Hochschild cohomology of Hilbert schemes of points

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(joint work with Lie Fu, Andreas Krug)

We will start from an algebro-geometric question which, a priori, has nothing to do with Hochschild cohomology. Yet, with the right approach, it turns out that (a generalization of) Hochschild cohomology is precisely the tool to answer this question, and at the same time the methods and tools also suggest interesting invariants to study outside this specific geometric setup.

Geometric motivation Let S be a smooth projective surface. Its *Hilbert scheme of points* $\operatorname{Hilb}^n S$ (where $n \geq 2$) is a smooth projective variety arising as an important example of a moduli space: the moduli space of length-n subschemes, whilst at the same time it is a crepant resolution of singularities of $\operatorname{Sym}^n S = S^n \times / \operatorname{Sym}_n$, through the Hilbert–Chow morphism

$$\operatorname{Hilb}^n S \to \operatorname{Sym}^n S. \tag{1}$$

Its geometry has been the topic of significant interest.

We are interested in its *deformation theory*, which we will approximate by trying to understand the vector space $\mathrm{H}^1(\operatorname{Hilb}^n S, \mathrm{T}_{\operatorname{Hilb}^n S})$ classifying first-order deformations of the Hilbert scheme. It always contains the first-order deformations $\mathrm{H}^1(S, \mathrm{T}_S)$ of the surface, but what else might be in there?

The following intermediate results exist:

(a) Fantechi [5] has shown that if $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ or $\mathrm{H}^0(S, \mathrm{T}_S) = 0$, and at the same time $\mathrm{H}^0(S, \omega_S^\vee) = 0$, then
$$\mathrm{H}^1(\operatorname{Hilb}^n S, \mathrm{T}_{\operatorname{Hilb}^n S}) = \mathrm{H}^1(S, \mathrm{T}_S),$$
i.e., they have the *same* deformation theory. These conditions hold, e.g., whenever S is a surface of general type.

(b) Hitchin [6] has shown that if $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, then
$$\mathrm{H}^1(\operatorname{Hilb}^n S, \mathrm{T}_{\operatorname{Hilb}^n S}) = \mathrm{H}^1(S, \mathrm{T}_S) \oplus \mathrm{H}^0(S, \omega_S^\vee),$$
thus linking the Poisson structures on S to the deformations of $\operatorname{Hilb}^n S$.

Both proofs are very geometric and heavily rely on the geometry of (1). A more categorical proof of Hitchin's result, moreover assuming that $\mathrm{H}^2(S, \mathcal{O}_S) = 0$ is given in [3], which uses Hochschild cohomology and its limited functoriality.

But in complete generality, by [2, Corollary B] the answer for an arbitrary surface is given by

$$\mathrm{H}^1(\operatorname{Hilb}^n S, \mathrm{T}_{\operatorname{Hilb}^n S}) = \mathrm{H}^1(S, \mathrm{T}_S) \oplus \mathrm{H}^0(S, \omega_S^\vee) \oplus \left(\mathrm{H}^1(S, \mathcal{O}_S \otimes \mathrm{H}^0(S, \mathrm{T}_S)\right). \tag{2}$$

Hochschild–Serre cohomology In order to prove (2) we (re)introduce a bigraded algebra that contains Hochschild cohomology and Hochschild homology as graded subspaces. This definition has an obvious analogue for an arbitrary smooth and proper dg category $\mathcal{A}$ (and we will come back to this later), with $\mathbf{D}^{\mathrm{b}}(X)$ for X a smooth projective variety (or Deligne–Mumford stack) recovering the geometric definition we make now.

The *Hochschild–Serre cohomology* of X is

$$\mathrm{HS}^*_\bullet(X) := \bigoplus_{j,k\in\mathbb{Z}} \mathrm{HS}^j_k(X)$$

where

$$\mathrm{HS}^j_k(X) := \mathrm{Ext}^{j+k\dim X}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^{\otimes k}).$$

One recognizes the powers of the Serre functor of $\mathbf{D}^{\mathrm{b}}(X)$, which explains how to define this for every dg category which admits a Serre functor.

We have that

(1) $k = 0$ recovers the Hochschild cohomology of X,
(2) $k = 1$ recovers the Hochschild homology of X.

There are some obvious questions one should ask about this object. But first we explain the relation to the deformation theory of $\mathrm{Hilb}^n S$.

