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Sheet 10

Solutions to be handed in before class on Wednesday June 19

Problem 45. Let k be a field of characteristic 0, and V a vector space over k.

1. Let $R \subseteq V$ be an irreducible root system. Show that at most 2 root lengths can appear. (2 points)
2. Show that without irreducibility an arbitrary number of root lengths is possible. (1 point)

Problem 46. Show that all roots of a given length are conjugate under the Weyl group action. (2 points)

Problem 47. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, with Cartan subalgebra $\mathfrak{h}$ and root system $R = R(\mathfrak{g}, \mathfrak{h})$. Choose a basis π of the root system, and denote R^+ the corresponding system of positive roots.

Let $\mathfrak{n}^+ := \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- := \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$, and consider the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Choose for each $\alpha \in R$ a root vector $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. Show that

1. $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent. (1 point)
2. $\mathfrak{n}^+$ is generated as a Lie algebra by the root vectors x_α, for $\alpha \in \pi$. (1 point)
3. $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$ is a maximal solvable Lie subalgebra of $\mathfrak{g}$. (1 point)
4. Let L be either $\mathfrak{b}$, $\mathfrak{n}^+$ or $\mathfrak{n}^-$. Then $\mathrm{U}(\mathfrak{g})$ is a free left (resp. right) $\mathrm{U}(L)$-module via left (resp. right) multiplication. Describe a basis of $\mathrm{U}(\mathfrak{g})$ as a left $\mathrm{U}(L)$-module. (2 points)

Problem 48. Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0. Let V be a representation of $\mathfrak{g}$. Show that $\bigwedge^i V$ and $\mathrm{Sym}^i V$ (also denoted $\mathrm{S}(V)_i$ in class, where $\mathrm{S}(V) = \bigoplus_{i \geq 0} \mathrm{S}(V)_i$ is the symmetric algebra) can be made into a representation which is a quotient (resp. subrepresentation) of $V^{\otimes i}$, for all $i \geq 0$. (2 points)

Problem 49. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Fix the standard Cartan subalgebra $\mathfrak{h}$ given by all diagonal matrices in $\mathfrak{sl}_2(\mathbb{C})$. Let V be a finite-dimensional representation of $\mathfrak{g}$. Then V decomposes into a direct sum of irreducible representations, where the highest weights and multiplicities are uniquely determined. This multiset is the highest weight decomposition.

Let $V = \mathbb{C}^2$ be the standard representation. Compute the highest weight decomposition of

1. $\bigwedge^3 V$ (0.5 point)
2. $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$ (0.5 point)
3. $\mathrm{Sym}^6(\mathrm{Sym}^2 V)$ (0.5 point)

4. $\mathrm{Sym}^8(\mathrm{Sym}^2 V)$ (0.5 point)

You are allowed to use Sage for this exercise, as suggested below.

Problem 50. Let $\phi\colon \mathfrak{b} \to \mathfrak{g}$ be a homomorphism of Lie algebras. Show that ind^ϕ is left adjoint to res^ϕ, but not right adjoint. (2 points)

The following exercise suggests an alternative approach to problem 49, not for extra credit, but to give you the choice between doing it by hand and doing it by computer.

Optional problem 3. Compute the decompositions in problem 49 using the functionality provided by `WeylCharacterRing` in Sage. You can make working with the weights easier passing it the option `style='coroots'`.

Optional problem 4. Let $\mathfrak{g}$ be the complex simple Lie algebra of type G_2. Let V be the adjoint representation. Compute the highest weight decomposition of

1. $\bigwedge^3 V$ (0.5 point)
2. $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$ (0.5 point)
3. $\mathrm{Sym}^6(\mathrm{Sym}^2 V)$ (0.5 point)
4. $\mathrm{Sym}^8(\mathrm{Sym}^2 V)$ (0.5 point)