The periodic tables of algebraic geometry

Pieter Belmans

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Abstract

To understand our world we classify things. A famous example is the periodic table of elements, describing the properties of all chemical elements known to humanity, a classification of the building blocks we can use in physics, chemistry, and biology. In mathematics, and algebraic geometry in particular, there are many instances of similar "periodic tables", describing fundamental classification results. We will go on a tour of some of these.

To get a grip on the complexity of the world around us and the objects—such as animals, or chemical elements, or stars—appearing in it, we want to classify these objects. This allows us to describe the relationships, similarities, and differences between things we might be interested in, and thus further understand our world.

An early, and somewhat cruel, effort to understand a class of living creatures lead to *lepidopterology*: the study of butterflies and moths, most famously performed by sticking needles through them and displaying them in nice wooden cases, cf. Figure 1. From the 17th century onwards this was an important feature of humanity's interest in biology, and it serves as a prime example of classification in biology. Another important example is Darwin's description and classification of the beaks of the finches on the Galápagos islands, which led him to formulate the theory of evolution.

In this snapshot I want to introduce you to the idea that *classification is an essential aspect of mathematics*, just like it is for biology (and other sciences). The mathematical objects we will discuss are truly as pretty as the butterflies from Figure 1. And whilst in some cases it takes a bit of training as a mathematician to fully grasp their beauty, at least no living creatures need to be harmed to study them.

In §1 we will recall the *periodic table of elements*: an essential tool in modern chemistry, and the result of a lengthy classification effort. The organisation of elements like hydrogen, carbon, and uranium is similar to how mathematical objects are catalogued and have their properties described in a systematic way. Luckily, the study of these mathematical objects requires less interaction with dangerous chemicals.

An important feature is that these classification efforts are an *ongoing process*: when mathematicians complete one classification, they will move on to the next and more challenging one. That is why we will discuss periodic tables in algebraic geometry, going from the 19th to the 21st century, and from completely known settings to cutting-edge research, in §3 up to §6.

1 The periodic table of elements

Every time one enters a chemistry classroom one is presented with a large poster, listing all the 118 known chemical elements together with their properties. This is the famous *periodic table*, and a very basic version is given in Figure 2. It lists elements like hydrogen, helium, and

Figure 1: The work of a lepidopterist

nitrogen in a specific shape which was essential for the development of chemistry in the 19th and 20th century, and continues to be used to explore and explain chemical elements.

The name periodic table refers to an experimentally observed periodicity in the chemical behavior of elements: certain elements tend to exhibit similar behavior. For example the atomic radius has a periodicity: decreasing from left to right, and going up when going down in the table. In Figure 2, the inert noble gases are listed on the very right in light green, with the halogens as main building blocks for salts next to them, and the alkali metals in the first column all being soft and reactive metals. These observations are what chemists tried to formalise into a system. In 1869 Mendeleev catalogued the then-known elements in terms of atomic mass, obtaining the periodic table we now know.

Originally there were gaps in the table: elements that were predicted to exist, but which were not yet discovered. The periodicity of the periodic table also predicted some of the properties that these elements were required to have. For example, Mendeleev predicted the existence of an element with atomic mass ± 72.5, a high melting point, and a gray color. This was element was subsequently found in 1887, and called *germanium*, in order to fill the gap which existed at position 32.

Invariants of elements The periodic table in Figure 2 is a simplification of the periodic table as you usually see it. For space reasons we only list the chemical symbol and its atomic number. But usually a periodic table contains *lots* more data, such as the atomic weight, the melting and boiling point, the electron configuration, etc. A beautiful interactive version can be found at `https://ptable.com`.

These are all examples of *invariants* of the objects being classified: properties of the chemical