Main result The Hochschild–Serre cohomology can be shown to be a categorical invariant. And we can extend (1) to include the stacky symmetric quotient

(3)
$$\begin{array}{ccc} \mathrm{Hilb}^n S & & [\mathrm{Sym}^n S] = [S^n/\mathrm{Sym}_n] \\ & \underset{\text{Hilbert–Chow}}{\searrow} \quad \underset{\text{coarse moduli space}}{\swarrow} & \\ & \mathrm{Sym}^n S & \end{array}$$

which also acts as a crepant resolution, it just happens to be a Deligne–Mumford stack. The Bridgeland–King–Reid–Haiman equivalence gives the equivalence

(4)
$$\mathbf{D}^{\mathrm{b}}(\mathrm{Hilb}^n S) \cong \mathbf{D}^{\mathrm{b}}([\mathrm{Sym}^n S])$$

making it possible to compute the Hochschild (and Hochschild–Serre) cohomology of $\mathrm{Hilb}^n S$ by computing it for $[\mathrm{Sym}^n S]$.

There is an orbifold Hochschild–Kostant–Rosenberg decomposition [4], which makes it possible to compute the Hochschild–Serre cohomology of any symmetric quotient stack $[\mathrm{Sym}^n X]$, which is where all the algebro-geometric work takes place. By suitably decomposing the computation using orbifold Hochschild–Kostant–Rosenberg, and then combining the components *for all n simultaneously* we can get a short answer. It depends on the parity of $\dim X$, so let us just give the conclusion for $\mathrm{Hilb}^n S$:

(5)
$$\bigoplus_{n\geq 0} \mathrm{HS}^*_k(\mathrm{Hilb}^n S)t^n \cong \mathrm{Sym}^*\left(\bigoplus_{i\geq 1} \mathrm{HS}^*_{1+(k-1)i}(S)t^i\right).$$

To prove (2) we take $k = 0$, so that *all* of the negative Hochschild–Serre cohomology of S is used, and subsequently we take $* = 2$ to compute $\mathrm{HH}^2(\mathrm{Hilb}^n S)$. To obtain the geometric deformations in the Hochschild–Kostant–Rosenberg decomposition of $\mathrm{HH}^2(\mathrm{Hilb}^n S)$, one bootstraps from earlier results which describe the components $\mathrm{H}^2(\mathrm{Hilb}^n S, \mathcal{O}_{\mathrm{Hilb}^n S})$ and $\mathrm{H}^2(\mathrm{Hilb}^{\mathrm{n}}\,\mathrm{S}, \bigwedge^2 \mathrm{T}_{\mathrm{Hilb}^n S})$, and cancels these contributions in the Hochschild–Serre calculation.

Questions We have the following obvious questions, which are of interest even if you do not care at all about (2):

(1) Equip the Hochschild–Serre cohomology of a smooth and proper dg category with the structure of a Gerstenhaber algebra (and also a Connes differential), recovering the usual Gerstenhaber calculus structure on the *pair* $(\mathrm{HH}^\bullet(X), \mathrm{HH}_\bullet(X))$.

(2) Relate this Gerstenhaber algebra structure to the geometric Gerstenhaber algebra structure on the Hochschild–Kostant–Rosenberg decomposition of the Hochschild–Serre cohomology, generalizing the work of Kontsevich, Căldăraru, Calaque–Van den Bergh, ... These two questions are work-in-progress by Lie Fu and collaborators.

(3) Extend the picture beyond smooth and proper dg categories.

(4) There is a Heisenberg algebra controlling the properties of symmetric quotient stacks (and symmetric powers of dg categories). This originates in the computation of Betti and Hodge numbers of Hilbert schemes of points. Is there a Heisenberg algebra controlling the Hochschild–Serre cohomology of symmetric quotient stacks?

    There are some obvious problems that arise: Hochschild–Serre cohomology is not very functorial (yet), and the description of the Hochschild–Serre cohomology does not obviously fit in the usual description of a Fock space.

(5) Compute the Hochschild–Serre cohomology of the symmetric power of a dg category $\mathcal{A}$ in terms of the Hochschild–Serre cohomology of $\mathcal{A}$ in geometrically meaningful examples, e.g., for the noncommutative projective planes and quadrics from [1].

References

[1] Belmans, P. Hochschild cohomology of noncommutative planes and quadrics. *J. Noncommut. Geom.*. **13**, 769-795 (2019)

[2] Belmans, P., Fu, L. and Krug, A. Hochschild cohomology of Hilbert schemes of points on surfaces. arXiv:2309.06244.

[3] Belmans, P., Fu, L. & Raedschelders, T. Hilbert squares: derived categories and deformations. *Selecta Math. (N.S.)*. **25**, Paper No. 37, 32 (2019)

[4] Arinkin, D., Căldăraru, A. & Hablicsek, M. Formality of derived intersections and the orbifold HKR isomorphism. *J. Algebra.* **540** pp. 100-120 (2019)

[5] Fantechi, B. Deformation of Hilbert schemes of points on a surface. *Compositio Math.*. **98**, 205-217 (1995)

[6] Hitchin, N. Deformations of holomorphic Poisson manifolds. *Mosc. Math. J.*. **12**, 567-591, 669 (2012)