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Sheet 4

Solutions to be handed in before class on Friday (!) May 3

Problem 19 (Jordan–Chevalley decomposition)**.** Let k be an algebraically closed field, and V a finite-dimensional vector space over it. We call an element $\varphi \in \mathrm{End}_k(V)$ *semisimple* if the roots of its minimal polynomial are all distinct, or equivalently if φ is diagonalisable.

1. If φ and ψ are commuting semisimple endomorphisms, then so are $\varphi + \psi$ and $\varphi - \psi$. (1 point)
2. For $\varphi \in \mathrm{End}_k(V)$ there exists unique (!) φ_{s} and φ_{n} such that $\varphi = \varphi_{\mathrm{s}} + \varphi_{\mathrm{n}}$, with φ_{s} semisimple, φ_{n} nilpotent and φ_{s} and φ_{n} commute. (3 points)

   Hint Decompose V into eigenspaces V_i on which φ has characteristic polynomial $(t-a_i)^{m_i}$. We wish to construct a polynomial $f(t)$ which you will evaluate on φ, such that this polynomial satisfies the congruences

   $$\begin{aligned} f(t) &\equiv a_i \bmod (t-a_i)^{m_i} \\ f(t) &\equiv 0 \bmod t. \end{aligned} \tag{32}$$

   Which properties does $f(\varphi)$ have, and how to continue from here?

Problem 20. Let D be a (not necessarily inner) derivation of a Lie algebra $\mathfrak{g}$. Construct a semidirect product $\mathfrak{g} \rtimes_f k$ (where k is the one-dimensional Lie algebra), such that D is the restriction to $\mathfrak{g}$ of an inner derivation on the semidirect product. (2 points)

Problem 21. Let $\mathfrak{g}$ be a Lie algebra. Let U, V, W be $\mathfrak{g}$-modules.

1. Give an isomorphism of $\mathfrak{g}$-modules

   $$\mathrm{Hom}_k(U \otimes_k V, W) \xrightarrow{\sim} \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)). \tag{33}$$

   (2 points)
2. Show that taking $\mathfrak{g}$-invariants gives an isomorphism of vector spaces

   $$\mathrm{Hom}_{\mathfrak{g}}(U \otimes_k V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(U, \mathrm{Hom}_k(V, W)). \tag{34}$$

   (2 points)

Problem 22. A *Borel subalgebra* of a Lie algebra is a maximal solvable Lie subalgebra. Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

1. Show that the set $\mathfrak{b}$ of upper triangular matrices forms a Borel subalgebra. (2 points)

2. Show that given any other Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{g}$ there exists an element $g \in \mathrm{SL}_2(\mathbb{C})$ such that $g\mathfrak{b}'g^{-1} = \mathfrak{b}$. (2 points)

3. Let $\mathfrak{b}$ now denote any Borel subalgebra of $\mathfrak{g}$. Consider the natural representation of $\mathfrak{b}$ on $\mathbb{C}^2$. Show that there exists a unique one-dimensional subrepresentation $U_{\mathfrak{b}}$ of $\mathbb{C}^2$, such that the assignment $\mathfrak{b} \mapsto U_{\mathfrak{b}}$ defines a bijection

$$\{\text{Borel subalgebras of } \mathfrak{g}\} \leftrightarrow \mathbb{P}^1(\mathbb{C}). \tag{35}$$

(2 points)