Representation Theory I — Prof. Dr. Catharina Stroppel
Bonn, summer term 2019 — Dr. Pieter Belmans

Sheet 2

Solutions to be handed in before class on Wednesday April 17.

Problem 10. True or false:

1. The Lie algebra $\mathfrak{so}_2(\mathbb{C})$ is an abelian Lie algebra. (1 point)
2. The Lie algebra $\mathfrak{sl}_2(k)$ is a nilpotent Lie algebra if the characteristic of k is different from 2. (1 point)
3. The Lie algebra $\mathfrak{sl}_2(k)$ is a nilpotent Lie algebra if the characteristic of k is equal to 2. (1 point)

Problem 11. Show that the class of nilpotent Lie algebras is closed under taking quotients and subalgebras. Show that the class of solvable Lie algebras is closed under taking quotients, subalgebras *and* extensions. (3 points)

Problem 12. Show that the class of nilpotent Lie algebras is *not* closed under extensions. In other words, find a short exact sequence of Lie algebras

$$0 \to \mathfrak{g}_1 \to \mathfrak{g} \to \mathfrak{g}_2 \to 0 \tag{18}$$

where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are nilpotent, but $\mathfrak{g}$ isn't. (2 points)

Problem 13. Show the following "exceptional isomorphisms":

1. $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C})$. (1 point)
2. $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C})$. (1 point)
3. $\mathfrak{sl}_4(\mathbb{C}) \cong \mathfrak{so}_6(\mathbb{C})$. (2 points)

   Hint Let V be a 4-dimensional vector space. Then $W := \bigwedge^2 V$ is 6-dimensional, and

   $$W \times W \to \bigwedge^4 V : (x, y) \mapsto x \wedge y \tag{19}$$

   defines a symmetric non-degenerate bilinear form β by choosing an isomorphism $\bigwedge^4 V \cong \mathbb{C}$. Then the Lie algebra $\mathfrak{g}_\beta$ (which is a subalgebra of $\mathfrak{gl}(W)$) defined in problem 8 was shown to be isomorphic to $\mathfrak{so}_6(\mathbb{C})$.

   On the other hand one constructs a homomorphism

   $$\mathfrak{sl}(V) \to \mathfrak{gl}(W) : X \mapsto (\rho(X) : v_1 \wedge v_2 \mapsto X(v_1) \wedge v_2 + v_1 \wedge X(v_2)). \tag{20}$$

Problem 14. Let $A = (a_{i,j})_{i,j=1}^n$ be the Cartan matrix in type A_n, i.e.

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & & & 0 \\ 0 & -1 & 2 & -1 & & & 0 \\ 0 & 0 & -1 & 2 & & & 0 \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & -1 \\ 0 & & & & \dots & -1 & 2 \end{pmatrix} \tag{21}$$

Let $\mathfrak{g}$ be the complex Lie algebra generated by the $3n$ elements $\{e_i, f_i, h_i\}_{i=1}^n$ subject to the relations

$$[h_i, h_j] = 0 \tag{22}$$
$$[e_i, f_j] = \delta_{i,j} h_i \tag{23}$$
$$[h_i, e_j] = a_{j,i} e_j \tag{24}$$
$$[h_i, f_j] = -a_{j,i} f_j \tag{25}$$
$$(\operatorname{ad} e_i)^{-a_{j,i}+1}(e_j) = 0 \qquad (i \neq j) \tag{26}$$
$$(\operatorname{ad} f_i)^{-a_{j,i}+1}(f_j) = 0 \qquad (i \neq j) \tag{27}$$

1. Show that there is a surjective homomorphism $\mathfrak{g} \to \mathfrak{sl}_{n+1}(\mathbb{C})$ of Lie algebras. (2 points)
2. Show that it is an isomorphism[1] for $n = 1$. (2 points)

[1]It is in fact an isomorphism for $n \geq 1$, but we lack the tools to prove this for the time being.