Representation Theory II
Bonn, winter term 2019–2020

Prof. Dr. Catharina Stroppel
Dr. Pieter Belmans

Sheet 3

Solutions to be handed in before class on Wednesday (!) October 30.

Problem 16. Let G be a linear algebraic group, and H a closed subgroup. Show that

$$(G \times G)/(H \times H) \cong G/H \times G/H \tag{3}$$

as varieties. (3 points)

Problem 17. Prove Rosenlicht's theorem, which states that if a unipotent closed subgroup H of GL_n acts on an affine variety $X = \operatorname{Spec} A$, then all of its orbits are necessarily closed. Here a (sub)group is unipotent if all of its elements are unipotent. U in $G = \mathrm{GL}_n$, unipotent subgroup

1. Show that if a linear algebraic group G acts on an affine variety $X = \operatorname{Spec} A$ we get that A is a G-representation, which is moreover locally finite-dimensional, i.e. every element $a \in A$ is contained in finite-dimensional subrepresentation of A. Here one uses the coaction $A \to A \otimes \mathbb{C}[G]$.
2. Prove that a unipotent closed subgroup H of GL_n can be conjugated into the (unipotent) subgroup U_n of upper triangular matrices.
3. Now replace the action of G on the closure of an orbit, and assume that the orbit isn't closed. How do elements of U_n act on the defining ideal?

(7 points)

Problem 18. Show that the projection map $\pi\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed.

Hints

1. A closed subset Z of $\mathbb{P}^n \times \mathbb{P}^m$ can be written as the vanishing locus of bihomogeneous polynomials of degree (a, b) such that $a + b = d$ is fixed. This the straightforward generalisation of the same statement for a closed subset of $\mathbb{P}^n$, starting from $\mathrm{V}(f) = \mathrm{V}(x_0^d f, \ldots, x_n^d f)$.
2. Prove that $a \in \mathbb{P}^m$ is *not* contained in $\pi(Z)$ if and only if the degree k part of $\mathbb{C}[x_0, \ldots, x_n]$ is contained in $(g_1, \ldots, g_r)$ for some $k \in \mathbb{N}$, where g_i is the partial evaluation of f_i, and Z is cut out by $(f_1, \ldots, f_r)$.
3. In that situation, explain why the map

$$F_k\colon (\mathbb{C}[x_0, \ldots, x_n]_{k-d})^{\oplus r} \to \mathbb{C}[x_0, \ldots, x_n]_k = (g_1, \ldots, g_r)_k \tag{4}$$

needs to be surjective, and how this leads to a Zariski-open condition.

(6 points)