha(1+x)^\beta$.

We will need to consider these polynomials only for the parameters $(\alpha, \beta) = (0, 1)$. In this case they are for low values of $n$ given

$$\text{(A.2)}\quad \begin{aligned} \mathrm{P}_1^{(0,1)}(t) &= \frac{3}{2}t - \frac{1}{2}, \\ \mathrm{P}_2^{(0,1)}(t) &= \frac{5}{2}t^2 - t - \frac{1}{2}, \\ \mathrm{P}_3^{(0,1)}(t) &= \frac{35}{8}t^3 - \frac{15}{8}t^2 - \frac{15}{8}t + \frac{3}{8}, \\ \mathrm{P}_4^{(0,1)}(t) &= \frac{63}{8}t^4 - \frac{7}{2}t^3 - \frac{21}{4}t^2 + \frac{3}{2}t + \frac{3}{8}, \\ \mathrm{P}_5^{(0,1)}(t) &= \frac{231}{16}t^5 - \frac{105}{16}t^4 - \frac{105}{8}t^3 + \frac{35}{8}t^2 + \frac{35}{16}t - \frac{5}{16}. \end{aligned}$$

These are obviously *not* the polynomials we are looking for to describe the Hilbert polynomial of $\mathrm{HH}^\bullet(\mathbb{P}^n)$ as they do not have positive integer coefficients. To remedy this, we will apply the transformation $t \mapsto 2t + 1$ to get the correct coefficients.

## A.3 The structure as a Lie algebra

By the Hochschild–Kostant–Rosenberg decomposition we need to determine the sheaf cohomology spaces $\mathrm{H}^i(\mathbb{P}^n, \bigwedge^j \mathrm{T}_{\mathbb{P}^n})$. It turns out that for $i + j$ fixed, i.e. when computing $\mathrm{HH}^k(\mathbb{P}^n)$ they are nonzero only for $i = 0$. The vanishing of $\mathrm{H}^i(\mathbb{P}^n, \bigwedge^j \mathrm{T}_{\mathbb{P}^n})$ for $i \geq 1$ can be proven using an induction on the Euler exact sequence. And this is the method used in [105, theorem 3.2]. A description of the non-zero term $\mathrm{H}^0(\mathbb{P}^n, \bigwedge^k \mathrm{T}_{\mathbb{P}^n})$ using *toric* methods can be found in [105, theorem 3.8].

In this section we will show how it is also possible to use the Borel–Weil–Bott theorem to describe the Hochschild cohomology. These have the benefit of explicitly describing the structure as a Lie algebra, and giving an explicit formula for the dimension of each graded piece.

### A.3.1 Cohomology of polyvector fields

Recall that $\mathbb{P}^n$ can be considered as the partial flag variety of the group $\mathrm{GL}_{n+1}$ associated to the parabolic subgroup

$$P = \left\{ \left( \begin{array}{c|ccc} g_{1,1} & g_{1,2} & \cdots & g_{1,n+1} \\ \hline 0 & g_{2,2} & \cdots & g_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n+1,2} & \cdots & g_{n+1,n+1} \end{array} \right) \in \mathrm{GL}_{n+1} \right\} \tag{A.3}$$

stabilising the flag $0 \subseteq \langle e_0 \rangle \subset \langle e_0, \ldots, e_n \rangle$.

To compute the Hochschild cohomology of $\mathbb{P}^n$, we first need to determine the vector spaces $\mathrm{H}^i(\mathbb{P}^n, \bigwedge^j \mathrm{T}_{\mathbb{P}^n})$ for all $i, j \leq n$. The Borel–Weil–Bott theorem which expresses the cohomology of line bundles of flag varieties in terms of $\mathrm{GL}_{n+1}$-representations, which can then be used to express the cohomology of equivariant vector bundles on Grassmannians, and in particular $\mathbb{P}^n$.

This description is stated as [237, exercise 4.4.(a)], and in the notation of §4.1 of op. cit. we will consider an $n+1$-dimensional vector space $E$, and take $r = n$. We will denote $\mathbb{P}(E) = \mathbb{P}^n$. Then the tautological sequence

$$0 \to \mathcal{R} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathcal{Q} \to 0 \tag{A.4}$$

is identified with (a twist of) the Euler exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n/k}(1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0. \tag{A.5}$$

We have that all $\mathrm{GL}(E)$-equivariant vector bundles on $\mathbb{P}^n$ can be written as

$$\mathcal{V}(\alpha) \coloneqq \mathrm{K}_\beta\, \mathcal{R}^\vee \otimes \mathrm{K}_\gamma\, \mathcal{Q}^\vee, \tag{A.6}$$

for appropriate $\beta = (\alpha_1, \ldots, \alpha_n)$ and $\gamma = (\alpha_{n+1})$, where $\mathrm{K}_\beta$ denotes the Weyl functor. Now using exercise 2.18(b) of op. cit. we can write the tangent bundle of $\mathbb{P}^n$, and its exterior powers, as

$$\bigwedge^j \mathrm{T}_{\mathbb{P}^n} \cong \mathrm{K}_{(1^j)}\, \mathrm{T}_{\mathbb{P}^n}(-1) \otimes \mathrm{K}_{-j}\, \mathcal{O}_{\mathbb{P}^n}(-1) \tag{A.7}$$

for $j = 0, \ldots, n$, where we denote $(1^j) = (1, \ldots, 1, 0, \ldots, 0)$ with 1 repeated $j$ times and padded with zeroes to length $n$.

**Proposition A.2.** We have that the Hochschild–Kostant–Rosenberg decomposition of the $j$th Hochschild cohomology only has a single summand

$$\mathrm{HH}^j(\mathbb{P}^n) \cong \mathrm{H}^0(\mathbb{P}^n, \bigwedge^j \mathrm{T}_{\mathbb{P}^n}), \tag{A.8}$$

which as a $\mathrm{GL}(E)$-representation is given by

$$\mathrm{H}^0(\mathbb{P}^n, \bigwedge^j \mathrm{T}_{\mathbb{P}^n}) \cong \mathrm{K}_{(1,\ldots,1,0,\ldots,0,-j)}\, E^\vee \tag{A.9}$$

for $j = 0, \ldots, n$.

*Proof.* By [237, corollary 4.1.9] we have that

$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{V}(1,\dots,1,0,\dots,0,-j)) \cong \mathrm{K}_{(1,\dots,1,0,\dots,0,-j)}\, E^\vee. \tag{A.10}$$

and all higher cohomology vanishes, because our choice of $\alpha$ is already non-increasing. Observe that we have described a *rational* $\mathrm{GL}(E)$-representation. □

In appendix A.3.3 we will use this description as $\mathrm{GL}_{n+1}$-representation to describe the structure as $\mathfrak{sl}_{n+1}$-representation, which is the Lie algebra part of the Gerstenhaber algebra structure.

### A.3.2 Hilbert series is a Jacobi polynomial

We can now show that the Hilbert series of $\mathrm{HH}^\bullet(\mathbb{P}^n)$ is actually a Jacobi polynomial for the appropriate parameters and an affine transformation of the variable. To compute the dimension of (A.9) we could use [237, proposition 2.2.1] to make it polynomial, hence it is described by a Young diagram. Then we can use proposition 2.1.15 of op. cit. to conclude that

$$\dim_k \mathrm{H}^0(\mathbb{P}^n, \textstyle\bigwedge^j \mathrm{T}_{\mathbb{P}^n}) = \#\,\mathrm{SST}((j+1,\dots,j+1,j,\dots,j,0),[1,n+1]) \tag{A.11}$$

where $\mathrm{SST}(\lambda,[1,n+1])$ is the set of semistandard Young tableaux of shape $\lambda$ with entries in $\{1,\dots,n+1\}$. Its cardinality can be computed using the hook-length formula.

But for the computation of the coefficients it will be easier to use a different collection of semistandard Young tableaux. That they are of the same cardinality follows from the following lemma.

**Lemma A.3.** We have that

$$\#\,\mathrm{SST}((j{+}1,\dots,j{+}1,j,\dots,j,0),[1,n{+}1]) = \#\,\mathrm{SST}((j{+}1,1,\dots,1,0,\dots,0),[1,n{+}1]) \tag{A.12}$$

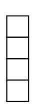
where $(j{+}1,1,\dots,1,0,\dots,0)$ is the *hook partition* with arm length $j$ and leg length $n-j$. We will use the shorthand $\lambda^j_{n+1}$ for this.

