Sheet 13

These exercises will not be graded, and solutions will be discussed in the tutoring session in the last week

Problem 62. Show the following properties.

1. If $A = k[x]$, then any k-derivation d of A is of the form $d(g) = f(x)\frac{\mathrm{d}}{\mathrm{d}x}g$ for some $f(x) \in k[x]$. In other words, the space of derivations is free of rank 1 as an A-module.

2. If $A = k[x_1, \ldots, x_n]$, then the derivations $\partial/\partial x_i$ form a free basis of the space of k-derivations of A.

3. The k-algebra homomorphisms $\phi\colon A \to A[\epsilon]/(\epsilon^2)$ are in bijective correspondence with the set of k-derivations on A. The derivation corresponding to ϕ is d where $\phi(a) = a + d(a)\epsilon$.

Problem 63. Let $X = \mathbb{V}(I) \subseteq \mathbb{A}_k^n$ be an affine algebraic variety given by the vanishing set $\mathbb{V}(I)$ of the (ideal generated by) the polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$. The *tangent space* at $x \in X$ is defined as $\mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$, where $\mathfrak{m}_x$ is the maximal ideal of $k[x_1, \ldots, x_n]$ which defines x.

1. Assume that $x = 0 \in X$. Show that $\mathfrak{m}_x/\mathfrak{m}_x^2$ is isomorphic to $(x_1, \ldots, x_n)/(I + (x_1, \ldots, x_n)^2)$, in other words, the tangent space of X at $0 \in X$ is identified with the degree 1 parts of the f_i's.

2. Show that the tangent space at $(0, 0)$ for the affine curve $x^2 - y^3$ is 2-dimensional.

3. Show that the tangent space at $(0, 0)$ for the affine curve $x^2 - y^2$ is 2-dimensional.

4. Show that the tangent space at $(0, 0, 0)$ for the affine surface $x - y^4 - z^7$ is 2-dimensional, spanned by $\partial/\partial y$ and $\partial/\partial z$.

Problem 64. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Denote by χ_λ the central character of the Verma module $M(\lambda)$, where $\lambda \in \mathfrak{h}^*$.

Let $\lambda, \mu \in \mathfrak{h}^*$. Show that if $\lambda \sim \mu$, then $\chi_\lambda = \chi_\mu$.