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Sheet 2

Solutions to be handed in before class on Friday October 25.

Problem 12. Give a description of the Schubert cells in $\mathrm{Gr}(2,4)$, by describing for each $I \in \mathcal{I}_{2,4}$

1. the set of all $A \in \mathrm{M}_{n,d}(\mathbb{C})$ such that their row span gives a point in the cell C_I;
2. the dimension;
3. the partition, respectively Young diagram associated with the cell C_I.

Write down the inclusion order for Schubert varieties, and determine which cells are closed.

Interpreting each Schubert cell as an orbit for a an affine algebraic group action, show by using properties about Schubert cells and Schubert varieties that each orbit is open and dense in its closure. Can you give a general proof which works for any Grassmannian?

Pick a set R of coset representatives for $\mathrm{S}_n/\mathrm{S}_d \times \mathrm{S}_{n-d}$. Take the identification of $\mathrm{S}_n/\mathrm{S}_d \times \mathrm{S}_{n-d}$ with torus fixed points in $\mathrm{Gr}(d,n)$ to give a bijection between the Schubert cells and the elements in R. How can the dimension of C_I be read off from the corresponding element in R? Explain the partial ordering as above now in terms of elements in R.

(8 points)

Problem 13. Show the following alternative description of Schubert cells claimed in the lectures: for $I \in \mathcal{I}_{d,n}$ show that

$$C_I = \left\{ V \in \mathrm{Gr}(d,n) \mid \begin{array}{c} \forall r = 1,\ldots,d\colon\ \dim(V \cap F_{i_r}^{\mathrm{st}}) = r, \\ \text{and } \forall s < i_r\colon\ \dim(V \cap F_s^{\mathrm{st}}) < r \end{array} \right\} \tag{2}$$

where $I = \{i_1 < i_2 < \ldots < i_d\}$ and C_I the Schubert cell associated to I.

(3 points)

Problem 14. Give an example of a group G acting on an affine variety X such that not all orbits are closed.

(2 points)

Problem 15. Show that $(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}})$ is a ringed space, where $f\colon U \to \mathbb{C}$ is regular at $x \in U$ if and only if locally around x we have that f is of the form g/h where $g, h \in \mathbb{C}[x_0,\ldots,x_n]$ are homogeneous of the same degree.

(3 points)