Representation Theory II
Bonn, winter term 2019–2020

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Sheet 4

Solutions to be handed in before class on Friday November 8.

Problem 19. We define $G \coloneqq \mathrm{O}_m(\mathbb{C})$ as the algebraic subgroup of $\mathrm{GL}_m(\mathbb{C})$ of matrices which preserve the symmetric bilinear form given by $(e_i, e_{m-j+1}) = \delta_{i,j}$. Write down a candidate Borel subgroup, denote it B, and prove that it is in fact a Borel subgroup. (5 points)

Hint Show that G/B is projective by identifying it with a closed subvariety of the usual (full) flag variety.

Problem 20. Prove that the character group of the standard torus T in GL_n is $\mathbb{Z}^n$.

(3 points)

Problem 21. Consider $G \coloneqq \mathrm{O}_m(\mathbb{C})$ where $m = 2k+1$, and let T be the subgroup of diagonal matrices in G.

1. Determine $\mathrm{N}_G(T)$ as a subgroup of G, determine an explicit set of coset representatives for the Weyl group $W = \mathrm{N}_G(T)/T$ and show that W is isomorphic to the Weyl group of type B_k. You can use that it has cardinality $2^k k!$.
2. Show that there exists a Bruhat decomposition for G.

(8 points)

Remark The Weyl group of type B_k has generators s_i, $i = 0, \ldots, k$ and defining relations

- $s_i^2 = e$ for all $i = 0, \ldots, k$;
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i = 1, \ldots, k-1$;
- $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

Hint For the second part, start from the Bruhat decomposition for $\mathrm{GL}_m(\mathbb{C})$ and restrict accordingly.

For extra credit you can do the following exercise.

Optional problem 1. Prove that the following subgroups of $\mathrm{GL}_2(\mathbb{C})$ are not closed, and compute their closures:

1. $\mathbb{Z} \subseteq \mathrm{GL}_2(\mathbb{C})$ where $n \mapsto \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$;
2. $\mathrm{S}^1 \subseteq \mathrm{GL}_2(\mathbb{C})$ where S^1 denotes the 1-dimensional sphere, and $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)$.

(3 points)