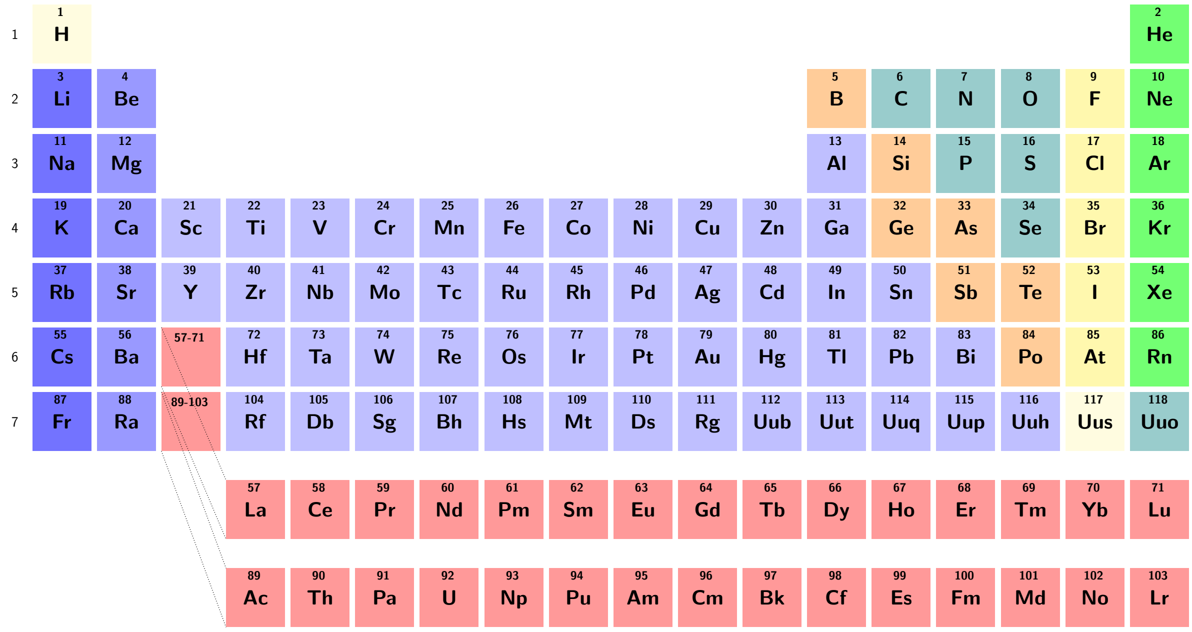


Figure 2: The periodic table of elements

elements that do not change over time, and that do not depend on who measures them. By measuring invariants we can identify which chemical element we are looking at, and distinguish different elements. This is an important idea in mathematics too: mathematicians love to study invariants of objects, and then use them to distinguish between different objects.

Stars and the Hertzsprung–Russell diagram In the first paragraph we also mentioned that one can try to classify the stars in the sky. To better understand an important aspect of classifications in mathematics we need to discuss how classifying stars is different from classifying chemical elements.

Astronomers observed that not all stars are equal: some are brighter than others (even when accounting for the distance), and some are hotter than others. Back in the early 1910s Hertzsprung and Russell made a plot of those two properties of stars, and they noticed that some types of stars are impossible. There are no super-bright cold stars, nor are there very faint hot stars. And there are many stars like our Sun: they all have roughly the same brightness and the same temperature. The interested reader is invited to read up more on the Hertzsprung–Russell diagram.

The main takeaway is that it is possible to *vary the parameters* of a star, subject to certain rules imposed by physics. This is a feature not present in the periodic table, but something similar will happen in mathematics, so one better keep this behavior in mind.

2 Classifications in algebraic geometry

We now turn to classifications in mathematics. One famous instance of such a result in mathematics is the *Classification of Finite Simple Groups* (CFSG). A group describes the symmetries of an object, and group theory is a fundamental subject in modern mathematics. Just like molecules are built using atoms, Jordan showed in 1870 that every finite group is built using simple groups.

The first (interesting) simple groups were already discovered by Galois in 1831, when he was studying solutions of polynomials of degree ≥ 5. The first example contains 60 elements. The last group to be discovered (in 1981) was the Monster group, and it has approximately $8 \cdot 10^{53}$ elements. Through a large effort of many mathematicians the CFSG was obtained, stating that all simple groups had been found in those 150 years. For more on this see, e.g., [Cra22, GT16] in this very Snapshots series.

We will instead focus on classifications in *algebraic geometry*, because the author is an algebraic geometer and not a group theorist, and because the story of classifications in algebraic geometry is less well-known than the CFSG. Algebraic geometry is the study of shapes described by polynomial equations. The shapes we will be interested in are *smooth projective varieties*, defined over the complex numbers. Let us unpack what this means.

Smooth projective varieties First of all, working over the *complex numbers* is a necessity to make things tractable, but it also makes it harder to make drawings. Usually we visualise the complex numbers as the complex plane, with one real axis and one imaginary axis. But from the point-of-view of an algebraic geometer the complex numbers are really a one-dimensional object! That is why an algebraic geometer will often draw an impression of an object when considered over the real numbers. More concretely, Figure 3a is what an algebraic geometer would draw when drawing a curve, whilst Figure 4b is what a complex geometer would think of, but they really are manifestations of the same object.