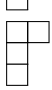
*Proof.* By [237, exercise 2.18(b) and proposition 2.2.1] we have that the Weyl modules associated to $(j+1,\dots,j+1,j,j,0)$ and $\lambda^j_{n+1}$ have the same dimension. □

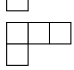
In corollary A.8 we will use the same results to actually interpret the representations considered in the proof of this lemma.

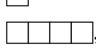
**Example A.4.** If we are considering $\mathbb{P}^3$, then the relevant hook partitions are

(A.13)

$\lambda^0_4 =$ 

$\lambda^1_4 =$ 

$\lambda^2_4 =$ 

$\lambda^3_4 =$ , and lemma A.3 tells us for instance that

$$\dim_k \mathrm{K}_{\ydiagram{3,1}}(E) = \dim_k \mathrm{K}_{\ydiagram{3,3,2}}(E^\vee). \tag{A.14}$$

**Lemma A.5.** We have that

$$\#\,\mathrm{SST}(\lambda^i_{n+1}, [1, n+1]) = \binom{n}{k}\binom{n+k+1}{n+1}. \tag{A.15}$$

*Proof.* By the hook-length formula of [202, (7.106)] it suffices to see that the product over the cells in the vertical part of the hook give rise to the first factor, whereas the product over the cells in the horizontal part give rise to the second factor. □

In [238, theorem 3.4] the same dimension count is obtained through algebraic methods, using the Beilinson quiver. Using either description of the dimension we get the following corollary.

**Corollary A.6.** We have that

$$\mathrm{HP}_{\mathrm{HH}^\bullet(\mathbb{P}^n)}(t) = \mathrm{P}_n^{0,1}(2t+1). \tag{A.16}$$

*Proof.* The final step is to show that the coefficients of $\mathrm{P}_n^{0,1}(2t+1)$ agree with the numbers from lemma A.5. The transformation $t \mapsto 2t+1$ ensures that the summation in (A.1) is homogeneous in $t$, and the coefficient $1/2^m$ is cancelled. Then it suffices to evaluate the coefficient for the piece in degree $k$ for $(\alpha, \beta) = (0, 1)$. □

These coefficients are given in [OEIS, sequence A178301].

**Example A.7.** As an example we describe the semistandard Young tableaux which are required for $\mathbb{P}^2$. The Weyl functors we need are indexed by the partitions $(1,1,1)$, $(2,1)$ and $(3)$. Their semistandard Young tableaux are

$$\ytableaushort{1,2,3} \tag{A.17}$$

$$\ytableaushort{11,2}, \ytableaushort{11,3}, \ytableaushort{12,2}, \ytableaushort{12,3}, \ytableaushort{13,2}, \ytableaushort{13,3}, \ytableaushort{22,3}, \ytableaushort{23,3} \tag{A.18}$$

$$\ytableaushort{111}, \ytableaushort{112}, \ytableaushort{113}, \ytableaushort{122}, \ytableaushort{123}, \ytableaushort{133}, \ytableaushort{222}, \ytableaushort{223}, \ytableaushort{233}, \ytableaushort{333} \tag{A.19}$$

From this we can read off the Hilbert polynomial, as in (A.20).

We can now list the Hilbert series for low values of $n$.

$$\begin{aligned} \mathrm{HP}_{\mathrm{HH}^\bullet(\mathbb{P}^1)} &= 3t + 1, \\ \mathrm{HP}_{\mathrm{HH}^\bullet(\mathbb{P}^2)} &= 10t^2 + 8t + 1, \\ \mathrm{HP}_{\mathrm{HH}^\bullet(\mathbb{P}^3)} &= 35t^3 + 45t^2 + 15t + 1, \\ \mathrm{HP}_{\mathrm{HH}^\bullet(\mathbb{P}^4)} &= 126t^4 + 224t^3 + 126t^2 + 24t + 1, \\ \mathrm{HP}_{\mathrm{HH}^\bullet(\mathbb{P}^5)} &= 462t^5 + 1050t^4 + 840t^3 + 280t^2 + 35t + 1. \end{aligned} \tag{A.20}$$

### A.3.3 Representation series

If we are interested in describing the Gerstenhaber algebra structure of $\mathrm{HH}^\bullet(\mathbb{P}^n)$, we can use the proof of lemma A.3 to understand how $\mathrm{GL}(E)$-representations can be interpreted as $\mathfrak{sl}_{n+1}$-representations by ignoring the determinant representation. We can then enhance the notion of a Hilbert series by not just considering the dimensions of the representations, but rather taking coefficients in the representation ring of $\mathfrak{sl}_{n+1}$. In this way we will consider the *representation series*

$$\text{(A.21)}\quad \mathrm{HP}^{\mathfrak{sl}_{n+1}}_{\mathrm{HH}^\bullet(\mathbb{P}^n)}(t) = \sum_{i=0}^{n} [\mathrm{HH}^i(\mathbb{P}^n)]t^i.$$

**Corollary A.8.** We have that

$$\text{(A.22)}\quad \mathrm{HP}^{\mathfrak{sl}_{n+1}}_{\mathrm{HH}^\bullet(\mathbb{P}^n)}(t) = \sum_{i=0}^{n} \mathrm{K}_{\lambda^i_{n+1}}(E)t^i.$$

*Proof.* As we have seen before, if we are only interested in $\mathrm{SL}_{n+1}$ and $\mathfrak{sl}_{n+1}$ we can ignore tensor powers of the determinant representation of $\mathrm{GL}_{n+1}$. Also observe that the structure as $\mathfrak{sl}_{n+1}$-representation is induced from the structure of $\mathrm{GL}(E)$-representation, because $\bigwedge^i \mathrm{T}_{\mathbb{P}^n}$ is an equivariant vector bundle and first Hochschild cohomology group can be considered as infinitesimal automorphisms arising from this. Now using the proof of lemma A.3 we then get the statement in (A.22). ◻

**Example A.9.** Continuing the example from example A.7 we get

$$\text{(A.23)}\quad \mathrm{HP}^{\mathfrak{sl}_3}_{\mathrm{HH}^\bullet(\mathbb{P}^2)}(t) = \mathrm{K}_{\yng(3)}(k^3)t^2 + \mathrm{K}_{\yng(2,1)}(k^2)t + \mathrm{K}_{\yng(1,1,1)}(k^3).$$

## A.4 The structure as an algebra

In this section we give a geometric proof for the fact that $\mathrm{HH}^\bullet(\mathbb{P}^n)$ *as an associative algebra* is generated by $\mathrm{HH}^1(\mathbb{P}^n)$. An algebraic proof for this fact can be found in [238], where it is done through explicit algebraic manipulations with the Beilinson quiver.

**Proposition A.10.** $\mathrm{HH}^\bullet(\mathbb{P}^n)$ is generated by $\mathrm{HH}^1(\mathbb{P}^n)$ under the cup product.

*Proof.* Because the Hochschild–Kostant–Rosenberg decomposition for $\mathbb{P}^n$ has such a particularly easy description, we can use the result of [57] which tells us that the cup product structure on the side of Hochschild cohomology agrees with the cup product structure on the side of polyvector fields. So we wish to show that the natural map

$$\text{(A.24)}\quad \bigwedge^i \mathrm{HH}^1(\mathbb{P}^n) = \bigwedge^i \mathfrak{sl}_{n+1} \to \mathrm{H}^0(\mathbb{P}^n, \bigwedge^i \mathrm{T}_{\mathbb{P}^n}) = \mathrm{HH}^i(\mathbb{P}^n)$$

is surjective.

Using the Euler sequence we see that $\mathfrak{sl}_{n+1}$ is generated by $E^\vee \otimes_k E$. The Schur complex describing $\bigwedge^i \mathrm{T}_{\mathbb{P}^n}$ is

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \ldots \to \bigwedge^{i-1} E^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(i-1) \to \bigwedge^i E^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(i) \to \bigwedge^i \mathrm{T}_{\mathbb{P}^n} \to 0. \tag{A.25}$$

By splicing up the sequence and taking global sections we get that $\bigwedge^i E^\vee \otimes E$ maps surjectively to $\mathrm{H}^0(\mathbb{P}^n, \mathrm{T}_{\mathbb{P}^n})$. □

*Appendix B*

# Categorical representability of central simple algebras and Brauer–Severi varieties

## B.1 Introduction

In theorem 1.20 we saw a full and strong exceptional collection on $\mathbb{P}^n_k$, and in theorem 1.34 we saw a relative version of this statement for projective bundles. More generally we can consider not only projective bundles, but also Brauer–Severi schemes which are twisted forms of projective bundles. The associated semiorthogonal decomposition is given in theorem 1.35.

