Representation Theory I — Prof. Dr. Catharina Stroppel
Bonn, summer term 2019 — Dr. Pieter Belmans

Sheet 1

Solutions to be handed in before class on Wednesday April 10.

Problem 6. Define

$$\mathfrak{sl}_n(\mathbb{C}) \coloneqq \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid \operatorname{tr}(A) = 0\}, \tag{11}$$

together with the bracket

$$[X, Y] \coloneqq XY - YX \tag{12}$$

for $X, Y \in \mathfrak{sl}_n(\mathbb{C})$.

1. Show that this defines a Lie algebra. (1 point)
2. Determine its dimension. (1 point)

Problem 7. Define

$$\mathfrak{so}_n(\mathbb{C}) \coloneqq \left\{A \in \mathfrak{gl}_n(\mathbb{C}) \mid A + A^\ddagger = 0\right\} \tag{13}$$

resp.

$$\mathfrak{sp}_{2n}(\mathbb{C}) \coloneqq \left\{A \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid A = \left(\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right), A_2 - A_2^\ddagger = A_3 - A_3^\ddagger = A_1 + A_4^\ddagger = 0\right\}. \tag{14}$$

where $A^\ddagger = (a_{i,j}^\ddagger)_{i,j=1}^n$ with $a_{i,j}^\ddagger = a_{n-j+1,n-i+1}$. The Lie bracket is in each case given by the commutator

$$[X, Y] \coloneqq XY - YX \tag{15}$$

for X, Y in the respective vector spaces.

1. Show that these define Lie algebras. (2 points)
2. Determine their dimensions. (2 points)

Problem 8. Let β be a symmetric (resp. skew-symmetric) non-degenerate bilinear form on $\mathbb{C}^n$.

1. Show that

   $$\mathfrak{g}_\beta \coloneqq \{\varphi \in \operatorname{End}_{\mathbb{C}}(\mathbb{C}^n) \mid \forall v, w \in V \colon \beta(\varphi(v), w) + \beta(v, \varphi(w)) = 0\} \tag{16}$$

   defines a Lie algebra, with Lie bracket $[\varphi, \psi] \coloneqq \varphi \circ \psi - \psi \circ \varphi$. (2 points)
2. Show that if β is symmetric, then $\mathfrak{g}_\beta \cong \mathfrak{so}_n(\mathbb{C})$. (2 points)
3. Show that if β is skew-symmetric (so that n is even), then $\mathfrak{g}_\beta \cong \mathfrak{sp}_n(\mathbb{C})$. (2 points)

Problem 9. Define the cross product on $\mathbb{R}^3$ as

$$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 : \left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) \mapsto v \times w := \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}. \tag{17}$$

1. Show that $(\mathbb{R}^3, \times)$ is a real Lie algebra. (2 points)
2. Show that $(\mathbb{R}^3, \times)$ is isomorphic to $\mathfrak{so}_3(\mathbb{R})$, where $\mathfrak{so}_3(\mathbb{R})$ is defined as in (13), replacing $\mathbb{C}$ by $\mathbb{R}$. (2 points)