Now, what does it mean to describe a shape *using polynomials*? If $f \in \mathbb{C}[x]$ is a polynomial, so $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, we define the variety associated to it as

$$\mathbb{V}(f) = \{\alpha \in \mathbb{C} \mid f(\alpha) = 0\},$$

the set of zeroes of f in the complex plane. This set is always finite if the polynomial is not constant zero and consists of at most d points if it is not constant. Over the complex numbers it consists of exactly d points counted with multiplicities. In general we will consider a finite collection of polynomials in n variables $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$, and define

$$\mathbb{V}(f_1, \ldots, f_r) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \mid \forall i = 1, \ldots, r\colon f_i(\alpha_1, \ldots, \alpha_n) = 0\},$$

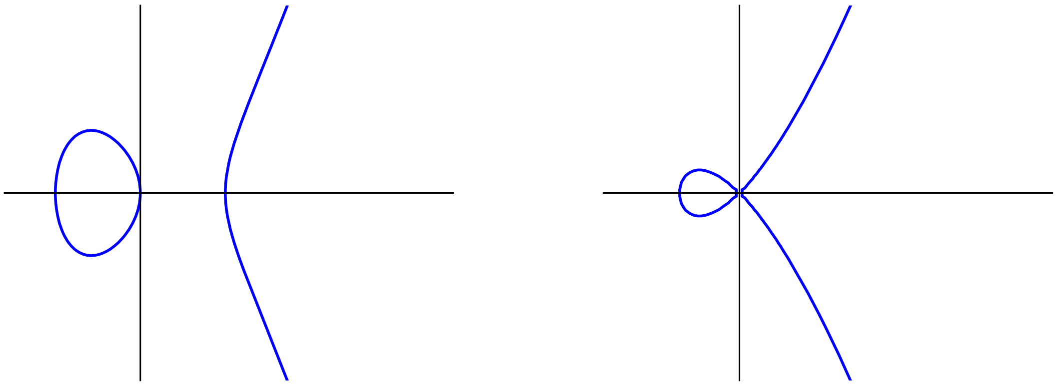
the set of points (inside the *affine space* $\mathbb{C}^n$), or zero locus, satisfying all polynomial equations simultaneously. These subsets are called *affine varieties*.

Instead of affine varieties we will be interested in *projective varieties*. In the affine plane two lines can be parallel, but this causes annoying situations in which we have to say that two distinct lines intersect in precisely one point unless they are parallel. That is why we extend our affine geometry: to make statements like the one on intersections of distinct lines more uniform, and get rid of the exceptions. In the projective world we have that our initial two parallel lines now intersect in a point at infinity, so any two distinct lines now always intersect.

For this we need to replace the affine space in which affine varieties live, by *projective space* $\mathbb{P}^n(\mathbb{C})$. It is defined by considering the set $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ of all points except the origin in the affine space one dimension up, up to the equivalence relation which says that $(\alpha_0, \ldots, \alpha_n) \sim (\beta_0, \ldots, \beta_n)$ if there exists some $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\alpha_i = \lambda \beta_i$ for all $i = 0, \ldots, n$.

To work with projective varieties using polynomials, we will only consider *homogeneous* polynomials: a polynomial in which every term has the same degree. For example $x^2 + y^2 + z^2$ is a homogeneous polynomial of degree 2, in 3 variables. The *projective variety* associated to a collection of homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$ is

$$\mathbb{V}(f_1, \ldots, f_r) = \{(\alpha_0, \ldots, \alpha_n) \in \mathbb{P}^n(\mathbb{C}) \mid \forall i = 1, \ldots, r\colon f_i(\alpha_0, \ldots, \alpha_n) = 0\}.$$



(a) Real points of the cubic curve $y^2 = x^3 - 2x$

(b) Real points of the singular cubic curve $y^2 = x^3 + x^2$

Figure 3: A smooth versus a singular cubic curve

This is well-defined because we restricted our attention to homogeneous polynomials, so that asking whether a polynomial is (non-)zero is independent of the scaling in the equivalence relation. The example $x^2 + y^2 + z^2$ thus defines a curve in $\mathbb{P}^2(\mathbb{C})$—a conic—corresponding to Figure 4a.

The final ingredient in order to describe the objects we are interested in is *smoothness*. This is best explained through an example: consider the following degree 3 polynomials

$$f = -y^2 + x^3 - 2x$$
$$g = -y^2 + x^3 + x^2$$

describing affine curves living in $\mathbb{C}^2$. If we draw these curves inside $\mathbb{R}^2 \subset \mathbb{C}^2$ we get the pictures as in Figure 3. We immediately see that on the right there is something funny happening at the origin: there is a *singularity*. For more on singularities we refer to another Snapshot [BF14].

