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Sheet 5

Solutions to be handed in before class on Friday November 15.

Problem 22. Show that $(G = \mathrm{S}_n, B = \mathrm{S}_1 \times \mathrm{S}_{n-1}, N = \mathrm{S}_2 \times \mathrm{S}_{n-2}, W \cong \mathrm{S}_2)$ is a Tits' system.

(2 points)

Problem 23. Let $s \in S = \{\text{simple transpositions}\}$. Show that $B \cup \mathcal{C}(s)$ is a minimal proper (i.e. not equal to B) parabolic subgroup of $\mathrm{GL}_n(\mathbb{C})$.

Hint Consider first the case $n = 2$.

(3 points)

Problem 24. Show that

$$\mathrm{GL}_n(k) = \dot{\bigcup_{w \in W(P,P')}} PwP' \tag{5}$$

for an appropriately chosen set $W(P, P')$ of permutation matrices (specify it!), where P and P' are standard parabolics.

(3 points)

Problem 25. Construct an action of the Hecke algebra $\mathrm{H}_n(q)$ on the vector space

$$\{f : B\backslash G/P \to \mathbb{C}\} = \{f : G \to \mathbb{C} \mid \forall b \in B, \forall \in P : f(g) = f(bg) = f(gp)\} \tag{6}$$

where $G = \mathrm{GL}_n(\mathbb{F}_q)$, B is the standard Borel subgroup and P is a standard parabolic subgroup. Give a basis of this vector space, and describe the action in terms of this basis.

(3 points)

Problem 26. We wish to prove that two length functions for the (Weyl) group $W = \mathrm{S}_n$ are the same. Here we define for $w \in W$

$$\begin{aligned} \ell(w) &:= \min\{k \mid w = s_1 \cdots s_k \text{ where } s_1, \ldots, s_k \text{ are simple transpositions}\}, \\ \ell'(w) &:= \#\{\alpha \in R^+ \mid w(\alpha) \in R^-\}. \end{aligned} \tag{7}$$

We set $\ell(e) = 0$ here.

1. Let $s = s_\alpha$ be a simple transposition. Show that

$$\begin{aligned} \ell'(sw) &= \begin{cases} \ell'(w) + 1 & \text{if } w^{-1}(\alpha) \in R^+ \\ \ell'(w) - 1 & \text{if } w^{-1}(\alpha) \in R^- \end{cases} \\ \ell'(ws) &= \begin{cases} \ell'(w) + 1 & \text{if } w(\alpha) \in R^+ \\ \ell'(w) - 1 & \text{if } w(\alpha) \in R^-. \end{cases} \end{aligned} \tag{8}$$

2. Let $s_1, \dots, s_k$ and $s = s_\alpha$ be simple transpositions. Denote $w = s_1 \cdots s_k$, and assume that $w(\alpha) \in R^-$. Show that there exists $j \in \{1, \dots, k\}$ such that
$$w = s_1 \cdots \widehat{s_j} \cdots s_k s. \tag{9}$$

3. Let $s_1, \dots, s_k$ be simple transpositions. Assume that $\ell'(s_1 \cdots s_k) < k$. Show that there exists $i, j \in \{1, \dots, k\}$ with $i < j$ such that
$$s_1 \cdots s_k = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k. \tag{10}$$

4. Show that $\ell(w) = \ell'(w)$ for all $w \in W$.

(5 points)

For extra credit you can do the following exercise.

Optional problem 2. Show that the derivative of $\mathrm{Ad}\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{gl}_n(\mathbb{C}))$ at the unit element is $\mathrm{ad}\colon \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{End}(\mathfrak{gl}_n(\mathbb{C}))$.