Sheet 13

Solutions to be handed in before class on Friday January 24.

Problem 57. Consider $\mathbb{P}^1 = \mathrm{GL}_2(\mathbb{C})/B$. Using the statement of the Borel–Weil–Bott theorem, describe the cohomology of the line bundles $\mathcal{O}_{\mathbb{P}^1}(d)$ for $d = -5, \ldots, 5$. To do so, describe $\mathcal{O}_{\mathbb{P}^1}(d)$ as a line bundle $\mathcal{L}_\lambda$ for the appropriate choice of $\lambda \in \mathrm{X}(T)$. Then make a table containing

1. the degree in which the cohomology $\mathrm{H}^i(\mathcal{L}_\lambda)$ is nonzero;
2. the representation of $\mathrm{GL}_2(\mathbb{C})$ which appears as the cohomology (if nonzero).

Explain which line bundles have no cohomology, and why.

(6 points)

Problem 58. Describe the fixed points of the following actions:

1. the torus $(\mathbb{C}^\times)^{n+1}$ acting on $\mathbb{P}^n_{\mathbb{C}}$ as
$$(\lambda_0, \ldots, \lambda_n) \cdot [x_0 : \ldots : x_n] = [\lambda_0 x_0 : \ldots : \lambda_n x_n]; \tag{34}$$
2. the torus $\mathbb{C}^\times$ acting on $\mathbb{P}^n_{\mathbb{C}}$ as
$$\lambda \cdot [x_0 : \ldots : x_n] = [\lambda x_0 : \lambda^2 x_1 : \ldots : \lambda^{n+1} x_n]; \tag{35}$$
3. the torus $\mathbb{C}^\times$ acting on $\mathbb{P}^n_{\mathbb{C}}$ as
$$\lambda \cdot [x_0 : \ldots : x_n] = [\lambda x_0 : \ldots : \lambda x_n]. \tag{36}$$

How many fixed points are there, if the number is finite? Determine their stabilisers.

(5 points)

Problem 59. Let $A \subseteq B$ an extension of rings, and let $b \in B$ denote an element. Show that the following are equivalent.

1. b is integral over A, i.e. there exists a monic polynomial $f \in A[t]$ such that $f(b) = 0$;
2. $A[b]$ is finitely generated as an A-module;
3. there exists a ring extension $A[b] \subseteq B'$ such that B' is finitely generated as an A-module.

Use this to show that:

1. If B is finitely generated as an A-module, then B is integral over A, i.e. every $b \in B$ is integral over A.
2. If $b_1, \ldots, b_n \in B$ are integral over A, then $A[b_1, \ldots, b_n]$ is finitely generated as an R-module.

In particular, conclude that if A is noetherian[1] and B is finitely generated as an A-algebra, then being integral is equivalent to being finitely generated as a module.

(5 points)

[1] Without noetherianity one replaces finite generated with finite type.