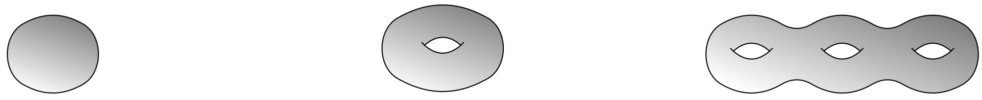
Classifying smooth projective varieties? In what follows next we discuss examples of classifications of smooth projective varieties. This will illustrate how the life of an algebraic geometer can be very similar to that of someone sticking needles through unsuspecting butterflies, or that of an experimental chemist inhaling noxious fumes in order to isolate an unknown chemical element.

Before we embark on our journey we need to point out that to an algebraic geometer classification can mean different things. Certainly, we are not just classifying polynomials, rather we are interested in classifying varieties independently of their realisation. This gives rise to classification we will mostly be talking about: that of varieties *up to isomorphism*, i.e. up to their realisation inside some projective space.

It turns out that already in dimension 2 this becomes impossible, so that we will only try to classify certain well-chosen objects. There is an entire branch of algebraic geometry, called birational geometry, devoted to understanding the precise relationships between smooth projective varieties which are different but nevertheless almost the same: we say that they are *birational*, but we will not discuss this further.

3 Curves: Riemann

The first classification we look at will be of the simplest objects we can try to classify: curves, or one-dimensional varieties.



(a) A curve of genus 0 (b) A curve of genus 1 (c) A curve of genus 3

Figure 4: Complex algebraic curves as Riemann surfaces

We have already seen an important recipe and some examples to describe curves: take a homogeneous polynomial $f \in \mathbb{C}[x,y,z]$ of degree d and consider its zero locus $\mathbb{V}(f) \subset \mathbb{P}^2(\mathbb{C})$. The pictures in Figure 3 really live in $\mathbb{C}^2$, but e.g., for Figure 3a we can remedy this affine vs. projective discrepancy, replacing $-y^2+x^3-2x$ by $-y^2z+x^3-2xz^2$.

Can we describe all curves using a single homogeneous polynomial in 3 variables? In other words, is every curve a *plane curve*, living in $\mathbb{P}^2(\mathbb{C})$? No. But one can show that every curve is a *space curve*: it can be described by several polynomials in 4 variables, so that it is a curve living in $\mathbb{P}^3(\mathbb{C})$.

But does this help in the classification of curves? Recall that we don't want to classify polynomials, because those are only the tools to describe the curves. So we need to talk about things which are *intrinsic* to the geometry of a curve, independent of the realisation.

A discrete invariant For this we can look at an algebraic curve as a so-called Riemann surface: because we work over the complex numbers, a topologist's picture of a curve is really 2-dimensional. This is also how they were originally studied by Riemann in the 1850s. We end up with drawings as in Figure 4. They correspond to plane curves of degree 2, 3 and 4 respectively. The obvious difference between these pictures is "the number of holes". Mathematicians refer to this number as the *genus*.

It is similar to what a biologist would use to distinguish a giraffe from a fish: the former has 4 legs, the latter has none. Therefore they must be different in a meaningful sense, and the biologist will say they belong to different species (even though they are both animals).

So for now the "periodic table" of the classification of curves looks pretty bland: it is just the sequence of integers $0, 1, 2, \ldots$ But biologists don't stop classifying animals after having counted their legs, and neither will we after counting holes.

On a Riemann surface mathematicians study certain additional structure which allows us to speak about the *curvature*. This curvature is either $+1$, 0 or -1. The case of a sphere (as in Figure 4a) has positive curvature, the case of a torus (as in Figure 4b) has zero curvature (we say it is *flat*), and all other cases (e.g., Figure 4c) have negative curvature. Thus, there is an additional *trichotomy* into $g=0$, $g=1$ and $g \geq 2$.

Continuous parametrisations But this is still not the end of the classification of algebraic curves. We have described a Riemann surface of genus 1 using a homogeneous polynomial of degree 3. What happens if we start varying the coefficients of this polynomial? For example, what if instead of the homogeneous version $y^2z = x^3 - 2xz^2$ of the equation in Figure 3a, we consider $y^2z = x^3 - 3xz^2$? We can show that they define "the same" curve: they are *isomorphic*.

