Mirror symmetry for moduli of rank 2 bundles on a curve

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(joint work with Sergey Galkin, Swarnava Mukhopadhyay)

1. MIRROR SYMMETRY FOR FANO VARIETIES

Mirror symmetry refers to identifications between certain geometric objects, where properties of the complex geometry on one side are identified with properties of the symplectic geometry on the other. For Calabi–Yau threefolds its origins lie in a symmetry for Hodge diamonds which was then enhanced to a relation between Gromov–Witten invariants and periods [1], and finally an equivalence between the derived category and the Fukaya category [9]. In (almost) all of these settings the mirror to a Calabi–Yau threefold is again a Calabi–Yau threefold.

For Fano varieties the origin story is slightly different, in that the mirror to a Fano variety X is not another variety, rather it is a *Landau–Ginzburg model*: a pair (Y, f) where Y is a smooth quasiprojective variety and $f\colon Y \to \mathbb{A}^1$ is a regular function. The origins of this can be traced back to [7]. Again, mirror symmetry can take on different manifestations, and we will discuss two.

A strong form of mirror symmetry for Fano varieties and Landau–Ginzburg models is *homological mirror symmetry*, which posits that X and (Y, f) are mirror if there are equivalences of triangulated categories as in the table.

A-side (symplectic)	B-side (complex)
$\mathrm{FS}(Y, f)$ Fukaya–Seidel category	$\mathbf{D}^{\mathrm{b}}(\operatorname{coh} X)$ derived category of coherent sheaves
$\mathrm{Fuk}(X)$ Fukaya category	$\mathrm{MF}(Y, f)$ matrix factorisation category

A weaker form of mirror symmetry, similar to the equality between Gromov–Witten invariants and Hodge-theoretic periods for Calabi–Yau varieties, is *enumerative mirror symmetry*. For us this will take the form of an equality between the quantum period (an A-side invariant for X) and the classical period (a B-side invariant for (Y, f)). The (regularised) *quantum period* is defined as the power series

$$\widehat{\mathrm{G}}_X(t) := 1 + \sum_{d=2}^{+\infty} d!\langle[\mathrm{pt}] \cdot \psi^{d-2}\rangle_{0,1,d}^{X} t^d \tag{1}$$

giving a slice of the (descendant) Gromov–Witten invariants of X. On the mirror side we consider a torus $\mathbb{G}_{\mathrm{m}}^{\dim X}$ (Y is expected to be glued together from different tori, so this restriction makes sense) so that $f \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ is a Laurent polynomial in $n = \dim X$ variables, and we can define its classical period as the power series

$$\pi_f(t) := \frac{1}{2\pi\mathrm{i}} \int_{(\mathrm{S}^1)^n} \frac{1}{1 - tf(x_1, \ldots, x_n)} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}. \tag{2}$$

When we are only given a Laurent polynomial, we will say that it is a *weak mirror* if this equality holds. This is a discrete invariant of Fano varieties, in that it does not depend on the complex structure of X, which forms the basis of the Fanosearch program [2].

These two aspects of mirror symmetry are well-studied in certain situations. One setting which is out of reach of the existing methods is that of the moduli space $\mathrm{M}_C(2, \mathcal{L})$ of stable vector bundles of rank two and fixed odd determinant on a curve of genus $g \geq 2$. The construction of a candidate weak Landau–Ginzburg mirror, the verification of mirror symmetry hypotheses, and the application to a conjectural semiorthogonal decomposition form the subject of the works I'm reporting on.

2. Graph potentials

In [6] we construct a family of Laurent polynomials from the datum of a trivalent graph $\gamma = (V, E)$ of genus g (so that $\#V = 2g - 2$ and $\#V = 3g - 3$) with a coloring $c\colon V \to \mathbb{Z}/2\mathbb{Z}$. For a vertex v with coloring $c(v) \in \mathbb{Z}/2\mathbb{Z}$ and the incident edges labelled as x, y, z we define the *vertex potential* as

$$W_{v,c(v)} := \begin{cases} xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} & c(v) = 0 \\ \frac{1}{xyz} + \frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x} & c(v) = 1 \end{cases} \tag{3}$$

and if we denote by $x_1, \ldots, x_{3g-3}$ the edges in γ we define the *graph potential* as

$$W_{\gamma,c} := \sum_{v \in V} W_{v,c(v)}. \tag{4}$$

There are many different choices of γ and c for a fixed genus, and using elementary operations between different trivalent graphs and colorings we can relate the classical periods: the periods only depend on the genus and the parity of the number of colored vertices. We can in fact prove the following theorem, by considering all genera simultaneously.

