Representation Theory II — Prof. Dr. Catharina Stroppel
Bonn, winter term 2019–2020 — Dr. Pieter Belmans

Sheet 8

Solutions to be handed in before class on Friday December 6.

Problem 35. Show that the dimension of the Schubert variety is the maximum of the dimensions of the Schubert cells contained in it.

(1 point)

Problem 36. Using problems 12 and 13, describe the Schubert variety for $s_1 s_3 s_2$ as the intersection of the Grassmannian $\mathrm{Gr}(2,4)$ (which is a quadric in $\mathbb{P}^5$) with the tangent space at the identity element. Conclude that this Schubert variety is singular.

(3 points)

Problem 37.

1. Show that a Schubert variety is irreducible.
2. Show that $X = \mathrm{Gr}(d,n)$ and $\mathrm{Fl}_{\mathbf{d}}$ have a stratification

$$X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_s = \emptyset \tag{15}$$

where X_i is closed, and $X_i \setminus X_{i+1}$ is the disjoint union of affine spaces (of possibly different dimension).

(2 points)

These properties show that the cohomology ring (which is an associative and commutative graded ring in degrees $0, \ldots, d$ where d is the dimension) of these flag varieties has as an additive basis the classes of Schubert varieties, with the class of a Schubert variety of codimension c in degree $2c$. In particular we also have from the general formalism that:

1. there is a unique class in the top degree (why is this the case in our setting?)
2. if two subvarieties of complementary dimension meet transversally in precisely t points, then the product of their classes is t times the unique class in top degree.

Problem 38. We want to understand the intersection of a Schubert variety Ω_λ and an opposite Schubert variety $\tilde{\Omega}_\mu$ in $\mathrm{Gr}(r,m)$. We will denote $F_\bullet$ the standard flag, and $\tilde{F}_\bullet$ the opposite flag. We denote

$$\begin{aligned} A_i &\coloneqq F_{n+i-\lambda_i} \\ B_i &\coloneqq \tilde{F}_{n+i-\mu_i} \\ C_i &\coloneqq A_i \cap B_{r+1-i} \end{aligned} \tag{16}$$

for $1 \leq i \leq r$, where $n \coloneqq m - r$.

1. Show that C_i is spanned by those vectors e_j for which

$$i + \mu_{r+1-i} \leq j \leq n + i - \lambda_i, \tag{17}$$

so that

$$\dim C_i = n + 1 - \lambda_i - \mu_{r+1-i} \tag{18}$$

if the right-hand side is nonnegative, and $C_i = 0$ otherwise.

(2 points)

2. If Ω_λ and $\widetilde{\Omega}_\mu$ are not disjoint, then

$$\lambda_i + \mu_{r+1-i} \leq n \tag{19}$$

for all $1 \leq i \leq r$.

(2 points)

3. Assume that $|\lambda|+|\mu| = r(m-r)$. Show that the intersection of Ω_λ and $\tilde{\Omega}_\mu$ is

$$\Omega_\lambda \cap \tilde{\Omega}_\nu = \begin{cases} \{\mathrm{pt}\} & \text{if } \lambda_i + \mu_{r+1-i} = n \text{ for all } 1 \leq i \leq r \\ \emptyset & \text{if } \lambda_i + \mu_{r+1-i} > n \text{ for some } i \end{cases} \tag{20}$$

In the language of cohomology rings (or Chow rings), this means for the Schubert classes $\sigma_\lambda = [\Omega_\lambda]$ and $\sigma_\mu = [\tilde{\Omega}_\mu]$ that

$$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} 1 & \lambda \text{ and } \mu \text{ dual} \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

for λ and μ as above, and dual meaning that $\mu_i = n - \lambda_{r+1-i} - i$.

(3 points)

Problem 39. Compute the ring structure of the cohomology ring of $\mathrm{Gr}(2,4)$, using Pieri's rule, which for a Grassmannian $\mathrm{Gr}(d,n)$ says that

$$\sigma_\lambda \cdot \sigma_k = \sum_{\lambda'} \sigma_{\lambda'} \tag{22}$$

where λ' runs over the Young diagrams obtained from λ obtained by adding k boxes, but never more than one in a single column. Here σ_k is the special Schubert variety, associated to $\lambda = (k)$.

Describe all non-zero products of Schubert classes.

(3 points)