Representation Theory I
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Prof. Dr. Catharina Stroppel
Dr. Pieter Belmans

Tutorial exercises

These exercises are to be done in class. By no means are you expected to solve all of them during class. The exercises marked $(*)$ are the most important.

Problem 1 $(*)$**.**

1. Classify all 1-dimensional complex Lie algebras.
2. Classify all 2-dimensional complex Lie algebras.
3. Construct an infinite family of pairwise non-isomorphic 3-dimensional complex Lie algebras.

   Hint Consider the bracket

   $$\begin{aligned} [x,y] &= 0 \\ [x,z] &= x \\ [y,z] &= cy \end{aligned} \tag{1}$$

   where $c \in \mathbb{C}$.

Problem 2 $(*)$**.** Let k be a field. Let A be a k-algebra. Show that the k-linear derivations

$$\mathrm{Der}_k(A) \coloneqq \{D \in \mathrm{End}_k(A) \mid \forall a,b \in A\colon D(ab) = D(a)b + aD(b)\} \tag{2}$$

equipped with the bracket

$$[D,D'] \coloneqq D \circ D' - D' \circ D \tag{3}$$

form a Lie algebra over k.

Problem 3. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. We define the product $\mathfrak{g}_1 \times \mathfrak{g}_2$ as the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, together with

$$[(x,y),(x',y')] \coloneqq ([x,x'],[y,y']) \tag{4}$$

for all $x,x' \in \mathfrak{g}_1$ and $y,y' \in \mathfrak{g}_2$.

1. Show that this is again a Lie algebra.
2. Show that it satisfies the universal property of the product.

Problem 4. Let $U \subseteq \mathbb{R}^n$ be an open subset, with coordinate functions $x_1,\ldots,x_n$, and let $A = C^\infty(U;\mathbb{R})$ be the $\mathbb{R}$-algebra of $\mathbb{R}$-valued smooth functions on U. Denote by $\mathfrak{g} = C^\infty(U;\mathbb{R}^n)$ the $\mathbb{R}$-vector space of $\mathbb{R}^n$-valued smooth functions (which form the vector fields on U). Define an operation on $\mathfrak{g}$ by

$$[X,Y] \coloneqq \sum_{i=1}^n \sum_{j=1}^n X_j \frac{\partial Y_i}{\partial x_j} - \sum_{j=1}^n Y_j \frac{\partial X_i}{\partial x_j}. \tag{5}$$

Define

$$\mathfrak{g} \times A \to A : (X,f) \mapsto \mathrm{Lie}_X(f) \coloneqq X(f) \coloneqq \sum_{i=1}^n X_i \frac{\partial f}{\partial x_i}. \tag{6}$$

1. Show that $[-,-]$ equips $\mathfrak{g}$ with the structure of a real Lie algebra.
2. Show that the morphism

$$\mathfrak{g} \to \mathrm{Der}_{\mathbb{R}}(A) : X \mapsto \mathrm{Lie}_X(-) \tag{7}$$

is a well-defined injective homomorphism of real Lie algebras.

Problem 5. Let $\mathfrak{g}$ be a Lie algebra. Define

$$\widetilde{\mathfrak{g}} \coloneqq \mathfrak{g} \oplus kc \tag{8}$$

where c is a formal basis vector. Let

$$\kappa \colon \mathfrak{g} \times \mathfrak{g} \to k \tag{9}$$

be a bilinear map such that for all $x, y \in \mathfrak{g}$ we have that

1. $\kappa(x, x) = 0$;
2. $\kappa(x, [y, z]) + \kappa(y, [z, x]) + \kappa(z, [x, y]) = 0$.

We then say that κ is a *2-cocycle.*

Show that $\widetilde{\mathfrak{g}}$ is a Lie algebra, when it is equipped with the Lie bracket

$$[x + \lambda c, y + \mu c] \coloneqq [x, y] + \kappa(x, y)c. \tag{10}$$

This is a *1-dimensional central extension.*