Representation Theory I
Bonn, summer term 2019

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Sheet 9

Solutions to be handed in before class on Wednesday June 5

Problem 42. Let R be a root system in a finite-dimensional $\mathbb{R}$-vector space V. Let π be a basis of the root system.

1. Prove that V^{reg} is non-empty, and that there exists a vector $\gamma \in V^{\mathrm{reg}}$ such that $(\gamma, \alpha) > 0$ for all $\alpha \in \pi$. (3 points)

2. Illustrate this in the rank 2 root systems, using solutions to earlier exercises. (1 point)

3. Take $\gamma \in V^{\mathrm{reg}}$. Define

$$\begin{aligned} R^+(\gamma) &\coloneqq \{\alpha \in R \mid (\alpha, \gamma) > 0\}, \\ \pi(\gamma) &\coloneqq \left\{\alpha \in R^+(\gamma) \mid \alpha \text{ is not a sum of roots in } R^+(\gamma)\right\}. \end{aligned} \tag{39}$$

   Show that $\pi(\gamma)$ is a basis of the root system. (2 points)

Problem 43. Let $R^\vee$ be the (Langlands) dual root system of R, and set $\pi^\vee \coloneqq \{\alpha^\vee \mid \alpha \in \pi\}$. Prove that $\pi^\vee$ is a basis of $R^\vee$. (2 points)

Problem 44.

1. Prove that the Weyl group of $\mathfrak{sl}_n(\mathbb{C})$ is isomorphic to S_n. (3 points)

2. Choose basis π for the root system. Prove that there is a unique element w_0 in the Weyl group sending R^+ to R^-. (2 points)

3. Show that any element in the Weyl group has a reduced expression, i.e. it is written as the product of s_α for $\alpha \in \pi$. (1 point)

4. Prove that any reduced expression for w_0 must involve all s_α for $\alpha \in \pi$. (2 points)

Optional problem 2. The length of a Weyl group element is the length of a reduced expression. Determine how many elements of each length there are, in the Weyl groups of type A_2, B_2, A_3 and D_4. (2 points)