Prof. Dr. Catharina Stroppel
Dr. Pieter Belmans

Solution to problem 36

Problem 36. Using problems 12 and 13, describe the Schubert variety for $s_1s_3s_2$ as the intersection of the Grassmannian $\mathrm{Gr}(2,4)$ (which is a quadric in $\mathbb{P}^5$) with the tangent space at the identity element. Conclude that this Schubert variety is singular.

The embedding of $\mathrm{Gr}(2,4)$ into $\mathbb{P}^5$ is given by the single Plücker equation

$$x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{2,3}x_{1,4} = 0 \tag{1}$$

in $\mathbb{C}[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}]$. (This is a quadric in $\mathbb{P}^5$).

Next we wish to describe the Schubert variety for $w = s_1s_3s_2$, which is the Schubert variety $X_{2,4}$ by considering the permutation $(1\,2)(3\,4)(2\,3)$ and writing it as $\left(\begin{smallmatrix}1&2&3&4\\2&4&1&3\end{smallmatrix}\right)$. We see that it consists of the subspaces $V \subset \mathbb{C}^4$ for which

$$\dim_{\mathbb{C}}(V \cap \mathbb{C}e_1 + \mathbb{C}e_2) \geq 1 \tag{2}$$

i.e. V must contain a vector of the form $ae_1 + be_2$ for a, b not both zero. Let us denote a second basis vector as $ke_1 + le_2 + me_3 + ne_4$. Translating this to Plücker coordinates we get that $X_{2,4}$ (as the locus of such subspaces) is then described by

$$\begin{aligned} &(ae_1 + be_2) \wedge (ke_1 + le_2 + me_3 + ne_4) \\ &= (al - bk)e_1 \wedge e_2 + am\, e_1 \wedge e_3 + an\, e_1 \wedge e_4 + bm\, e_2 \wedge e_3 + bn\, e_2 \wedge e_4 \end{aligned} \tag{3}$$

and hence $X_{2,4}$ is given as the intersection of the Plücker quadric with the hyperplane $x_{3,4} = 0$ as this coefficient is 0. Hence we can consider $X_{2,4}$ also as a subspace in $\mathbb{P}^4$ with homogeneous coordinate ring $\mathbb{C}[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}]$ cut out by the restriction of the equation (1) to this hyperplane.

The identity element of the group $\mathrm{GL}_4(\mathbb{C})$ defines a point in $\mathrm{GL}_4(\mathbb{C})/P$, which equals the Schubert cell given by the Weyl group element $w = e$. In other words, we are considering the Schubert variety $X_{1,2}$ (which is again just a point) on $\mathrm{Gr}(2,4)$ and wish to show that its tangent space is equal to $X_{2,4}$. In the Plücker coordinates this point has coordinates $[1 : 0 : 0 : 0 : 0 : 0]$, i.e. $x_{1,2} = 0$ and all other coordinates are zero.

Now the tangent space to a hypersurface $X = \mathbb{V}(f)$ in $\mathbb{P}^n$ at a point $p = [p_0 : \ldots : p_n]$, with $f \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous, is given by

$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(p_i)x_i. \tag{4}$$

We then see that the tangent space at the Schubert variety $X_{1,2}$ is given by the equation

$$x_{3,4} = 0 \tag{5}$$

in the Plücker coordinates. Hence these two subspaces are equal.

The equation of the intersection with the tangent space $\mathrm{T}_e \operatorname{Gr}(2,4)$ (or equivalently the tangent space at the Schubert variety $X_{1,2}$) is therefore again a quadric, now in $\mathbb{P}^4$, given by the equation

$$-x_{1,3}x_{2,4} + x_{2,3}x_{1,4} = 0 \tag{6}$$

given by a homogeneous polynomial in $\mathbb{C}[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}]$. One could now argue using simply the rank of this quadratic form (which is 4, not 5) that the corresponding quadric is necessarily singular.

More generally we can also use the Jacobian criterion. Let us denote the restricted equation (6) as g. We can see that it is singular precisely in the point $[1:0:0:0:0]$ (which is contained in the quadric), as

$$\left(\frac{\partial g}{\partial x_{i,j}}\right) = \begin{pmatrix} 0 & -x_{2,4} & x_{2,3} & x_{1,3} & -x_{1,3} \end{pmatrix} \tag{7}$$

where $x_{i,j}$ does not include $x_{3,4}$. Evaluating it in the point $[1:0:0:0:0]$ gives the zero matrix, hence by the Jacobian criterion the variety defined by the equation g is singular at this point.

Finally, $[1:0:0:0:0]$ is the image of the point $[1:0:0:0:0:0]$ from the intersection of the Grassmannian in the ambient $\mathbb{P}^5$ to the hyperplane $\mathbb{P}^4$ cut out by $x_{3,4} = 0$.