We will only consider the case where $S = \operatorname{Spec} k$, and $\mathcal{A}$ is a central simple algebra algebra that we will denote $A$. Then this semiorthogonal decomposition is the same as a full and strong *w-exceptional* collection in $\mathbf{D}^{\mathrm{b}}(\mathrm{BS}(A))$, a concept introduced in [178]. The endomorphism algebras of the w-exceptional objects are Morita equivalent to $A^{\otimes i}$. The Brauer–Severi scheme is called the *Brauer–Severi variety* in this case, and it is a twisted form of $\mathbb{P}^n_k$. One can show that there is a correspondence between Brauer–Severi varieties and central simple algebras [87].

Inspired by the indecomposability of the derived category of a curve [173], and the obstruction results of chapter 2 we can ask for a lower bound on the dimension of the varieties such that $\mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(X)$, where $A$ is a division algebra over $k$ (which we can assume by Morita theory).

Closely related is the question of categorical representability: central simple algebras arise as the endomorphism algebras of w-exceptional objects, but having a w-exceptional collection does not imply that the derived category is categorically representable in dimension 0 like it does for an exceptional collection. Indeed, if the central simple algebra has (Schur) index $\geq 2$ its derived category is not equivalent to $\mathbf{D}^{\mathrm{b}}(k)$.

Recently this analogous question regarding the categorical representability of Brauer–Severi varieties was studied in [172]. Using the semiorthogonal decomposition of the Brauer–Severi variety, it is not hard to show the following inequality [172, proposition 5.1].

**Proposition B.1** (Novaković)**.** Let $X$ be a Brauer–Severi variety. Then

$$\mathrm{rdim}_{\mathrm{cat}}(X) \leq \mathrm{ind}(X) - 1. \tag{B.1}$$

Based on results for central simple algebras of index 2 and 3 (where the statement is indeed true) Novaković conjectured that the inequality (B.1) for the categorical representability of a Brauer–Severi variety in the previous proposition is actually an equality. In the next section we outline a counterexample of index 4, obtained during discussions with Marcello Bernardara and Michel Van den Bergh on a related question, showing that the inequality (B.1) can be strict. In the example the index is 4, whilst $\mathrm{rdim}_{\mathrm{cat}}$ is 2.

Based on the idea behind the counterexample it is possible to improve proposition B.1 in the following way.

**Proposition B.2.** Let $A = A_1 \otimes_k \ldots \otimes_k A_n$ be the primary decomposition of $A$. Then

$$\mathrm{rdim}_{\mathrm{cat}}(\mathrm{BS}(A)) \leq \sum_{i=1}^{n} \mathrm{ind}(A_i). \tag{B.2}$$

But the counterexample has a trivial primary decomposition because the index has only 1 prime factor, so the inequality can still be strict.

## B.2 The counterexample

Consider a biquaternion algebra $A$, given by the tensor product of two quaternion algebras $Q_1$ and $Q_2$. This is a central simple algebra of degree 4. Its exponent is either 1 (in which case it is split) or 2. Its index is equal to 1, 2 or 4. In any case, we can construct the following embedding.

**Proposition B.3.** There exists a fully faithful embedding

$$(\mathrm{B.3})\quad \mathbf{D}^{\mathrm{b}}(A) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathrm{BS}(Q_1) \times \mathrm{BS}(Q_2)).$$

*Proof.* The derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{BS}(Q_i))$ is a Brauer–Severi curve, for which we have a semiorthogonal decomposition

$$(\mathrm{B.4})\quad \mathbf{D}^{\mathrm{b}}(\mathrm{BS}(Q_i)) = \langle \mathbf{D}^{\mathrm{b}}(k), \mathbf{D}^{\mathrm{b}}(Q_i)\rangle$$

induced by a w-exceptional collection $(E_{i,1}, E_{i,2})$. Using [136, §5] we have an induced semiorthogonal decomposition for $\mathbf{D}^{\mathrm{b}}(\mathrm{BS}(Q_1) \times \mathrm{BS}(Q_2))$, given by a w-exceptional collection. The endomorphism algebra of $p_1^*(E_{1,2}) \otimes p_2^*(E_{2,2})$ is isomorphic to $A$. □

The variety $\mathrm{BS}(Q_1) \times \mathrm{BS}(Q_2)$ is a twisted form of $\mathbb{P}^1 \times \mathbb{P}^1$.

**Corollary B.4.** We have that

$$(\mathrm{B.5})\quad \mathrm{rdim}_{\mathrm{cat}}(\mathrm{BS}(A)) = 2.$$

*Proof.* Using theorem 1.35 we have a semiorthogonal decomposition

$$(\mathrm{B.6})\quad \mathbf{D}^{\mathrm{b}}(\mathrm{BS}(A)) = \left\langle \mathbf{D}^{\mathrm{b}}(k), \mathbf{D}^{\mathrm{b}}(A), \mathbf{D}^{\mathrm{b}}(k), \mathbf{D}^{\mathrm{b}}(A)\right\rangle$$

because the period of $A$ is at most 2. The result follows from proposition B.3. □

**Remark B.5.** An analogous counterexample can be found in [17, table 3] using the language of involution varieties in the first two rows, i.e. whenever $\mathrm{ind}(S) = 4$. The example from (8.2) in op. cit. is the one we have discussed here. More generally one can use involution varieties in higher dimensions, which are twisted forms of quadrics, to construct fully faithful functors in higher dimensions for which the inequality (B.1) is strict.

# Bibliography


[1] Tarig Abdelgadir, Shinnosuke Okawa, and Kazushi Ueda. *Compact moduli of noncommutative projective planes.* 2014. arXiv: `1411.7770v1 [math.AG]`.

[2] Milton Abramowitz and Irene Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Vol. 55. National Bureau of Standards Applied Mathematics Series. 1964, pp. xiv+1046.

[3] Valery Alexeev and Dmitri Orlov. "Derived categories of Burniat surfaces and exceptional collections". In: *Mathematische Annalen* 357.2 (2013), pp. 743–759.

[4] Alexey Ananyevskiy, Asher Auel, Skip Garibaldi, and Kirill Zainoulline. "Exceptional collections of line bundles on projective homogeneous varieties". In: *Advances in Mathematics* 236 (2013), pp. 111–130.

[5] Benjamin Antieau. "On a theorem of Hazrat and Hoobler". In: *Proceedings of the American Mathematical Society* 141.8 (2013), pp. 2609–2613.

[6] Benjamin Antieau and Kenneth Chan. "Maximal orders in unramified central simple algebras". In: *Journal of Algebra* 452 (2016), pp. 94–105.

[7] Michela Artebani and Igor Dolgachev. "The Hesse pencil of plane cubic curves". In: *L'Enseignement Mathématique. Revue Internationale. 2e Série* 55.3-4 (2009), pp. 235–273.

[8] Michael Artin. "Geometry of quantum planes". In: *Azumaya algebras, actions, and modules (Bloomington, IN, 1990).* Vol. 124. Contemporary Mathematics. American Mathematical Society, 1992, pp. 1–15.

[9] Michael Artin, Daniel Chan, Aise Johan de Jong, and Max Lieblich. "Terminal orders over surfaces". unpublished. 2014.

[10] Michael Artin and David Mumford. "Some elementary examples of unirational varieties which are not rational". In: *Proceedings of the London Mathematical Society. Third Series* 25 (1972), pp. 75–95.

[11] Michael Artin and William Schelter. "Graded algebras of global dimension 3". In: *Advances in Mathematics* 66.2 (1987), pp. 171–216.

[12] Michael Artin, John Tate, and Michel Van den Bergh. “Modules over regular algebras of dimension 3”. In: *Inventiones Mathematicae* 106.2 (1991), pp. 335–388.

[13] Michael Artin, John Tate, and Michel Van den Bergh. “Some algebras associated to automorphisms of elliptic curves”. In: *The Grothendieck Festschrift, I*. Vol. 86. Progress in Mathematics. Birkhäuser, 1990, pp. 33–85.

[14] Michael Artin and James J. Zhang. “Noncommutative projective schemes”. In: *Advances in Mathematics* 109.2 (1994), pp. 228–287.

[15] Michael Atiyah and Ian Macdonald. *Introduction to commutative algebra*. Addison-Wesley, 1969, pp. ix+128.

[16] Marc Aubry and Jean-Michel Lemaire. “Zero divisors in enveloping algebras of graded Lie algebras”. In: *Journal of Pure and Applied Algebra* 38.2-3 (1985), pp. 159–166.