Theorem 1 (B–Galkin–Mukhopadhyay)**.** *There are two topological quantum field theories in the functorial sense of Atiyah (depending on the parity of the coloring)* $\mathrm{Z}_{\mathrm{gp},0}(t)$ *and* $\mathrm{Z}_{\mathrm{gp},1}(t)$ *for every sufficiently small value of* t *which allow us to compute efficiently compute the value at* t *of (the inverse Laplace transform of) the classical period* $\pi_{W_{\gamma,c}}(t)$ *from* $\mathrm{Z}_{g,\epsilon}(t)(\Sigma_g)$ *if* $g(\gamma) = g$ *and* $\epsilon(c) := \sum_{v \in V} c(v) = \epsilon$.

3. Enumerative mirror symmetry

In [5] we prove the form of enumerative mirror symmetry introduced above relating graph potentials to the moduli space $\mathrm{M}_C(2, \mathcal{L})$ of rank 2 stable vector bundles with fixed determinant $\mathcal{L}$ of odd degree, a smooth projective Fano variety of dimension $3g-3$, Picard rank 1, and index 2 (so that $\omega^{\vee}_{\mathrm{M}_C(2,\mathcal{L})} \cong \mathcal{O}_{\mathrm{M}_C(2,\mathcal{L})}(2)$). We have the following.

Theorem 2 (B–Galkin–Mukhopadhyay)**.** *We have*

$$\widehat{\mathrm{G}}_{\mathrm{M}_C(2,\mathcal{L})}(t) = \pi_{W_{\gamma,c}}(t) \tag{5}$$

where C is a curve of genus $g \geq 2$, γ is a trivalent graph of genus g, and c is an odd coloring, i.e. $\sum_{v\in V} c(v) = 1$.

The proof is somewhat involved, and combines

- toric degenerations for $\mathrm{M}_C(2,\mathcal{L})$ introduced by Manon [10] (using vector bundles of conformal blocks on $\overline{\mathcal{M}}_{g,n}$);
- monotone Lagrangian tori defined by integrable systems coming from toric degenerations, an idea introduced by Nishinou–Nohara–Ueda [11] and Bondal–Galkin and Mikhalkin (unpublished) and suitably generalized in op. cit.;
- an identification of the period of the Landau–Ginzburg potential for a monotone Lagrangian with the quantum period, due to Tonkonog [13];
- an identification of the Landau–Ginzburg potential for these specific monotone Lagrangians with the graph potential, so that the Newton polytope of the graph potential agrees with the fan polytope of the toric degeneration.

4. On (semi)orthogonal decompositions and homological mirror symmetry

In 2018 we proposed the following conjecture (independently formulated by Narasimhan around the same time) [3].

Conjecture 2.1 (B–Galkin–Mukhopadhyay, Narasimhan)**.** *Let C be a smooth projective curve of genus g. Then there exists a semiorthogonal decomposition*

$$\begin{aligned}\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L})) = \langle \mathcal{O}_{\mathrm{M}_C(2,\mathcal{L})}, \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^2 C), \ldots, \mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^{g-1} C),\\ \mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^{g-2} C), \ldots, \mathbf{D}^{\mathrm{b}}(C), \mathcal{O}_{\mathrm{M}_C(2,\mathcal{L})}(1)\rangle\end{aligned} \tag{6}$$

In [4] we give various types of evidence for this conjecture, most importantly an identity in the Grothendieck ring of categories. The state-of-the-art for this conjecture is achieved in [12, 14], which both exhibit the presence of all components in the semiorthogonal decomposition (and their semiorthogonality), whilst the fullness is still open.

We can also study it from the point-of-view of homological mirror symmetry:

- by computing decompositions of the Fukaya–Seidel category of the (so far hypothetical) mirror to obtain evidence for the conjecture;
- taking the conjecture at face value and considering the decompositions on the mirror side as evidence for the fact that graph potentials are indeed building blocks of the homological mirror.

In [4] we show that the critical locus of an appropriate graph potential has exactly the right shape for the Fukaya–Seidel category of the (yet-to-be-constructed) mirror to have a semiorthogonal decompositions with pieces as in (6), and also for the orthogonal decompositions that should exist for the Fukaya category and matrix factorisation category respectively.

In the spirit of Dubrovin's conjecture, and a generalisation to the case of non-semisimple quantum cohomology (see also [8]), we also describe the identification of the eigenvalue decomposition (due to Muñoz) for quantum multiplication with $\mathrm{c}_1(\mathrm{M}_C(2,\mathcal{L}))$ to the critical value decomposition of the graph potential. The different graph potentials can thus be seen as restrictions of the Landau–Ginzburg potential on the true mirror to various cluster charts, and the mutation rules are the gluing procedure. It would be interesting to understand more of gluing and the global picture that emerges from it.

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