But what if we consider $y^2z = x^3 - 3xz^2 + 2z^3$ instead? Then we can actually show that the curves are different, even though the genus is 1 in both cases. For this we'd have to use something called the j-invariant: a number attached to every genus 1 curve. In the former case it is equal to 1728, in the latter it is 864 (quick word of warning: the j-invariant is not a count of anything, in general it can be any complex number). Thus to a trained algebraic geometer these are in fact different curves. Hence we can use parameters in our equations and get truly distinct answers. In our example changed the coefficient of z^3 from 0 to 2, and we could in fact have considered all intermediate values (including complex numbers) to get all kinds of non-isomorphic curves of genus 1. They are usually referred to as *elliptic curves*, and they are an important tool in modern cryptography.

This continuous behavior is not something that happens in the periodic table of elements: you can't move from hydrogen to helium by adding tiny fractions of neutrons, electrons and protons. The closest analogy in science is the Hertzsprung–Russell diagram, where you can vary the luminosity and temperature of a star continuously.

What about other degrees? Here the trichotomy into $g = 0$, $g = 1$ and $g \geq 2$ comes back into play. We can show that for $g = 0$ there are no parameters possible (so there is a single curve of genus 0), whilst for $g \geq 2$ there are in fact $3g - 3$ parameters (so there are *many* curves of genus g, and describing them all in a suitable sense is an interesting problem). This result for $g \geq 2$ is what Riemann obtained back in 1853, effectively introducing the notion of a "moduli space" to mathematics: a parameter space to describe all curves of a given genus.

More precisely, a *moduli space* is a new geometric object, that acts as a record-keeping device to describe a classification that involves continuous parameters. Every point in the moduli space corresponds to a certain element in the classification, and if we move a tiny bit in the moduli space we modify the element accordingly by a small amount to get a new element in the classification. If we move around in a different direction, we get another element in the classification.

This turns a classification of shapes into a new shape, and thus we can use all the tools from algebraic geometry to study it. Many questions about a classification can be phrased in terms of geometric properties of the moduli space, and mathematicians study moduli spaces of all kinds.

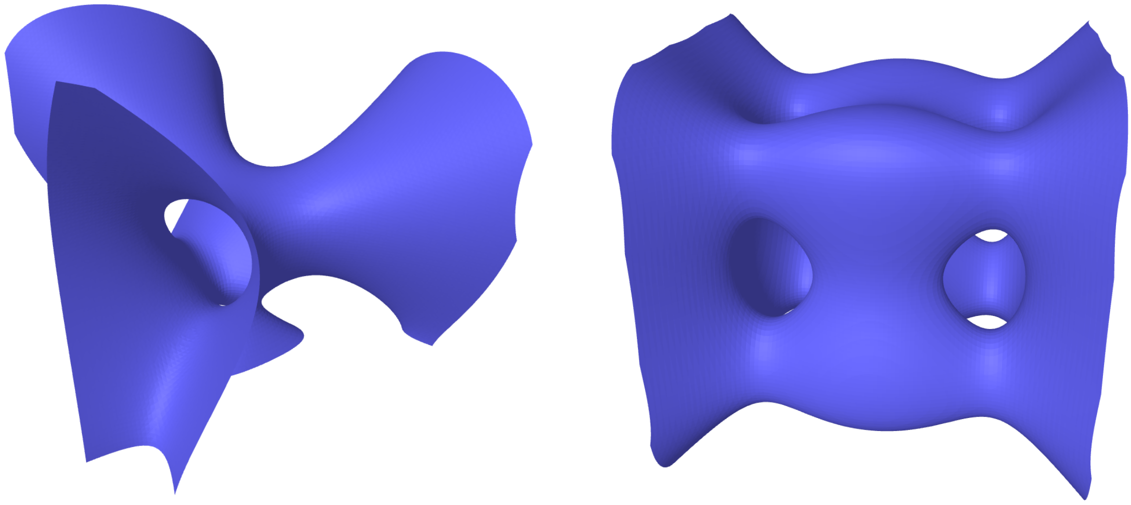
4 Smooth projective surfaces: Enriques

Going up one dimension we end up with surfaces. They have been at the forefront of algebraic geometry since the 19th century. The easiest algebraic surfaces we can produce are by taking a homogeneous polynomial in 4 variables, and consider $\mathbb{V}(f)$ in $\mathbb{P}^3(\mathbb{C})$. Because we are working over the complex numbers this would require a 4-dimensional drawing, which goes beyond what we can do here. But in Figure 5 we do what algebraic geometers often do: make a picture of an affine piece over the real numbers. IMAGINARY in fact offers software to do this easily: `https://www.imaginary.org/program/surfer`.