[17] Asher Auel and Marcello Bernardara. *Semiorthogonal decompositions and birational geometry of del Pezzo surfaces over arbitrary fields*. 2015. arXiv: `1511.07576v1 [math.AG]`.

[18] Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov. “Mirror symmetry for weighted projective planes and their noncommutative deformations”. In: *Annals of Mathematics. Second Series* 167.3 (2008), pp. 867–943.

[19] Maurice Auslander and Oscar Goldman. “Maximal orders”. In: *Transactions of the American Mathematical Society* 97 (1960), pp. 1–24.

[20] Maurice Auslander and Oscar Goldman. “The Brauer group of a commutative ring”. In: *Transactions of the American Mathematical Society* 97 (1960), pp. 367–409.

[21] Dagmar Baer. “Tilting sheaves in representation theory of algebras”. In: *Manuscripta Mathematica* 60.3 (1988), pp. 323–347.

[22] Paul Balmer. “Presheaves of triangulated categories and reconstruction of schemes”. In: *Mathematische Annalen* 324.3 (2002), pp. 557–580.

[23] Paul Balmer. “Tensor triangular geometry”. In: *Proceedings of the International Congress of Mathematicians. Volume II*. Hindustan Book Agency, 2010, pp. 85–112.

[24] Paul Balmer and Giordano Favi. “Generalized tensor idempotents and the telescope conjecture”. In: *Proceedings of the London Mathematical Society. Third Series* 102.6 (2011), pp. 1161–1185.

[25] Paul Balmer and Marco Schlichting. “Idempotent completion of triangulated categories”. In: *Journal of Algebra* 236.2 (2001), pp. 819–834.

[26] Hanno Becker. “Models for singularity categories”. In: *Advances in Mathematics* 254 (2014), pp. 187–232.

[27] Alexander Beilinson. “Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra”. In: *Akademiya Nauk SSSR. Funktsional’nyĭ Analiz i ego Prilozheniya* 12.3 (1978), pp. 68–69.

[28] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne. "Faisceaux pervers". In: *Analysis and topology on singular spaces, I (Luminy, 1981)*. Vol. 100. Astérisque. Société Mathématique de France, 1982, pp. 5–171.

[29] Pieter Belmans. *Hochschild cohomology of noncommutative planes and quadrics*. arXiv: 1705.06098 [math.AG].

[30] Pieter Belmans, Kevin De Laet, and Lieven Le Bruyn. "The point variety of quantum polynomial rings". In: *Journal of Algebra* 463 (2016), pp. 10–22. arXiv: 1509.07312 [math.RA].

[31] Pieter Belmans and Sebastian Klein. *Relative tensor triangular Chow groups for coherent algebras*. Accepted for publication in Journal of Algebra. 2016. arXiv: 1607.03423 [math.AG].

[32] Pieter Belmans and Dennis Presotto. *Construction of noncommutative surfaces with exceptional collections of length 4*. arXiv: 1705.06943 [math.AG].

[33] Pieter Belmans, Dennis Presotto, and Michel Van den Bergh. "Comparing two constructions of noncommutative del Pezzo surfaces of rank 4".

[34] Pieter Belmans and Theo Raedschelders. *Embeddings of quivers in derived categories of surfaces*. Accepted for publication in Proceedings of the American Mathematical Society. 2015. arXiv: 1501.04197 [math.AG].

[35] Pieter Belmans and Theo Raedschelders. *Noncommutative quadrics and Hilbert schemes of points*. 2016. arXiv: 1605.02795 [math.AG].

[36] Marcello Bernardara. "A semiorthogonal decomposition for Brauer-Severi schemes". In: *Mathematische Nachrichten* 282.10 (2009), pp. 1406–1413.

[37] Marcello Bernardara and Michele Bolognesi. "Derived categories and rationality of conic bundles". In: *Compositio Mathematica* 149.11 (2013), pp. 1789–1817.

[38] Marcello Bernardara and Gonçalo Tabuada. "Relations between the Chow motive and the noncommutative motive of a smooth projective variety". In: *Journal of Pure and Applied Algebra* 219.11 (2015), pp. 5068–5077.

[39] Joseph Bernstein, Israel Gel'fand, and Sergei Gel'fand. "Algebraic vector bundles on $\mathbb{P}^n$ and problems of linear algebra". In: *Akademiya Nauk SSSR. Funktsional'nyĭ Analiz i ego Prilozheniya* 12.3 (1978), pp. 66–67.

[SGA6] Pierre Berthelot, Alexander Grothendieck, and Luc Illusie, eds. *Séminaire de Géométrie Algébrique du Bois Marie — 1966–67 — Théorie des intersections et théorème de Riemann–Roch — (SGA6)*. Lecture notes in mathematics 225. Springer-Verlag, 1971, pp. xii+700.

[40] Walter Bichsel and Max-Albert Knus. "Quadratic forms with values in line bundles". In: *Recent advances in real algebraic geometry and quadratic forms*. Vol. 155. Contemporary Mathematics. American Mathematical Society, 1994, pp. 293–306.

[41] Anthony Blanc. "Topological K-theory of complex noncommutative spaces". In: *Compositio Mathematica* 152.3 (2016), pp. 489–555.

[42] Christian Böhning, Hans-Christian Graf von Bothmer, Ludmil Katzarkov, and Pawel Sosna. “Determinantal Barlow surfaces and phantom categories”. In: *Journal of the European Mathematical Society (JEMS)* 17.7 (2015), pp. 1569–1592.

[43] Alexei Bondal and Dmitri Orlov. “Reconstruction of a variety from the derived category and groups of autoequivalences”. In: *Compositio Mathematica* 125.3 (2001), pp. 327–344.

[44] Alexey Bondal. “Helices, representations of quivers and Koszul algebras”. In: *Helices and vector bundles*. Vol. 148. London Mathematical Society Lecture Note Series. Cambridge University Press, 1990, pp. 75–95.

[45] Alexey Bondal. *Noncommutative deformations and Poisson brackets on projective spaces*. MPIM: `93-67`. URL: `http://www.mpim-bonn.mpg.de/preblob/4912`.

[46] Alexey Bondal. “Representations of associative algebras and coherent sheaves”. In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 53.1 (1989), pp. 25–44.

[47] Alexey Bondal and Mikhail Kapranov. “Representable functors, Serre functors, and reconstructions”. In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 53.6 (1989), pp. 1183–1205.

[48] Alexey Bondal and Dmitri Orlov. “Derived categories of coherent sheaves”. In: *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*. Higher Education Press, 2002, pp. 47–56.

[49] Alexey Bondal and Dmitri Orlov. *Semiorthogonal decomposition for algebraic varieties*. 1995. arXiv: `alg-geom/9506012 [math.AG]`.

[50] Alexey Bondal and Alexander Polishchuk. “Homological properties of associative algebras: the method of helices”. In: *Rossiĭskaya Akademiya Nauk. Izvestiya. Seriya Matematicheskaya* 57.2 (1993), pp. 3–50.

[51] Alexey Bondal and Michel Van den Bergh. “Generators and representability of functors in commutative and noncommutative geometry”. In: *Moscow Mathematical Journal* 3.1 (2003), pp. 1–36, 258.

[52] Ragnar-Olaf Buchweitz and Lutz Hille. “Hochschild (co-)homology of schemes with tilting object”. In: *Transactions of the American Mathematical Society* 365.6 (2013), pp. 2823–2844.

[53] Ragnar-Olaf Buchweitz, Graham Leuschke, and Michel Van den Bergh. “On the derived category of Grassmannians in arbitrary characteristic”. In: *Compositio Mathematica* 151.7 (2015), pp. 1242–1264.

[54] Igor Burban and Yuriy Drozd. “Tilting on non-commutative rational projective curves”. In: *Mathematische Annalen* 351.3 (2011), pp. 665–709.

[55] Igor Burban, Yuriy Drozd, and Volodymyr Gavran. “Minors and resolutions of non-commutative schemes”. In: *European Journal of Mathematics* (2017).

[56] Stefaan Caenepeel and Freddy Van Oystaeyen. “Quadratic forms with values in invertible modules”. In: *K-Theory* 7.1 (1993), pp. 23–40.

[57] Damien Calaque, Carlo Rossi, and Michel Van den Bergh. "Hochschild (co)homology for Lie algebroids". In: *International Mathematics Research Notices* 21 (2010), pp. 4098–4136.

