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Sheet 3

Solutions to be handed in before class on Wednesday April 24.

Problem 15.

1. Show that nilpotent implies solvable. (1 point)
2. Show that solvable does not imply nilpotent. (2 points)

Problem 16 (Invariance lemma)**.** Let V be a finite-dimensional representation of a complex Lie algebra $\mathfrak{g}$ and let $I \subseteq \mathfrak{g}$ be an ideal. Let $\lambda \in I^*$ be a weight. Then the space λ-weight space

$$V_\lambda \coloneqq \{v \in V \mid \forall x \in I\colon xv = \lambda(x)v\} \tag{28}$$

considered as a representation of I is a subrepresentation of $\mathfrak{g}$, i.e. we have that $\mathfrak{g}(V_\lambda) \subseteq V_\lambda$. (4 points)

Hint Consider the action of $[x, y]$ on the subspace spanned by $v, yv, \ldots, y^n v$.

Problem 17. Throughout we let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic zero.

1. Prove that if $\mathfrak{g}$ is nilpotent, then the Killing form is identically zero. (1 point)
2. Prove that $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ lies in the radical of the Killing form. (2 points)
3. Show that the radical of $\mathfrak{g}$ is the orthogonal of $[\mathfrak{g}, \mathfrak{g}]$ with respect to the Killing form, i.e. that

$$\operatorname{rad}\mathfrak{g} = \{X \in \mathfrak{g} \mid \forall Y \in [\mathfrak{g}, \mathfrak{g}]\colon \kappa(X, Y) = 0\}. \tag{29}$$

(1 point)

4. Show that $[\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$ is a nilpotent Lie algebra. (1 point)
5. Let $\mathfrak{g}$ be the non-abelian two-dimensional Lie algebra you constructed in problem 1. Show that $\mathfrak{g}$ has non-trivial Killing form. (1 point)

Problem 18. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic zero.

1. Let $x \in \mathfrak{g}$ be an ad-nilpotent element. Show that

$$\exp(\operatorname{ad} x) = \sum_{n=0}^{+\infty} \frac{(\operatorname{ad} x)^n}{n!} \in \operatorname{End}_k(\mathfrak{g}) \tag{30}$$

is a Lie algebra automorphism of $\mathfrak{g}$, i.e. that $\exp(\operatorname{ad} x) \in \operatorname{Aut}_k(\mathfrak{g})$. (1 point)

2. The subgroup of inner automorphisms is the subgroup of $\mathrm{Aut}_k(\mathfrak{g})$ generated by all the elements $\exp(\operatorname{ad} x)$ with $x \in \mathfrak{g}$ ad-nilpotent. Show that this subgroup is a normal subgroup. (1 point)

3. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, and use the standard basis $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Given an explicit description of

$$s \coloneqq \exp(\operatorname{ad} e) \circ \exp(\operatorname{ad}(-f)) \circ \exp(\operatorname{ad} e) \in \mathrm{Aut}_k(\mathfrak{g}). \tag{31}$$

What is the order of this element? (1 point)