The classification of smooth projective surfaces is due to Enriques, as his life's work, posthumously published in 1949. We will necessarily have to gloss over many details, but we will highlight some of its most interesting features. Without further mention we will classify *minimal* surfaces: every surface can be reduced to such a surface in a controlled way, and we know how to go back. Thus it suffices to classify these.

Trichotomy Just like for Riemann surfaces we have a notion of curvature, introducing an important trichotomy between positive, flat and negative. With curves we had that *positively curved* case was the easiest, there being a unique such curve. For surfaces this is still the easiest case, but the uniqueness no longer holds: there are now 10 families of so-called *del Pezzo surfaces*, named after the mathematician who classified them in 1887. Some of these are unique in their family, for others there are continuous parameters.

If we consider a single homogeneous polynomial in 4 variables, the cases of degree $d = 1, 2$ and 3 give rise to del Pezzo surfaces. In Figure 5a we have given an impression of an important cubic surface, where $d = 3$. The geometry of these del Pezzo surfaces is already rich enough



(a) The Clebsch cubic surface (b) A quartic K3 surface

Figure 5: Two algebraic surfaces considered in $\mathbb{R}^3$

to fill entire books—even though their classification is relatively straightforward—and their higher-dimensional analogues will be important for what follows.

What about the flat case, the 2-dimensional analogue of elliptic curves? There are now *two* distinct families. The closest analogue of elliptic curves are *abelian surfaces*. But there are also *K3 surfaces*, named so by Weil in 1958 after the recently climbed K2 mountain in the Himalayas, and the three mathematicians Kummer, Kähler, and Kodaira, who had been building the tools to study algebraic geometry and these surfaces in particular.

If we again consider a single homogeneous polynomial in 4 variables, the case $d = 4$ gives rise to a K3 surface, thus in Figure 5b we see an impression of an example. As with del Pezzo surfaces, their geometry is rich enough to fill entire volumes, and their higher-dimensional analogues will again be important for what follows.

The geography of surfaces: surfaces of general type We now come to the analogue of curves of genus $g \geq 2$. There are such curves for every value of g, in fact there is an entire moduli space of dimension $3g - 3$ of them describing their classification.

Unlike for curves, a single integer is no longer enough to describe the crude classification of surfaces. Two important integers we can assign to a surface are the *Chern numbers* c_1^2 and c_2. For the genus we only had the inequality $g \geq 0$ because we were counting something. For surfaces the situation is more complicated, and the possible values depend on the curvature. In the flat case $\mathrm{c}_1^2 = 0$, whilst in the negatively curved case $\mathrm{c}_1^2 > 0$. But what other conditions do we have?

We also have $\mathrm{c}_2 > 0$, but more interesting we have that $\mathrm{c}_1^2 + \mathrm{c}_2 \equiv 0 \bmod 12$. Besides this, there are two inequalities that need to be satisfied, the easier inequality to describe saying that $\mathrm{c}_1^2 \leq 3\mathrm{c}_2$. In Figure 6 we have drawn the "legal" values for sufficiently small Chern numbers. These conditions are similar to a law in biology saying that the number of legs on an animal is always even, but starfish are obvious counterexamples, so this particular universal biological law does not exist.

Now we have discussed *necessary* conditions on the Chern numbers. Are these also *sufficient* conditions, i.e., can we always find a surface with these allowed numbers? This leads to the problem of understanding the *geography of surfaces*, in particular those with negative curvature,