[58] Fabrizio Catanese and Lothar Gœttsche. "$d$-very-ample line bundles and embeddings of Hilbert schemes of 0-cycles". In: *Manuscripta Mathematica* 68.3 (1990), pp. 337–341.

[59] Daniel Chan and Kenneth Chan. "Rational curves and ruled orders on surfaces". In: *Journal of Algebra* 435 (2015), pp. 52–87.

[60] Daniel Chan and Colin Ingalls. "Conic bundles and Clifford algebras". In: *New trends in noncommutative algebra*. Vol. 562. Contemporary Mathematics. American Mathematical Society, 2012, pp. 53–75.

[61] Daniel Chan and Colin Ingalls. "Non-commutative coordinate rings and stacks". In: *Proceedings of the London Mathematical Society. Third Series* 88.1 (2004), pp. 63–88.

[62] Daniel Chan and Colin Ingalls. "The minimal model program for orders over surfaces". In: *Inventiones Mathematicae* 161.2 (2005), pp. 427–452.

[63] Daniel Chan and Rajesh S. Kulkarni. "del Pezzo orders on projective surfaces". In: *Advances in Mathematics* 173.1 (2003), pp. 144–177.

[64] Denis-Charles Cisinski and Gonçalo Tabuada. "Symmetric monoidal structure on non-commutative motives". In: *Journal of K-Theory* 9.2 (2012), pp. 201–268.

[65] David Cox, Joh Little, and Henry Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, 2011, pp. xxiv+841. ISBN: 978-0-8218-4819-7.

[66] Charles Curtis and Irving Reiner. *Methods of representation theory. Vol. II.* Pure and Applied Mathematics. John Wiley & Sons, Inc., 1987, pp. xviii+951. ISBN: 0-471-88871-0.

[67] Vladimir Danilov. "Decomposition of some birational morphisms". In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 44.2 (1980), pp. 465–477, 480.

[68] Kevin De Laet. "Graded Clifford algebras of prime global dimension with an action of $H_p$". In: *Communications in Algebra* 43.10 (2015), pp. 4258–4282.

[69] Koen De Naeghel and Nicolas Marconnet. "Ideals of cubic algebras and an invariant ring of the Weyl algebra". In: *Journal of Algebra* 311.1 (2007), pp. 380–433.

[70] Koen De Naeghel and Michel Van den Bergh. "Ideal classes of three-dimensional Sklyanin algebras". In: *Journal of Algebra* 276.2 (2004), pp. 515–551.

[71] Louis de Thanhoffer de Völcsey. "Non-commutative projective geometry and Calabi–Yau algebras". PhD thesis. Universiteit Hasselt, 2015.

[72] Pierre Deligne and Luc Illusie. "Relèvements modulo $p^2$ et décomposition du complexe de de Rham". In: *Inventiones Mathematicae* 89.2 (1987), pp. 247–270.

[73] Michel Demazure. "Automorphismes et déformations des variétés de Borel". In: *Inventiones Mathematicae* 39.2 (1977), pp. 179–186.

[74] Hoang Dinh Van, Liyu Liu, and Wendy Lowen. "Non-commutative deformations and quasi-coherent modules". In: *Selecta Mathematica. New Series* 23.2 (2017), pp. 1061–1119.

[75] Igor Dolgachev. *Classical algebraic geometry: A modern view.* Cambridge University Press, 2012, pp. xii+639. ISBN: 978-1-107-01765-8.

[76] Vasily Dolgushev, Christopher Rogers, and Thomas Willwacher. "Kontsevich's graph complex, GRT, and the deformation complex of the sheaf of polyvector fields". In: *Annals of Mathematics. Second Series* 182.3 (2015), pp. 855–943.

[77] David Eisenbud and Joe Harris. *3264 and all that.* Cambridge University Press, 2016, pp. xvi+516. ISBN: 9781107602724.

[78] Alexey Elagin. "Semi-orthogonal decompositions for derived categories of equivariant coherent sheaves". In: *Rossiĭskaya Akademiya Nauk. Izvestiya. Seriya Matematicheskaya* 73.5 (2009), pp. 37–66.

[79] Alexey Elagin and Valery Lunts. "On full exceptional collections of line bundles on del Pezzo surfaces". In: *Moscow Mathematical Journal* 16.4 (2016), pp. 691–709.

[80] John Fogarty. "Algebraic families on an algebraic surface". In: *American Journal of Mathematics* 90 (1968), pp. 511–521.

[81] John Fogarty. "Algebraic families on an algebraic surface. II. The Picard scheme of the punctual Hilbert scheme". In: *American Journal of Mathematics* 95 (1973), pp. 660–687.

[82] Anton Fonarev and Alexander Kuznetsov. *Derived categories of curves as components of Fano manifolds.* 2016. arXiv: `1612.02241v1 [math.AG]`.

[83] William Fulton. *Intersection theory.* Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag, 1984, pp. xi+470. ISBN: 3-540-12176-5.

[84] Pierre Gabriel. "Des catégories abéliennes". In: *Bulletin de la Société Mathématique de France* 90 (1962), pp. 323–448.

[85] Christof Geiß and Detlef Voigt. "Non-reduced automorphism schemes". In: *Journal of Pure and Applied Algebra* 193.1-3 (2004), pp. 123–127.

[86] Murray Gerstenhaber and Samuel Schack. "A Hodge-type decomposition for commutative algebra cohomology". In: *Journal of Pure and Applied Algebra* 48.3 (1987), pp. 229–247.

[87] Philippe Gille and Tamás Szamuely. *Central simple algebras and Galois cohomology.* Vol. 101. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006, pp. xii+343. ISBN: 978-0-521-86103-8.

[88] Victor Ginzburg. *Lectures on noncommutative geometry.* 2005. arXiv: `math/0506603v1 [math.AG]`.

[89] Sergey Gorchinskiy and Dmitri Orlov. “Geometric phantom categories”. In: *Publications Mathématiques. Institut de Hautes Études Scientifiques* 117 (2013), pp. 329–349.

[90] Alexey Gorodentsev. “Exceptional bundles on surfaces with a moving anticanonical class”. In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 52.4 (1988), pp. 740–757, 895.

[91] Dieter Happel. “A family of algebras with two simple modules and Fibonacci numbers”. In: *Archiv der Mathematik* 57.2 (1991), pp. 133–139.

[92] Dieter Happel. “Hochschild cohomology of piecewise hereditary algebras”. In: *Colloquium Mathematicum* 78.2 (1998), pp. 261–266.

[93] Dieter Happel. “On the derived category of a finite-dimensional algebra”. In: *Commentarii Mathematici Helvetici* 62.3 (1987), pp. 339–389.

[94] Dieter Happel. “The trace of the Coxeter matrix and Hochschild cohomology”. In: *Linear Algebra and its Applications* 258 (1997), pp. 169–177.

[95] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics 52. Springer-Verlag, 1977, pp. xvi+496.

[96] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics 52. Springer-Verlag, 1977, pp. xvi+496.

[97] Robin Hartshorne. *Residues and duality*. Lecture Notes in Mathematics 20. Springer-Verlag, 1966, pp. vii+423.

[98] Roozbeh Hazrat. *Graded rings and graded Grothendieck groups*. Vol. 435. London Mathematical Society Lecture Note Series. Cambridge University Press, 2016, pp. vii+235.

[99] *Helices and vector bundles*. Vol. 148. London Mathematical Society Lecture Note Series. Cambridge University Press, 1990, pp. iv+143. ISBN: 0-521-38811-2.

[100] Alex Heller and Irving Reiner. “Grothendieck groups of integral group rings”. In: *Illinois Journal of Mathematics* 9 (1965), pp. 349–360.

[101] Lutz Hille and Markus Perling. “Exceptional sequences of invertible sheaves on rational surfaces”. In: *Compositio Mathematica* 147.4 (2011), pp. 1230–1280.

[102] Nigel Hitchin. “Deformations of holomorphic Poisson manifolds”. In: *Moscow Mathematical Journal* 12.3 (2012), pp. 567–591, 669.

[103] Gerhard Hochschild, Bertram Kostant, and Alex Rosenberg. “Differential forms on regular affine algebras”. In: *Transactions of the American Mathematical Society* 102 (1962), pp. 383–408.

[104] Timothy Hodges. “A noncommutative Bass-Tate sequence”. In: *Bulletin de la Société Mathématique de Belgique. Série B* 39.3 (1987), pp. 255–257.

