Representation Theory II
Bonn, winter term 2019–2020

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Sheet 11

Solutions to be handed in before class on Friday January 10.

Problem 47. Show that the fibres of a vector bundle are indeed vector spaces. (1 point)

Problem 48. Let $X = \bigcup_{i\in I} U_i$ be an open cover, and let $A_{i,j}$ be transition matrices (i.e. each $A_{i,j}$ is a square matrix with entries $(A_{i,j})_{k,l}$ in $\mathcal{O}_X(U_i \cap U_j)$) satisfying the cocycle condition, i.e.

1. $A_{i,i} = \mathrm{id}$
2. $A_{j,k}A_{i,j} = A_{i,k}$ (seen as matrices with coefficients in $\mathcal{O}_X(U_i \cap U_j \cap U_k)$).

Show how one can use this to define a vector bundle on X.

Conversely, show how our definition of a vector bundle gives rise to such a collection of transition matrices satisfying the cocycle condition.

(3 points)

Problem 49. Consider the projective variety $\mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj}\mathbb{C}[x_0,\ldots,x_n]$. Let $U_i = \{x_i \neq 0\} \cong \mathbb{A}^n_{\mathbb{C}}$ be a standard affine open subset. Recall that on U_i the regular functions are of the form f/x_i^k where f is homogeneous and $k \in \mathbb{Z}$. A *line bundle* is a vector bundle of rank 1.

1. Consider the transition matrices (they are in fact just scalars as we are working with 1×1-matrices) $C_{i,j} = (x_i/x_j)^{d_{i,j}}$, where $d_{i,j} \in \mathbb{Z}$. Spell out the cocycle condition, and give a complete description of all line bundles on $\mathbb{P}^n_{\mathbb{C}}$ in terms of an integer $d \in \mathbb{Z}$.
2. Explain how we can use the cocycle condition and the equality

$$(x_i/x_j)^d \sigma_i = \sigma_j \tag{32}$$

on $U_i \cap U_j$ for σ_i a regular function on U_i, considering it as an element of $\mathbb{C}[x_0/x_i,\ldots,x_{i-1}/x_i,x_{i+1}/x_i,\ldots,x_n/x_i]$ to show that

$$\mathrm{H}^0(\mathbb{P}^n_{\mathbb{C}},\mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{C}[x_0,\ldots,x_n]_d, \tag{33}$$

the vector space of homogeneous degree d polynomials, where $\mathcal{O}^n_{\mathbb{P}_{\mathbb{C}}}(d)$ for $d \in \mathbb{Z}$ are the line bundles constructed in the previous point.

Fact You can use without proof that line bundles on $\mathbb{A}^n_{\mathbb{C}}$ are trivial.

(5 points)

Problem 50. Assume that X is irreducible. Let E be a vector bundle on X. Let U be an open subset of X.

1. Explain how we can restrict E to U. For every open subset U we get the set of sections $\mathrm{H}^0(U,E) \coloneqq \mathrm{H}^0(U,E|_U)$.

2. Show that $\mathrm{H}^0(U, E)$ is naturally an $\mathcal{O}_X(U)$-module.
3. Explain how we have defined a sheaf of $\mathcal{O}_X$-modules, which is locally free.

Conversely, let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module. This is a sheaf on X, such that

- on every open set U the sections $\mathcal{F}(U)$ have the structure of an $\mathcal{O}_X(U)$-module;
- for every inclusion $V \subseteq U$ of open sets the restrictions $\mathrm{res}^{\mathcal{F}}_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ are compatible with the restrictions $\mathrm{res}^{\mathcal{O}_X}_{U,V} \colon \mathcal{O}_X(U) \to \mathcal{O}_X(V)$, in the sense that $\mathrm{res}^{\mathcal{F}}_{U,V}(rf) = \mathrm{res}^{\mathcal{O}_X}_{U,V}(r)\,\mathrm{res}^{\mathcal{F}}_{U,V}(f)$ for every $r \in \mathcal{O}_X(U)$ and $f \in \mathcal{F}(U)$;
- there exists an open cover $X = \bigcup_{i\in I} U_i$ such that $\mathcal{F}$ restricted to U_i is isomorphic to a free $\mathcal{O}_{U_i}$-module.

Consider an open cover $X = \bigcup_{i\in I} U_i$ such that $\mathcal{F}$ restricted to U_i is isomorphic to a free $\mathcal{O}_{U_i}$-module.

4. Explain how we can use this data to construct a vector bundle.

We can conclude that we have set up a bijective correspondence between

- locally free $\mathcal{O}_X$-modules and vector bundles;
- free $\mathcal{O}_X$-modules and trivial vector bundles.

Fact You can use without proof that morphisms of varieties can be defined on an open cover (provided they are compatible on the intersection).

(5 points)

Problem 51. Consider the algebraic group $\mathrm{GL}_2(\mathbb{C})$. Show that for each non-negative integer n there exists a finite-dimensional irreducible representation V_n of dimension $n+1$.

(2 points)