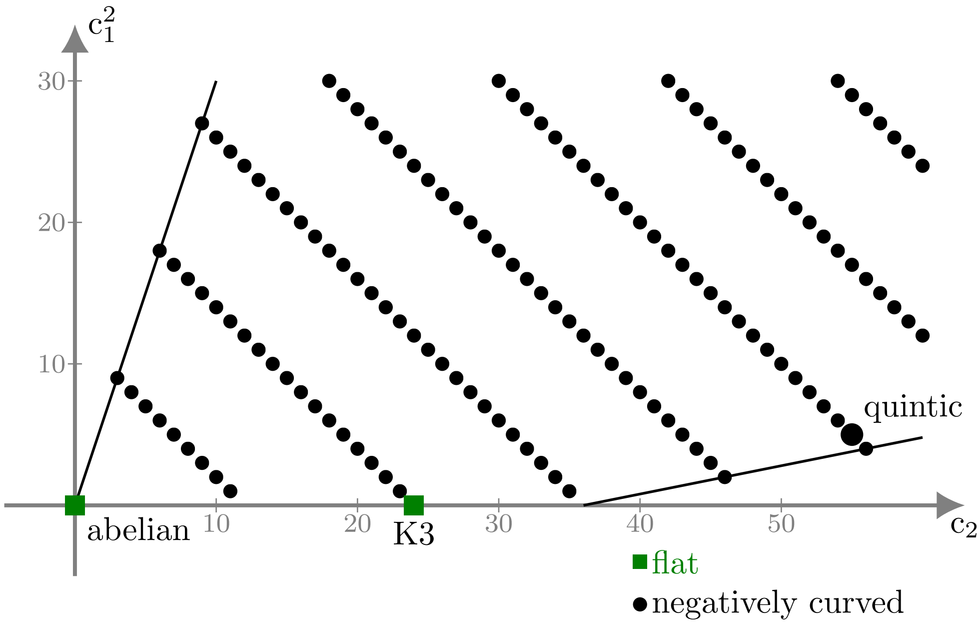


Figure 6: The geography of algebraic surfaces

which are called *of general type*,

We can already fill in two positions in our “atlas”: abelian and K3 surfaces have $c_2 = 0$ resp. 24. Next, if we take a homogeneous polynomial of degree 5, we get a quintic surface, for which $c_1^2 = 5$ and $c_2 = 55$. There are still *many* other legal values in Figure 6, and it is an interesting challenge to find a construction for a surface with given coordinates. We don't know yet whether every pair of coordinates corresponds to a surface for instance, but luckily algebraic geometers know of *many* interesting and beautiful examples.

The important takeaway is that the classification of surfaces of general type is still an open problem, and it has been giving mathematicians enough material to work on for over a century.

5 Fano 3-folds: Mori–Mukai

In our journey through smooth projective varieties we now reach dimension 3. From this point on it is impossible to make good pictures (although algebraic geometers do develop an intuition for these objects, and make drawings which are hard, if not downright impossible, to be interpreted by outsiders).

For curves and surfaces we saw that the trichotomy between positive, flat and negative curvature gave very different flavours to the classification problem. This pattern continues in higher dimension. The analogue of the $g = 0$ case for curves (so with positive curvature) and del Pezzo surfaces are so-called *Fano varieties*, and thus in dimension 3 these are called *Fano 3-folds*. We will first talk about these, as a full classification indeed exists. In dimension 2 we already saw that there are 10 families of del Pezzo surfaces. So, what about dimension 3?

Again, there exists a full classification, due to Mori and Mukai in 1981 (with important preliminary work by Iskovskikh), building on the results for which Mori eventually won the Fields medal in 1990. There are 105 families in the classification: originally they listed 104, but back in 2003 they found a missing case.

The geometry of Fano 3-folds is truly a treasure trove of interesting algebraic geometry, with lots of ongoing work which falls outside the scope of this snapshot. Having a classification of the objects is after all not the end of the work, but rather the beginning of the systematic study.

Fano 4-folds Having classified Fano varieties in dimensions 1, 2 and 3 we can turn to the classification of Fano varieties in higher dimension. An important result from 1992 by Kollár–Miyaoka–Mori is that in any dimension n, the number of deformation families is *finite* (we continue to only consider smooth projective varieties). But we have no idea how large this number really is. If we try to make the bound "effective" we end up with an upper bound of

$$(n+2)^{(n+2)^{n2^{3n}}}$$

for the number of families of n-dimensional Fano varieties. For $n = 1$ this gives a number with 3131 digits, which is *very* far off from the true value, which we saw is 1.

The first open case is the classification of Fano 4-folds. This is a large undertaking with many people working on it from different perspectives, and this topic alone would make another great Snapshot. Currently we have found about 700 families of Fano 4-folds, but we have no idea how close we are to a full classification. We know that there is only a finite number, but we don't know we are close (say the total number is 1000) or still very far off (say the total number is 100000). Hopefully within a few years we will know how to continue the sequence $1, 10, 105$.

One could also try to classify singular Fano varieties. The essential ingredient for this—the finiteness of the classification problem—was provided recently by Birkar, for which he received the Fields medal in 2018.

Calabi–Yau 3-folds Instead of going to higher-dimensional varieties with positive curvature, we could consider 3-dimensional varieties with flat curvature: *Calabi–Yau 3-folds*. They are the analogues of the K3 surfaces and abelian surfaces we saw before. These objects have played a tremendously important role in theoretical physics and string theory, and given their importance mathematicians have been constructing more and more of these. Their beautiful properties and ongoing classification would form an excellent subject for yet another Snapshot.