[105] Wei Hong. *Poisson cohomology of holomorphic toric Poisson manifolds*. 2016. arXiv: `1611.08485v1 [math-ph]`.

[106] Wei Hong and Ping Xu. "Poisson cohomology of del Pezzo surfaces". In: *Journal of Algebra* 336 (2011), pp. 378–390.

[107] Mark Hovey, John Palmieri, and Neil Strickland. "Axiomatic stable homotopy theory". In: *Memoirs of the American Mathematical Society* 128.610 (1997), pp. x+114.

[108] Daniel Huybrechts. "The global Torelli theorem: classical, derived, twisted". In: *Algebraic geometry—Seattle 2005. Part 1*. Vol. 80. Proceedings Symposia in Pure Mathematics. American Mathematical Society, 2009, pp. 235–258.

[109] Diel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. Oxford University Press, 2006, pp. viii+307. ISBN: 978-0-19-929686-6.

[110] Osamu Iyama. "Finiteness of representation dimension". In: *Proceedings of the American Mathematical Society* 131.4 (2003), pp. 1011–1014.

[111] Aise Johan de Jong. *A result of Gabber*. URL: `http://www.math.columbia.edu/~dejong/papers/2-gabber.pdf`.

[112] Dmitri Kaledin. "Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie". In: *Pure and Applied Mathematics Quarterly* 4.3, Special Issue: In honor of Fedor Bogomolov. Part 2 (2008), pp. 785–875.

[113] Dmitri Kaledin. *Spectral sequences for cyclic homology*. 2016. arXiv: `1601.00637v1 [math.AG]`.

[114] Mikhail Kapranov. "Derived category of coherent bundles on a quadric". In: *Akademiya Nauk SSSR. Funktsional'nyĭ Analiz i ego Prilozheniya* 20.2 (1986), p. 67.

[115] Mikhail Kapranov. "Derived category of coherent sheaves on Grassmann manifolds". In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 48.1 (1984), pp. 192–202.

[116] Mikhail Kapranov. "On the derived categories of coherent sheaves on some homogeneous spaces". In: *Inventiones Mathematicae* 92.3 (1988), pp. 479–508.

[117] Mikhail Kapranov. "On the derived category and K-functor of coherent sheaves on intersections of quadrics". In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 52.1 (1988), pp. 186–199.

[118] Kotaro Kawatani and Shinnosuke Okawa. *Nonexistence of semiorthogonal decompositions and sections of the canonical bundle*. 2015. arXiv: `1508.00682v1 [math.AG]`.

[119] Bernhard Keller. "Derived categories and tilting". In: *Handbook of tilting theory*. Vol. 332. London Mathematical Society Lecture Note Series. Cambridge University Press, 2007, pp. 49–104.

[120] Bernhard Keller. *Derived invariance of higher structures on the Hochschild complex*. URL: `https://webusers.imj-prg.fr/~bernhard.keller/publ/dih.pdf`.

[121] Bernhard Keller. “Hochschild cohomology and derived Picard groups”. In: *Journal of Pure and Applied Algebra* 190.1-3 (2004), pp. 177–196.

[122] Bernhard Keller. “Invariance and localization for cyclic homology of DG algebras”. In: *Journal of Pure and Applied Algebra* 123.1-3 (1998), pp. 223–273.

[123] Bernhard Keller. “On the cyclic homology of exact categories”. In: *Journal of Pure and Applied Algebra* 136.1 (1999), pp. 1–56.

[124] Bernhard Keller and Wendy Lowen. “On Hochschild cohomology and Morita deformations”. In: *International Mathematics Research Notices* 17 (2009), pp. 3221–3235.

[125] Sebastian Klein. “Chow groups of tensor triangulated categories”. In: *Journal of Pure and Applied Algebra* 220.4 (2016), pp. 1343–1381.

[126] Sebastian Klein. *Relative tensor triangular Chow groups, singular varieties and localization.* 2015. arXiv: `1510.00211v1 [math.CT]`.

[127] Maxim Kontsevich. “Deformation quantization of algebraic varieties”. In: *Letters in Mathematical Physics* 56.3 (2001). EuroConférence Moshé Flato 2000, Part III (Dijon), pp. 271–294.

[128] Maxim Kontsevich. “Deformation quantization of Poisson manifolds”. In: *Letters in Mathematical Physics* 66.3 (2003), pp. 157–216.

[129] Maxim Kontsevich. “Homological algebra of mirror symmetry”. In: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994).* Birkhäuser, Basel, 1995, pp. 120–139.

[130] Maxim Kontsevich and Yan Soibelman. “Notes on $A_\infty$-algebras, $A_\infty$-categories and non-commutative geometry”. In: *Homological mirror symmetry.* Vol. 757. Lecture Notes in Physics. Springer, Berlin, 2009, pp. 153–219.

[131] Henning Krause. “Localization theory for triangulated categories”. In: *Triangulated categories.* Vol. 375. London Mathematical Society Lecture Note Series. Cambridge University Press, 2010, pp. 161–235.

[132] Andreas Krug and Pawel Sosna. “On the derived category of the Hilbert scheme of points on an Enriques surface”. In: *Selecta Mathematica. New Series* 21.4 (2015), pp. 1339–1360.

[133] S. A. Kuleshov. “Rigid sheaves on surfaces”. In: *Algebraic geometry and its applications (Yaroslavlʹ, 1992).* Aspects Math., E25. Friedrich Vieweg, 1994, pp. 85–104.

[134] S. A. Kuleshov and Dmitri Orlov. “Exceptional sheaves on Del Pezzo surfaces”. In: *Rossiĭskaya Akademiya Nauk. Izvestiya. Seriya Matematicheskaya* 58.3 (1994), pp. 53–87.

[135] Dirk Kussin. “Weighted noncommutative regular projective curves”. In: *Journal of Noncommutative Geometry* 10.4 (2016), pp. 1465–1540.

[136] Alexander Kuznetsov. “Base change for semiorthogonal decompositions”. In: *Compositio Mathematica* 147.3 (2011), pp. 852–876.

[137] Alexander Kuznetsov. "Calabi-Yau and fractional Calabi-Yau categories". In: *Journal für die Reine und Angewandte Mathematik* (2017).

[138] Alexander Kuznetsov. "Derived categories of Fano threefolds". In: *Trudy Matematicheskogo Instituta Imeni V. A. Steklova. Rossiĭskaya Akademiya Nauk* 264 (2009), pp. 116–128.

[139] Alexander Kuznetsov. "Derived categories of quadric fibrations and intersections of quadrics". In: *Advances in Mathematics* 218.5 (2008), pp. 1340–1369.

[140] Alexander Kuznetsov. "Derived categories view on rationality problems". In: *Rationality problems in algebraic geometry*. Lecture Notes in Mathematics 2172. 2016, pp. 67–104.

[141] Alexander Kuznetsov. "Height of exceptional collections and Hochschild cohomology of quasiphantom categories". In: *Journal für die Reine und Angewandte Mathematik* 708 (2015), pp. 213–243.

[142] Alexander Kuznetsov. *Hochschild homology and semiorthogonal decompositions*. 2009. arXiv: `0904.4330v1 [math.AG]`.

[143] Alexander Kuznetsov. "Homological projective duality". In: *Publications Mathématiques de l'Institut de Hautes Études Scientifiques* 105 (2007), pp. 157–220.

[144] Alexander Kuznetsov. "Hyperplane sections and derived categories". In: *Rossiĭskaya Akademiya Nauk. Izvestiya. Seriya Matematicheskaya* 70.3 (2006), pp. 23–128.

[145] Alexander Kuznetsov. "Semiorthogonal decompositions in algebraic geometry". In: *Proceedings of the International Congress of Mathematicians—Seoul 2014*. Vol. 2. 2015, pp. 635–660.

[146] Alexander Kuznetsov and Alexander Perry. "Derived categories of cyclic covers and their branch divisors". In: *Selecta Mathematica. New Series* 23.1 (2017), pp. 389–423.

[147] Kevin De Laet. *Character series and Sklyanin algebras at points of order 2*. 2014. arXiv: `1412.7001v2 [math.RT]`.

[148] Tsit Yuen Lam. *A first course in noncommutative rings*. Vol. 131. Graduate Texts in Mathematics. Springer-Verlag, 1991, pp. xvi+397. ISBN: 0-387-97523-3.

[149] Lieven Le Bruyn. "Central singularities of quantum spaces". In: *Journal of Algebra* 177.1 (1995), pp. 142–153.

