Representation Theory II
Bonn, winter term 2019–2020

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Sheet 6

Solutions to be handed in before class on Friday November 22.

Problem 27. Let (W, S) be a Coxeter group. Prove that any parabolic subgroup W_P generated by $S_P \subset S$ is a Coxeter group and that the length function agrees with the induced length function.

(2 points)

Problem 28. Consider the symmetry group W of a square in the plane.

1. Show that W is a group generated by a set S of reflections such that (W, S) becomes a Coxeter group.

   (It is in fact isomorphic to the Weyl group associated to the root system of type B_2.)

2. Compute the Kazhdan–Lusztig basis, the Kazhdan–Lusztig polynomials and Kazhdan–Lusztig patterns for the Hecke algebra $\mathcal{H}_v((W, S))$ associated to (W, S).

   If you wish, you are allowed to first explain the Weyl chambers and compute graphically.

(3 points)

Problem 29.

1. For the symmetric group S_n consider $H = \sum_{y \in S_n} v^{r-\ell(y)} H_y$ where $r = \ell(w_0)$ is the length of the longest element $w_0 \in S_n$. Show that $\underline{H}_{w_0} = H$.

   Hint It might be helpful to verify that

   $$\mathbb{Z}[v, v^{-1}]H = \{h \in \mathcal{H}_v(S_n) \mid hC_i = (v + v^{-1})h\} \tag{11}$$

   for any $1 \leq i \leq n_1$, and use this to prove the claim.

2. Consider the Lie algebra $\mathfrak{sl}_3$ and the Verma modules $M(w \cdot 0)$ for $w \in S_3$. Try to verify the Kazhdan–Lusztig conjectures for all $M(w \cdot 0)$ for $w \in \{w_0, st, ts, s, t\}$ for the standard generators $s, t \in W$.

(5 points)

Problem 30. Consider the symmetric group S_n with its standard generators $S = \{\text{simple transpositions}\}$. Let $S_P \subset S$ and $P \leq S_n$ the corresponding parabolic subgroup. Show that the following holds

1. The modules $\mathcal{N}^P$ and $\mathcal{M}^P$ have a Kazhdan–Lusztig basis with respect to the compatible bar involution given in the lectures.

2. For $P = S_2 \times S_2 \leq S_4$ compute the respective Kazhdan–Lusztig bases. Compute the Bruhat graph of S_4^P. That is, the graph with vertices S_4^P and arrows from vertex x to vertex y if $y \leq x$ in the Bruhat ordering.

(4 points)

Problem 31.

1. Show that the Schubert cells C_w of $\mathrm{GL}_n(\mathbb{C})/B$ for $w \in \mathrm{S}_n$ are isomorphic to $\mathbb{C}^{\ell(w)}$.

2. Show that $\epsilon(\mathbf{1}_w) = q^{\ell(w)}$ for the function ϵ defined in the lecture.

(2 points)