

## Sheet 8

Solutions to be handed in before class on Wednesday May 29

**Problem 37.** Let  $R$  be a fixed root system of rank  $r$  in the space  $V$ . Let  $\alpha, \beta$  be non-zero elements of  $V$ , and  $\theta = \theta_{\alpha, \beta}$  the corresponding angle. Then  $(\alpha, \beta) = \|\alpha\| \|\beta\| \cos \theta$ . Denoting  $\langle \beta, \alpha^\vee \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$  we get  $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4 \cos^2 \theta$ .

1. Using the axioms of a root system, make a table of the possible values of  $\langle \alpha, \beta^\vee \rangle$ ,  $\langle \beta, \alpha^\vee \rangle$ ,  $\theta$  and  $\|\beta\|^2/\|\alpha\|^2$ . (1 point)

Notice that one can assume that  $\alpha \neq \pm\beta$ , and  $\|\beta\| \geq \|\alpha\|$ .

2. Using the pictures from problem 35, explain how the maximal  $\langle \beta, \alpha^\vee \rangle$  is realised by drawing the root system and explaining which choice of  $\alpha$  and  $\beta$  realises this scenario. (1 point)
3. Let  $\alpha$  and  $\beta$  be non-proportional roots. If  $(\alpha, \beta) > 0$  then  $\alpha - \beta$  is again a root. (2 points)

**Problem 38.** We continue the setup of the previous problem. Let  $\alpha$  and  $\beta$  be non-proportional roots. The  $\alpha$ -string through  $\beta$  is the set of roots of the form  $\beta + n\alpha$  for  $n \in \mathbb{Z}$ .

1. Denote  $r, q \geq 0$  the largest integers for which  $\beta - r\alpha$  (resp.  $\beta + q\alpha$ ) is still a root. Show that the string is unbroken, i.e. that it contains all elements between  $\beta - r\alpha$  and  $\beta + q\alpha$ . (2 points)
2. Show that  $r - q = \langle \beta, \alpha^\vee \rangle$ . (1 point)
3. What is the maximal length of a root string? (1 point)

**Problem 39.** An element of a basis  $\pi$  for a root system  $R$  is called a *simple* root.

1. Let  $\alpha$  be such a simple root. Show that the reflection  $s_\alpha$  permutes the positive roots other than  $\alpha$ . (2 points)

**Hint** Consider the coefficient of  $\alpha$  versus the coefficients of the other simple roots before and after applying  $s_\alpha$ .

2. Let  $\rho = \frac{1}{2} \sum_{\beta \in R^+} \beta$  be the half-sum of positive roots. Show that  $s_\alpha(\rho) = \rho - \alpha$  for  $\alpha$  a simple root. (1 point)

**Problem 40.** Let  $k$  be of characteristic zero. Assuming the existence of bases for root systems, show that there exists no semisimple Lie algebras of dimension 4 or 5. (2 points)

**Problem 41.** Assume  $R \subset V$  is a root system of rank 2, with basis  $\alpha, \beta$ . Consider the space  $V_{\mathbb{R}}$ , with the hyperplanes  $H_{s_\alpha}$  given by the reflections  $s_\alpha$ , with  $\alpha \in R$ . Draw a picture.

Let  $\mathcal{A}$  be the set of connected components of  $V_{\mathbb{R}} \setminus \{H_{s_{\alpha}} \mid \alpha \in R\}$ . Show that the Weyl group  $W = W(R, V)$  acts transitively on  $\mathcal{A}$ . Deduce from this a bijection between  $W$  and  $\mathcal{A}$  by picking an arbitrary element in  $\mathcal{A}$  corresponding to  $e \in W$ . Illustrate this in a picture! (3 points)

We will give occasionally some optional problems for additional credit. These can be solved using computer algebra, in particular using Sage. You can use the online version at <https://cocalc.com>.

The documentation for root systems can be found at [http://doc.sagemath.org/html/en/reference/combinat/sage/combinat/root\\_system/\\_\\_init\\_\\_.html](http://doc.sagemath.org/html/en/reference/combinat/sage/combinat/root_system/__init__.html).

**Optional problem 1.** Fill in the following table. We use  $\alpha_1, \dots, \alpha_n$  the simple roots in the usual construction of the root system as implemented in Sage, so that  $\beta \in R$  is  $\sum_{i=1}^n k_i \alpha_i$ , and the height  $\text{ht}(\beta)$  of a root  $\beta$  is the sum of the coefficients in the simple roots. (1 point)

Dynkin type	$\#\{\beta \in R^+ \mid k_1 \geq 1\}$	$\#\{\beta \in R^+ \mid \text{ht}(\beta) \geq 10\}$
A <sub>10</sub>		
F <sub>4</sub>		
E <sub>6</sub>		
E <sub>7</sub>		
E <sub>8</sub>		