Representation Theory II
Bonn, winter term 2019–2020

Prof. Dr. Catharina Stroppel
Dr. Pieter Belmans

Sheet 12

Solutions to be handed in before class on Friday January 17.

Problem 52. Let $\pi\colon E \to X$ and $\pi'\colon F \to X$ be vector bundles. Construct

1. the direct sum bundle $E \oplus F$, whose fibres are the direct sum of the fibres;
2. the tensor product bundle $E \otimes F$, whose fibres are the tensor product of the fibres;
3. the Hom-bundle $\mathcal{H}\mathrm{om}(E, F)$, whose fibres are the Hom of the fibres.

(2 points)

Problem 53. Let G be an algebraic group, and B a Borel subgroup. Let V be a finite-dimensional representation of B. Show that $G \times_B V$ is a G-equivariant vector bundle on G/B.

(3 points)

Problem 54. Consider the algebraic Peter–Weyl theorem from the lecture for $\mathbb{C}[\mathrm{SL}_2(\mathbb{C})] = \mathbb{C}[a, b, c, d]/(ad - bc - 1)$.

1. Equip this algebra with a filtration by assigning degree 1 to the generators, and prove that the left and right actions of $\mathrm{SL}_2(\mathbb{C})$ preserve the filtration, and therefore pass to the associated graded.
2. Determine the Hilbert series of the associated graded.
3. Using the Peter–Weyl theorem, give a complete classification (up to isomorphism) of finite-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$, by mimicking the construction used in Problem 51.

(4 points)

Problem 55. Consider $\mathbb{P}^1$ as $\mathrm{GL}_2(\mathbb{C})/B$. By the *Birkhoff–Grothendieck theorem* we have that all vector bundles on $\mathbb{P}^1$ split as a direct sum of line bundles.

1. Show that all line bundles on $\mathbb{P}^1$ considered in Problem 49 arise as equivariant line bundles from the description as the quotient $\mathrm{GL}_2(\mathbb{C})/B$, i.e. exhibit a 1-dimensional representation of B which gives rise to $\mathcal{O}_{\mathbb{P}^1}(d)$ for all $d \in \mathbb{Z}$.

   These are in fact *all* line bundles on $\mathbb{P}^1$. So by Birkhoff–Grothendieck, every vector bundle of rank n on $\mathbb{P}^1$ is isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(d_i)$ for the appropriate choice of $d_i \in \mathbb{Z}$.
2. Is the standard 2-dimensional representation of $\mathrm{GL}_2(\mathbb{C})$, seen as a representation of B irreducible?
3. Explain why there is no contradiction between the Birkhoff–Grothendieck theorem and the classification of equivariant vector bundles in terms of finite-dimensional representations of B.

4. Consider the equivariant vector bundle associated to the standard representation. Write it as an extension of line bundles, by writing the standard representation in terms of a short exact sequence of 1-dimensional representations of B.

As an additional problem: Is the extension constructed in (4) split as an extension of vector bundles?

(4 points)

Problem 56.

1. Show that a rank n vector bundle $E \to X$ is trivial if and only if $\mathrm{H}^0(X, E)$ is n-dimensional, spanned by sections $s_1, \ldots, s_n$ which are linearly independent in every fibre.
2. Compute $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. How many linearly independent sections are there?
3. Explain why the condition of part 1 does not apply.

(3 points)