Representation Theory I — Prof. Dr. Catharina Stroppel
Bonn, summer term 2019 — Dr. Pieter Belmans

Sheet 6

Solutions to be handed in before class on Friday (!) May 17

Problem 30. Show that the set of diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$ forms a Cartan subalgebra. (3 points)

Problem 31. Let $\mathfrak{h}$ be a Lie subalgebra of a complex Lie algebra $\mathfrak{g}$ such that $\operatorname{ad}_{\mathfrak{g}} h$ is semisimple for all $h \in \mathfrak{h}$. Show that $\mathfrak{h}$ is abelian. (4 points)

In other words, one of the conditions in our definition of a Cartan subalgebra for a semisimple Lie algebra is redundant.

Problem 32. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ a subalgebra. The *normaliser* of $\mathfrak{h}$ is

$$\mathrm{N}_{\mathfrak{g}}(\mathfrak{h}) \coloneqq \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subseteq \mathfrak{h}\}. \tag{38}$$

1. Show that $\mathrm{N}_{\mathfrak{g}}(\mathfrak{h})$ is a Lie algebra. (2 points)
2. Show that $\mathfrak{h} \subseteq \mathrm{N}_{\mathfrak{g}}(\mathfrak{h})$ is an ideal. (2 points)
3. Assume that $\mathfrak{g}$ is nilpotent, and let the inclusion $\mathfrak{h} \subseteq \mathfrak{g}$ be strict. Then the inclusion $\mathfrak{h} \subset \mathrm{N}_{\mathfrak{g}}(\mathfrak{h})$ is also strict. (2 points)

In other words, for the general definition of Cartan subalgebra (i.e. nilpotent and self-normalising) we obtain that nilpotent Lie algebras are equal to their Cartan subalgebras.

Problem 33. Show by means of a counterexample that the dimension of the Cartan subalgebra in a semisimple Lie algebra can be strictly less than the maximal dimension of an abelian subalgebra (so not necessarily satisfying the other conditions of a Cartan subalgebra). (3 points)