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Sheet 7

Solutions to be handed in before class on Friday November 29.

Problem 32. Consider a Coxeter group (W, S).

1. Show that the Bruhat order on W is a *directed* poset, i.e. for all $x, y \in W$ there exists a $z \in W$ such that $x \leq z$ and $y \leq z$.
2. If W is finite, show that there exists a longest element w_0, and describe it for $W = \mathrm{S}_n$ and S the simple transpositions.
3. Define the *interval* $[x, y]$ for $x, y \in W$ as
$$[x, y] \coloneqq \{z \mid x \leq z \leq y\}. \tag{12}$$
Show that $[x, y]$ is finite, by giving an explicit upper bound.
4. Show that there exists a unique dense B-orbit in G/B, where $G = \mathrm{GL}_n(\mathbb{C})$.
5. Show that this orbit is open.

(8 points)

Problem 33. Let G be an algebraic group, and $B \subseteq G$ a closed subgroup (feel free to assume $G = \mathrm{GL}_n(\mathbb{C})$ and B the standard Borel). Show that there exists a bijection between

1. G-orbits of the diagonal G-action on $G/B \times G/B$ (i.e. $g(x, y) = (gx, gy)$);
2. B-orbits of the G-action on G/B.

Using this, describe the fibres of the projection
$$\mathrm{pr}_1 \colon G/B \times G/B \to G/B. \tag{13}$$

(3 points)

Problem 34. Consider the *Hilbert scheme of* n *points on* $\mathbb{A}^2_{\mathbb{C}}$, which as a set is given by the ideals in $\mathbb{C}[x, y]$ of colength n, i.e.
$$\mathrm{Hilb}^n \mathbb{A}^2_{\mathbb{C}} = \{I \lhd \mathbb{C}[x, y] \mid \dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n\}. \tag{14}$$
We will use without proof that it is a quasiprojective variety.

There exists an action of the torus $T = (\mathbb{C}^\times)^2$ on $\mathbb{A}^2_{\mathbb{C}}$ by $(t_1, t_2) \cdot (x, y) = (t_1 x, t_2 y)$ for $(t_1, t_2) \in (\mathbb{C}^\times)^2$ and $(x, y) \in \mathbb{A}^2_{\mathbb{C}}$. This induces an action on the regular functions $\mathbb{C}[x, y]$ in the usual way.

1. Show that this induces an action of T on $\mathrm{Hilb}^n \mathbb{A}^2_{\mathbb{C}}$.
2. Show that the torus fixed points are precisely the monomial ideals, i.e. ideals generated by monomials.
3. Set up a correspondence between monomial ideals and partitions of n, by considering the generators of the monomial ideal.

(5 points)