Representation Theory II
Bonn, winter term 2019–2020

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Sheet 1

Solutions to be handed in before class on Friday October 18.

Problem 7. Consider $\mathbb{C}^n$ with basis $e_1, \ldots, e_n$. Let V be a d-dimensional subspace (so a point of $\mathrm{Gr}(d, n)$). If $v_1, \ldots, v_d$ is a basis for V we can express it in terms of the $e_1, \ldots, e_n$ and obtain an $d \times n$-matrix A. Show that

$$v_1 \wedge \ldots \wedge v_d = \sum_I p_I e_I \tag{1}$$

where p_I are the d-minors of A. (3 points)

Problem 8. Consider the group S_n with its standard generators $s_i = (i, i+1)$ for $1 \leq i \leq n-1$.

1. Show that the set $\{e, s_{n-1}, s_{n-2}s_{n-1}, \ldots, s_1 s_2 \ldots s_{n-1})$ is a system of shortest coset representatives for $\mathrm{S}_n/(\mathrm{S}_{n-1} \times \mathrm{S}_1)$.
2. Give a similar description for the cosets of $\mathrm{S}_n/(\mathrm{S}_d \times \mathrm{S}_{n-d})$ for any $1 \leq d \leq n-1$.

(5 points)

Problem 9. Show that $\mathrm{Gr}(d, n) \cong \mathrm{Gr}(n-d, n)$ as varieties. (3 points)

Hint: Use the notion of orthogonal complement to construct a set-theoretic map. You can use without proof that it is enough for a globally defined map to be an isomorphism that it is locally an isomorphism.

Problem 10 (Segre embedding)**.** Consider $\mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$ and $\mathbb{P}^m_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[y_0, \ldots, y_m]$. Set $N := (n+1)(m+1)-1$ and consider $\mathbb{P}^N_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[z_{0,0}, \ldots, z_{n,m}]$. Consider the set-theoretic map $f\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ given by $z_{i,j} = x_i y_j$.

1. Show that the image is again a projective variety, cut out by the ideal generated by $z_{i,j}z_{k,l} - z_{i,l}z_{k,j}$ for all $0 \leq i, k \leq n$ and $0 \leq j, l \leq m$.
2. Show that this map is in fact an isomorphism (if you are uncomfortable, it suffices to show it is a bijection).

In this way we obtain that products of projective varieties are again projective varieties.

(3 points)

Problem 11. Give a presentation of the homogeneous coordinate of $\mathrm{Gr}(3, 6)$ in terms of generators and relations. (2 point)