Representation Theory I — Prof. Dr. Catharina Stroppel
Bonn, summer term 2019 — Dr. Pieter Belmans

Sheet 12

Solutions to be handed in before class on Wednesday July 3

Problem 56. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the standard choice of Cartan subalgebra $\mathfrak{h}$ and positive roots $R^+ = \{\alpha\}$. Determine explicitly for which weights $\lambda \in \mathfrak{h}^*$ the representation $L(\lambda)$ is finite-dimensional, and compute its character and dimension using Weyl's formulas. (2 points)

Problem 57. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ with the standard choice of Cartan subalgebra. Let $\alpha = \epsilon_1 - \epsilon_2$ and $\beta = \epsilon_2 - \epsilon_3$, where ϵ_i picks out the ith entry of the diagonal. Determine the character and dimension of $L = L(\alpha + \beta)$ using Weyl's formulas, and give an explicit isomorphism of representations between L and the adjoint representation. (2 points)

Problem 58. Show that

$$\begin{aligned} \mathrm{char}(M \oplus N) &= \mathrm{char}(M) + \mathrm{char}(N) \\ \mathrm{char}(M \otimes N) &= \mathrm{char}(M)\,\mathrm{char}(N) \end{aligned} \tag{43}$$

for representations M, N with a weight space decomposition, such that the character is defined. (2 points)

Problem 59. Show that $\widehat{\mathbb{Z}\mathfrak{h}^*}$ is a ring with pointwise addition and multiplication. (2 points)

Problem 60. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, with the standard Cartan subalgebra.

1. Show that $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ is equal to the sum of the fundamental weights ω_i, where the fundamental weights ω_i are

$$\omega_i \coloneqq (\epsilon_1 + \ldots + \epsilon_i) - \frac{i}{n}\sum_{j=1}^{n} \epsilon_j \tag{44}$$

   and the positive roots are $\epsilon_i - \epsilon_j$ for $1 \leq i < j \leq n$ (with ϵ_i the projection on the ith entry of the diagonal).

   Moreover show that the weight lattice X is equal to $\bigoplus_{i=1}^{n} \mathbb{Z}\omega_i$ (and in fact X^+ is the cone given by those sums of ω_i's with positive coefficients). (2 point)

Using highest weight theory and Weyl's formulas, show that

2. $\bigwedge^k V$ is irreducible, and isomorphic to $L(w_i)$, where V is the standard n-dimensional representation, and $w_i = \sum_{j=1}^{i} \epsilon_j - \frac{i}{n+1}\sum_{j=1}^{n+1} \epsilon_j$; (2 points)

3. every finite-dimensional representation of $\mathfrak{g}$ is a direct summand of some finite tensor product of exterior powers of V. (2 points)

   Hint Show using highest weight theory that it is a quotient.

Problem 61. Show that a submodule of a Verma module cannot be finite-dimensional. (2 points)

Hint Prove that a Verma module is torsion-free as $\mathrm{U}(\mathfrak{n}^-)$-module.

Optional problem 5. Let $\mathfrak{g} = \mathfrak{sl}_2$.

1. The Casimir element C is central in $\mathrm{U}(\mathfrak{g})$. Compute its projection to $\mathrm{S}(\mathfrak{h}) = \mathrm{U}(\mathfrak{h})$ via the decomposition $\mathrm{U}(\mathfrak{g}) = (\mathfrak{n}^-\mathrm{U}(\mathfrak{g}) + \mathrm{U}(\mathfrak{g})\mathfrak{n}^+) \oplus \mathrm{U}(\mathfrak{h})$.(2 points)

2. Show that the center $\mathrm{Z}(\mathrm{U}(\mathfrak{g}))$ of the universal enveloping algebra is generated by C. (1 point)

3. Show that the projection introduced above defines an isomorphism of algebras $\mathrm{Z}(\mathrm{U}(\mathfrak{g})) \to \mathrm{S}(\mathfrak{h})^W$, where the Weyl group W acts on $\mathrm{S}(\mathfrak{h})$ by the dot action induced from the dot action on $\mathfrak{h}^*$ under the identification of $\mathrm{S}(\mathfrak{h})$ with the polynomial maps on $\mathfrak{h}^*$. In other words, if $f\colon \mathfrak{h}^* \to \mathbb{C}$ is a polynomial map, then $(w \cdot f)(x) = f(w^{-1} \cdot x)$ for $w \in W$ and $x \in \mathfrak{h}^*$. (2 points)