Representation Theory II
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Tutorial exercises

These exercises are to be done in class. By no means are you expected to solve all of them during class.

Problem 1 (The Proj construction)**.** Let A be a finitely generated $\mathbb{N}$-graded $\mathbb{C}$-algebra such that $A_0 = \mathbb{C}$. Define $\operatorname{Proj} A$ as the topological space on the set of homogeneous prime ideals of A not containing the irrelevant ideal $A_+ \coloneqq \bigoplus_{i\geq 1} A_i$, with the Zariski topology defined by the closed subsets of the form $\mathrm{V}(I)$ where I is a homogeneous ideal and $\mathrm{V}(I)$ is the set of homogeneous prime ideals containing I.

Show that the distinguished open sets $\mathrm{D}(f) \coloneqq \operatorname{Proj} A \setminus \mathrm{V}(f)$ for $f \in A_+$ form a basis for the topology, and that $\mathrm{D}(f) \cap \mathrm{D}(g) = \mathrm{D}(f \cap g)$.

In this way A is the *homogeneous coordinate ring* for the projective variety $\operatorname{Proj} A$. We will see that if we write A as the quotient of a polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$, so that in particular it is generated by the (classes of) the elements $x_0, \ldots, x_n$ of degree 1, we have realised $\operatorname{Proj} A$ as a closed subvariety of $\mathbb{P}^n_{\mathbb{C}}$.

We say that f vanishes on a point $\mathfrak{p} \in \operatorname{Proj} A$ if f is zero in $A/\mathfrak{p}$.

Problem 2. Let A be as before.

1. Let I be a homogeneous ideal, and f a homogeneous element of A. Show that $f = 0$ on $\mathrm{V}(I)$ if and only if $f^n \in I$ for some n
2. Let Z be a subset of $\operatorname{Proj} A$. Show that $\mathrm{V}(\mathrm{I}(Z)) = \operatorname{cl} Z$, where $\mathrm{I}(Z)$ is the homogeneous ideal of A generated by the homogeneous polynomials that vanish on Z.

Problem 3. Let A be as before. Show that the following are equivalent for a homogeneous ideal I:

1. $\mathrm{V}(I) = \emptyset$.
2. For every set of homogeneous generators $\{f_1, \ldots, f_n\}$ of I we have that $\bigcup_{i=1}^n \mathrm{D}(f_i) = \operatorname{Proj} A$.
3. $A_+ \subseteq \operatorname{rad} I$,

This explains why A_+ is called irrelevant.

Problem 4. Let $X = \bigcup_{i\in I} U_i$ be an open cover of a topological space X, such that $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. Show that

1. if U_i is connected for all $i \in I$, then so is X;
2. if U_i is irreducible for all $i \in I$, then so is X.

Conclude that the Grassmannian $\mathrm{Gr}(d, n)$ is connected, irreducible and of dimension $d(n - d)$.

Problem 5 (Veronese embedding)**.** Let A be a finitely graded $\mathbb{N}$-graded $\mathbb{C}$-algebra. Let $d \geq 1$. The *dth Veronese subalgebra* of A is the algebra $A^{(d)} := \bigoplus A_{n \geq 0} A_{dn}$. If so desired, we can rescale the degree so that the part in degree d sits in degree 1, etc.

1. Show that $\operatorname{Proj} A \cong \operatorname{Proj} A^{(d)}$.

2. Assume that A is generated by homogeneous elements $f_1, \ldots, f_n$. Show that we can find a d such that $A^{(d)}$ is generated in degree 1. This way we can realise $\operatorname{Proj} A$ inside $\mathbb{P}^m_{\mathbb{C}}$, for some m.

3. Explain how we can embed $\mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$ as a closed subvariety of $\mathbb{P}^N_{\mathbb{C}}$. Determine N in terms of n and d.

4. Determine the ideal for the dth Veronese embedding of $\mathbb{P}^1_{\mathbb{C}}$ for $d = 2$ and $d = 3$.

Problem 6. Let $v \in \bigwedge^2 V$ be a non-zero element. Then v is a pure wedge (i.e. $v = x \wedge y$ for some $x, y \in V$) if and only if $v \wedge v = 0$ in $\bigwedge^4 V$.

Hint One possibility is to do an induction on the dimension.