[150] Lieven Le Bruyn. "Towards a noncommutative version of the Bass-Tate sequence". In: *Bulletin de la Société Mathématique de Belgique. Série B* 35.1 (1983), pp. 45–67.

[151] Cesar Lecoutre and Susan Sierra. *A new family of Poisson algebras and their deformations*. 2016. arXiv: `1611.04306v1 [math.RA]`.

[152] Joseph Lipman and Amnon Neeman. “Quasi-perfect scheme-maps and boundedness of the twisted inverse image functor”. In: *Illinois Journal of Mathematics* 51.1 (2007), pp. 209–236.

[153] Jean-Louis Loday. *Cyclic homology*. Vol. 301. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, 1992, pp. xviii+454. ISBN: 3-540-53339-7.

[154] Wendy Lowen. “Obstruction theory for objects in abelian and derived categories”. In: *Communications in Algebra* 33.9 (2005), pp. 3195–3223.

[155] Wendy Lowen and Michel Van den Bergh. “Deformation theory of abelian categories”. In: *Transactions of the American Mathematical Society* 358.12 (2006), pp. 5441–5483.

[156] Wendy Lowen and Michel Van den Bergh. “Hochschild cohomology of abelian categories and ringed spaces”. In: *Advances in Mathematics* 198.1 (2005), pp. 172–221.

[157] Valery Lunts. “Categorical resolution of singularities”. In: *Journal of Algebra* 323.10 (2010), pp. 2977–3003.

[158] Valery Lunts and Dmitri Orlov. “Uniqueness of enhancement for triangulated categories”. In: *Journal of the American Mathematical Society* 23.3 (2010), pp. 853–908.

[159] Yuri Manin. *Quantum groups and noncommutative geometry*. Université de Montréal, Centre de Recherches Mathématiques, 1988, pp. vi+91. ISBN: 2-921120-00-3.

[160] Nicolas Marconnet. “Homologies of cubic Artin-Schelter regular algebras”. In: *Journal of Algebra* 278.2 (2004), pp. 638–665.

[161] Alexander Merkurjev. “Adams operations and the Brown-Gersten-Quillen spectral sequence”. In: *Quadratic forms, linear algebraic groups, and cohomology*. Vol. 18. Developments in Mathematics. Springer, 2010, pp. 305–313.

[162] Izuru Mori. “The center of some quantum projective planes”. In: *Journal of Algebra* 204.1 (1998), pp. 15–31.

[163] Shigeru Mukai. “Duality between $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(\hat{X})$ with its application to Picard sheaves”. In: *Nagoya Mathematical Journal* 81 (1981), pp. 153–175.

[164] Shigeru Mukai. “On the moduli space of bundles on K3 surfaces. I”. In: *Vector bundles on algebraic varieties (Bombay, 1984)*. Vol. 11. Tata Institute of Fundamental Research Studies in Mathematics, 1987, pp. 341–413.

[165] Shigeru Mukai. “Symplectic structure of the moduli space of sheaves on an abelian or K3 surface”. In: *Inventiones Mathematicae* 77.1 (1984), pp. 101–116.

[166] David Mumford. “Rational equivalence of 0-cycles on surfaces”. In: *Journal of Mathematics of Kyoto University* 9 (1968), pp. 195–204.

[167] Constantin Năstăsescu and Fred Van Oystaeyen. *Graded ring theory*. Vol. 28. North-Holland Mathematical Library. North-Holland, 1982, pp. ix+340. ISBN: 0-444-86489-X.

[168] Amnon Neeman. "The Grothendieck duality theorem via Bousfield's techniques and Brown representability". In: *Journal of the American Mathematical Society* 9.1 (1996), pp. 205–236.

[169] Amnon Neeman. *Triangulated categories*. Vol. 148. Annals of Mathematics Studies. Princeton University Press, 2001, pp. viii+449. ISBN: 0-691-08685-0.

[170] Tom Nevins and Toby Stafford. "Sklyanin algebras and Hilbert schemes of points". In: *Advances in Mathematics* 210.2 (2007), pp. 405–478.

[171] Douglas Northcott. *Lessons on rings, modules and multiplicities*. Cambridge University Press, 1968, pp. xiv+444.

[172] Saša Novaković. *No phantoms in the derived category of curves over arbitrary fields, and derived characterizations of Brauer-Severi varieties*. 2017. arXiv: 1701.03020v2 [math.AG].

[173] Shinnosuke Okawa. "Semi-orthogonal decomposability of the derived category of a curve". In: *Advances in Mathematics* 228.5 (2011), pp. 2869–2873.

[174] Shinnosuke Okawa and Kazushi Ueda. *Noncommutative quadric surfaces and noncommutative conifolds*. 2014. arXiv: 1403.0713 [math.AG].

[175] Dmitri Orlov. "Derived categories of coherent sheaves, and motives". In: *Rossiĭskaya Akademiya Nauk. Moskovskoe Matematicheskoe Obshchestvo. Uspekhi Matematicheskikh Nauk* 60.6(366) (2005), pp. 231–232.

[176] Dmitri Orlov. "Geometric realizations of quiver algebras". In: *Proceedings of the Steklov Institute of Mathematics* 290.1 (2015), pp. 70–83.

[177] Dmitri Orlov. "Projective bundles, monoidal transformations, and derived categories of coherent sheaves". In: *Rossiĭskaya Akademiya Nauk. Izvestiya. Seriya Matematicheskaya* 56.4 (1992), pp. 852–862.

[178] Dmitri Orlov. "Smooth and proper noncommutative schemes and gluing of DG categories". In: *Advances in Mathematics* 302 (2016), pp. 59–105.

[179] Giorgio Ottaviani. "Spinor bundles on quadrics". In: *Transactions of the American Mathematical Society* 307.1 (1988), pp. 301–316.

[180] Alexander Polishchuk. "Lefschetz type formulas for dg-categories". In: *Selecta Mathematica. New Series* 20.3 (2014), pp. 885–928.

[181] Dalide Pontoni. "Quantum cohomology of $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ and enumerative applications". In: *Transactions of the American Mathematical Society* 359.11 (2007), pp. 5419–5448.

[182] Daniel Quillen. "Higher algebraic K-theory. I". In: (1973), 85–147. Lecture Notes in Math., Vol. 341.

[183] Irving Reiner. *Class groups and Picard groups of group rings and orders*. Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics 26. American Mathematical Society, 1976, pp. iv+44.

[184] Irving Reiner. *Maximal orders.* London Mathematical Society Monographs 5. Academic Press, 1975, pp. xii+395.

[185] Idun Reiten and Michel Van den Bergh. "Grothendieck groups and tilting objects". In: *Algebras and Representation Theory* 4.1 (2001), pp. 1–23.

[186] Idun Reiten and Michel Van den Bergh. "Two-dimensional tame and maximal orders of finite representation type". In: *Memoirs of the American Mathematical Society* 80.408 (1989), pp. viii+72.

[187] Alexander Rosenberg. "Noncommutative schemes". In: *Compositio Mathematica* 112.1 (1998), pp. 93–125.

[188] Raphaël Rouquier. "Derived categories and algebraic geometry". In: *Triangulated categories.* Vol. 375. London Mathematical Society Lecture Note Series. Cambridge University Press, 2010, pp. 351–370.

[189] Sean Sather-Wagstaff and Richard Wicklein. "Support and adic finiteness for complexes". In: *Communications in Algebra* 45.6 (2017), pp. 2569–2592.

[190] Jean-Pierre Serre. "Faisceaux algébriques cohérents". In: *Annals of Mathematics. Second Series* 61 (1955), pp. 197–278.

[191] Jean-Pierre Serre. *Linear representations of finite groups.* Springer-Verlag, 1977, pp. x+170. ISBN: 0-387-90190-6.

[192] Susan Sierra. "$G$-algebras, twistings, and equivalences of graded categories". In: *Algebras and Representation Theory* 14.2 (2011), pp. 377–390.

[193] Evgeny Sklyanin. "Some algebraic structures connected with the Yang-Baxter equation". In: *Akademiya Nauk SSSR. Funktsional'nyĭ Analiz i ego Prilozheniya* 16.4 (1982), pp. 27–34, 96.

[194] Evgeny Sklyanin. "Some algebraic structures connected with the Yang-Baxter equation. Representations of a quantum algebra". In: *Akademiya Nauk SSSR. Funktsional'nyĭ Analiz i ego Prilozheniya* 17.4 (1983), pp. 34–48.

[195] S. Paul Smith. "Point modules over Sklyanin algebras". In: *Mathematische Zeitschrift* 215.2 (1994), pp. 169–177.

