Prof. Dr. Catharina Stroppel
Dr. Pieter Belmans

Sheet 5

Solutions to be handed in before class on Wednesday May 8

Problem 23. Let $\mathfrak{g}$ be a semisimple complex Lie algebra.

1. Show that every finite-dimensional $\mathrm{U}(\mathfrak{g})$-module is semisimple. (0.5 point)
2. Give an example of a $\mathrm{U}(\mathfrak{g})$-module which is not semisimple. (0.5 point)

Problem 24. Give an example of a Lie algebra $\mathfrak{g}$ and a finite-dimensional representation of $\mathfrak{g}$ that is not semisimple. (2 points)

Problem 25. In the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2(k))$, express the elements ehf, $e^2h^2f^2$, hf^3 and h^3f in the standard Poincaré–Birkhoff–Witt basis given by $\{f^lh^me^n\}_{l,m,n\in\mathbb{N}}$. (3 points)

Problem 26 (Clebsch–Gordan)**.** Let k be algebraically closed of characteristic zero. Decompose the $\mathfrak{sl}_2(k)$-representation

$$V(n)\otimes_k V(m) \tag{36}$$

into irreducible representations, where $V(n)$ is the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2(k)$. (4 points)

Hint If V is a representation, use the *formal character* of V

$$\mathrm{ch}(V) \coloneqq \sum_{n\in\mathbb{Z}} \dim_k V_n t^n, \tag{37}$$

where V_n denotes the weight space of weight n, and notice that $\mathrm{ch}(V\oplus W) = \mathrm{ch}(V)\oplus\mathrm{ch}(W)$, and that the formal character determines the representation up to isomorphism. Then show that $\mathrm{ch}(V\otimes_k W) = \mathrm{ch}(V)\,\mathrm{ch}(W)$.

Problem 27. Show that if A is a filtered algebra such that the associated graded is a domain, then A is itself a domain. From this one can conclude that $\mathrm{U}(\mathfrak{g})$ is a domain. (1 point)

Problem 28. Show that if A is a filtered algebra such that the associated graded is left (resp. right) noetherian, then A is itself left (resp. right) noetherian. From this conclude that $\mathrm{U}(\mathfrak{g})$ is left (resp. right) noetherian, provided that $\mathfrak{g}$ is finite-dimensional. (2 points)

Hint You can define left (resp. right) noetherian as every left (resp. right) ideal being finitely generated. Consider the homogeneous left (resp. right) ideal generated by the leading term of a given left (resp. right) ideal of A.

Problem 29. Let $\mathfrak{g}$ be a Lie algebra. Show that there are unique algebra homomorphisms

1. $\Delta\colon \mathrm{U}(\mathfrak{g})\to\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g})$
2. $\eta\colon \mathrm{U}(\mathfrak{g})\to k$

3. $S\colon \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})^{\mathrm{op}}$

for which

1. $\Delta(X) = X \otimes 1 + 1 \otimes X$
2. $\eta(X) = 0$
3. $S(X) = -X$

for all $X \in \mathfrak{g} \subseteq \mathrm{U}(\mathfrak{g})$. (3 points)