But frustratingly enough, we don't know whether the final classification in this case is a finite classification or not! To give a precise number of the number of currently known families of Calabi–Yau 3-folds is hard, because it requires a careful comparison of all the different constructions. Let us just point out that one important type of construction (using reflexive 4-dimensional polytopes, of which there are a whopping 473 800 776) gives rise to 30 108 families of Calabi–Yau 3-folds which are guaranteed to be different.

6 Hyperkähler varieties

One important theme that has been present is that the higher we go in dimension, the more restrictive we want our class of varieties in order to have any hope of classification. The final periodic table in algebraic geometry we want to discuss is one of the most mysterious.

Amongst the varieties of flat curvature there exists a decomposition into building blocks, just like we can decompose molecules into atoms (for arbitrary varieties there is nothing like such a decomposition). There are 3 types:

- abelian varieties of arbitrary dimension;
- Calabi–Yau varieties of dimension ≥ 3;

- hyperkähler varieties.

So the classification problem of varieties with flat curvature reduces to three different classification problems.

We already discussed the classification of Calabi–Yau 3-folds, and in arbitrary dimension the situation is the same: we don't know whether the classification is finite, but we can construct *many* (really, many!) examples. On the other hand, although they possess lots of interesting geometry, the classification of abelian varieties is straightforward: in every dimension there is a single family.

That leaves us with hyperkähler varieties. These are necessarily even-dimensional, and possess an extremely rich and beautiful geometry. Amongst all the varieties we discussed so far, the only hyperkähler varieties are K3 surfaces. Are there any others?

The first examples of dimension ≥ 4 were obtained by Beauville in 1983. Using K3 surfaces he constructed a family of hyperkähler varieties of dimension $2n$. We will call varieties of this type $\mathrm{K3}^{[n]}$. Similarly using abelian surfaces he constructed another family of hyperkähler varieties of dimension $2n$. We will call varieties of this type Kum^n. By computing a numerical invariant of elements in these two families Beauville could moreover show that they are *different* families.

Mathematicians have found other constructions of hyperkähler varieties, but they all were similar to $\mathrm{K3}^{[n]}$ and Kum^n (in the precise sense that they are deformation equivalent). Is this then the end of the classification? **No**!

In 1999 and 2003 O'Grady constructed two new families of hyperkähler varieties, again using K3 surfaces and abelian surfaces, but sufficiently different from the previous construction in order to be truly new. Because they are 6- resp. 10-dimensional we will call them OG_6 and OG_{10}. Currently they look "exceptional", in the sense that they are seemingly not part of a construction that works in arbitrary dimension.

Is *this* then the end of the classification? Are all hyperkähler varieties of type $\mathrm{K3}^{[n]}$, Kum^n, OG_6 and OG_{10}? We have **absolutely no idea**! Already in dimension 4 we don't know whether $\mathrm{K3}^{[2]}$ and Kum^2 are all the types we need. There might be a type of hyperkähler variety that has been hiding from us, like a beautiful butterfly deep within the rain forest. In other words, we are still far from understanding the periodic table of hyperkähler varieties.

Or are we? Mathematicians have come up with a curious conjecture that would describe the periodic table of hyperkähler varieties. There is some similarity between this classification and that of (certain) simple Lie algebras. The latter classification is a famous result from the 19th century, involving 2 infinite families, and 3 exceptional cases. So provided the speculation is correct, we might be close to finding all the necessary objects. Maybe in a few years time someone will be able to write a Snapshot about a proof of this conjecture.

Interactive periodic tables in algebraic geometry Do you want to see some "periodic tables" in algebraic geometry in action? The author has created various interactive interfaces for some of classification results:

- `https://superficie.info`: Enriques–Kodaira classification of surfaces (joint with Johan Commelin)
- `https://fanography.info`: Mori–Mukai classification of Fano 3-folds
- `https://hyperkaehler.info`: classification of hyperkähler varieties
- `https://grassmannian.info`: generalised Grassmannians (not discussed)

It might be hard to really understand what is happening there, but hopefully it is clear that mathematicians are truly interested in classifications.

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Image credits

Figure 1 "Collection drawer with butterflies in Upper Silesian Museum in Bytom, Poland". Author: Marek Ślusarczyk. Licensed under Creative Commons Attribution 3.0 Unported, visited on August 4, 2022.

Figure 2 "Periodic table of elements". Original authors: Ryan Griffin and Janosh Riebesell. Licensed under MIT License, modified by the author from `https://tikz.net/periodic-table`, visited on August 4, 2022.

Figures 3–6 Created by the author using SageMath or Ti*k*Z.

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