[196] S. Paul Smith. "Some finite-dimensional algebras related to elliptic curves". In: *Representation theory of algebras and related topics (Mexico City, 1994).* Vol. 19. CMS Conference Proceedings. American Mathematical Society, 1996, pp. 315–348.

[197] S. Paul Smith. "The four-dimensional Sklyanin algebras". In: *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part I (Antwerp, 1992).* Vol. 8. 1. 1994, pp. 65–80.

[198] S. Paul Smith and John Tate. "The center of the 3-dimensional and 4-dimensional Sklyanin algebras". In: *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part I (Antwerp, 1992).* Vol. 8. 1. 1994, pp. 19–63.

[199] S. Paul Smith and Michel Van den Bergh. "Noncommutative quadric surfaces". In: *Journal of Noncommutative Geometry* 7.3 (2013), pp. 817–856.

[200] The Stacks Project Authors. *Stacks Project*. 2017. URL: http://stacks.math.columbia.edu.

[201] Toby Stafford and Michel Van den Bergh. "Noncommutative curves and noncommutative surfaces". In: *American Mathematical Society. Bulletin. New Series* 38.2 (2001), pp. 171–216.

[202] Richard Stanley. *Enumerative combinatorics. Vol. 2.* Vol. 62. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999, pp. xii+581. ISBN: 0-521-56069-1.

[203] Richard Stanley. "Hilbert functions of graded algebras". In: *Advances in Mathematics* 28.1 (1978), pp. 57–83.

[204] Darin R. Stephenson and Michaela Vancliff. "Constructing Clifford quantum $\mathbb{P}^3$s with finitely many points". In: *Journal of Algebra* 312.1 (2007), pp. 86–110.

[205] Greg Stevenson. "Filtrations via tensor actions". In: *International Mathematics Research Notices* (2017).

[206] Greg Stevenson. "Subcategories of singularity categories via tensor actions". In: *Compositio Mathematica* 150.2 (2014), pp. 229–272.

[207] Greg Stevenson. "Support theory via actions of tensor triangulated categories". In: *Journal für die Reine und Angewandte Mathematik* 681 (2013), pp. 219–254.

[208] Richard Swan. "Hochschild cohomology of quasiprojective schemes". In: *Journal of Pure and Applied Algebra* 110.1 (1996), pp. 57–80.

[209] Gonçalo Tabuada. *Noncommutative motives.* Vol. 63. University Lecture Series. American Mathematical Society, 2015, pp. x+114. ISBN: 978-1-4704-2397-1.

[210] Gonçalo Tabuada and Michel Van den Bergh. "Noncommutative motives of Azumaya algebras". In: *Journal de l'Institut de Mathématiques de Jussieu* 14.2 (2015), pp. 379–403.

[211] Dmitry Tamarkin and Boris Tsygan. "Noncommutative differential calculus, homotopy BV algebras and formality conjectures". In: *Methods of Functional Analysis and Topology* 6.2 (2000), pp. 85–100.

[212] Louis de Thanhoffer de Völcsey and Michel Van den Bergh. *On an analogue of the Markov equation for exceptional collections of length 4.* 2016. arXiv: 1607.04246v1 [math.AG].

[213] Louis de Thanhoffer de Volcsey and Dennis Presotto. "Some generalizations of preprojective algebras and their properties". In: *Journal of Algebra* 470 (2017), pp. 450–483.

[214] Louis de Thanhoffer de Völcsey and Dennis Presotto. *Homological properties of a certain noncommutative del Pezzo surface.* 2015. arXiv: 1503.03992v4 [math.AG].

[OEIS] *The On-Line Encyclopedia of Integer Sequences.* URL: http://oeis.org.

[215] Robert Thomason. "The classification of triangulated subcategories". In: *Compositio Mathematica* 105.1 (1997), pp. 1–27.

[216] Yukinobu Toda. "Deformations and Fourier-Mukai transforms". In: *Journal of Differential Geometry* 81.1 (2009), pp. 197–224.

[217] Bertrand Toën. "The homotopy theory of dg-categories and derived Morita theory". In: *Inventiones Mathematicae* 167.3 (2007), pp. 615–667.

[218] Bertrand Toën and Michel Vaquié. "Moduli of objects in dg-categories". In: *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série* 40.3 (2007), pp. 387–444.

[219] Bertrand Toën and Gabriele Vezzosi. *Géométrie non-commutative, formule des traces et conducteur de Bloch*. 2017. arXiv: `1701.00455v1 [math.AG]`.

[220] *Triangulated categories and geometry*. Course at École Normale Superieure, Paris. 1998. URL: `http://www.math.uchicago.edu/~mitya/langlands/kontsevich.ps`.

[221] Michel Van den Bergh. "A relation between Hochschild homology and cohomology for Gorenstein rings". In: *Proceedings of the American Mathematical Society* 126.5 (1998), pp. 1345–1348.

[222] Michel Van den Bergh. "Blowing up of non-commutative smooth surfaces". In: *Memoirs of the American Mathematical Society* 154.734 (2001), pp. x+140.

[223] Michel Van den Bergh. "Division algebras on $\mathbb{P}^2$ of odd index, ramified along a smooth elliptic curve are cyclic". In: *Algèbre non commutative, groupes quantiques et invariants (Reims, 1995)*. Vol. 2. Sémin. Congr. Société Mathématique de France, 1997, pp. 43–53.

[224] Michel Van den Bergh. "Noncommutative homology of some three-dimensional quantum spaces". In: *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992)*. Vol. 8. 3. 1994, pp. 213–230.

[225] Michel Van den Bergh. "Non-commutative $\mathbb{P}^1$-bundles over commutative schemes". In: *Transactions of the American Mathematical Society* 364.12 (2012), pp. 6279–6313.

[226] Michel Van den Bergh. "Noncommutative quadrics". In: *International Mathematics Research Notices* 17 (2011), pp. 3983–4026.

[227] Michel Van den Bergh. "Notes on formal deformations of abelian categories". In: *Derived categories in algebraic geometry*. EMS Series of Congress Reports. European Mathematical Society, 2012, pp. 319–344.

[228] Michel Van den Bergh and Jan Van Geel. "A duality theorem for orders in central simple algebras over function fields". In: *Journal of Pure and Applied Algebra* 31.1-3 (1984), pp. 227–239.

[229] Martine Van Gastel. "On the center of the Proj of a three dimensional regular algebra". In: *Communications in Algebra* 30.1 (2002), pp. 1–25.

[230] Michaela Vancliff, Kristel Van Rompay, and Luc Willaert. "Some quantum $\mathbb{P}^3$s with finitely many points". In: *Communications in Algebra* 26.4 (1998), pp. 1193–1208.

[231] Jean-Louis Verdier. "Des catégories dérivées des catégories abéliennes". In: *Astérisque* 239 (1996), xii+253 pp. (1997).

[232] Angelo Vistoli. "Intersection theory on algebraic stacks and on their moduli spaces". In: *Inventiones Mathematicae* 97.3 (1989), pp. 613–670.

[233] Jorge Vitoria. *Equivalences for noncommutative projective spaces.* 2010. arXiv: 1001.4400v3 [math.RA].

[234] Charles Wall. "Nets of conics". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 81.3 (1977), pp. 351–364.

[235] Charles Wall. "Pencils of binary quartics". In: *Rendiconti del Seminario Matematico della Università di Padova* 99 (1998), pp. 197–217.

[236] Marc Wambst. "Complexes de Koszul quantiques". In: *Université de Grenoble. Annales de l'Institut Fourier* 43.4 (1993), pp. 1089–1156.

[237] Jerzy Weyman. *Cohomology of vector bundles and syzygies.* Vol. 149. Cambridge Tracts in Mathematics. Cambridge University Press, 2003, pp. xiv+371. ISBN: 0-521-62197-6.

[238] Yun Ge Xu, Chao Zhang, Xiao Jing Ma, and Qing Feng Hu. "Hochschild cohomology of Beilinson algebra of exterior algebra". In: *Science China. Mathematics* 55.6 (2012), pp. 1153–1170.

[239] Amnon Yekutieli. "The continuous Hochschild cochain complex of a scheme". In: *Canadian Journal of Mathematics. Journal Canadien de Mathématiques* 54.6 (2002), pp. 1319–1337.

[240] Amnon Yekutieli and James J. Zhang. "Dualizing complexes and perverse sheaves on noncommutative ringed schemes". In: *Selecta Mathematica. New Series* 12.1 (2006), pp. 137–177.