Projectivity of good moduli spaces of semistable quiver representations and vector bundles

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1. Parallels between quivers and curves

There are many parallels between moduli spaces $\mathrm{M}_Q^{\theta\text{-ss}}(\mathbf{d})$ of semistable quiver representations and moduli spaces $\mathrm{M}_C(r,d)$ of semistable vector bundles on a curve C. Here we consider a quiver $Q=(Q_0,Q_1)$ and finite-dimensional representations of Q with dimension vector $\mathbf{d}$, which are semistable with respect to a stability function $\theta\colon \mathbb{Z}^{Q_0}\to\mathbb{Z}$ such that $\theta(\mathbf{d})=0$ (resp. semistable bundles of rank r and degree d, where C has genus $g\geq 2$). Some of these parallels are discussed in [6], and there are others (such as similarities in the structure of their Brauer groups, or rationality questions).

The parallel I wish to focus on is their (usual) construction in algebraic geometry via geometric invariant theory (GIT). Using the notion of (semi)stability for quiver representations [7] (resp. for vector bundles [8]) one considers

- the open locus of $\mathrm{Rep}(Q,\mathbf{d})=\prod_{i\in Q_0}\mathbb{A}_k^{\mathbf{d}(i)}$
- resp. a suitable Quot scheme (obtained from making all bundles globally generated)

corresponding to semistable objects, and then quotients out

- the conjugation action by $\mathrm{GL}_{\mathbf{d}}=\prod_{i\in Q_0}\mathrm{GL}_{\mathbf{d}(i)}$
- resp. the action of GL_N, where N is the dimension of the global sections after twisting.

The moduli space of semistable quiver representations is *projective-over-affine*, the base affine variety being the spectrum of the invariant ring (from the zero stability function), which by Le Bruyn–Procesi is generated by traces along cycles. In particular, if Q is acyclic the resulting moduli space of semistable representations is *projective*. The moduli space of semistable vector bundles is always projective.

Nowadays there are many (more complicated) moduli spaces being studied, e.g. moduli spaces of Bridgeland-semistable objects [9]. For these *no* GIT-construction is available, so a GIT-free construction is needed. The general program is to:

(1) interpret the moduli problem of interest as an algebraic stack $\mathcal{M}$ of finite type;
(2) prove that there exists a good moduli space M, and show it is a proper algebraic space via the valuative criterion for universal closedness for $\mathcal{M}$;
(3) descend a line bundle to M and check its ampleness.

For step (2), if $\mathcal{M}$ has finite stabilisers, then the Keel–Mori theorem provides coarse moduli space M. If $\mathcal{M}$ has infinite stabilisers, one can use the recent Alper–Halpern-Leistner–Heinloth existence criterion [2, Theorem A].

For step (3) one uses the moduli-theoretic interpretation of the space and the line bundle to produce sections thereof, in order to check ampleness.

By implementing this program for well-known moduli spaces one can start to understand more complicated constructions, and also obtain additional results in these classical cases. The program is explained for $\mathrm{M}_C(r,d)$ in [1]. For the Deligne–Mumford compactification $\overline{\mathrm{M}}_g$ of the moduli space of curves (which is also usually constructed using GIT) it is explained in [4].

In the next section I will briefly explain the structure of the program in the case of quiver representations, which is the novel joint work I'm reporting on [3]. In this abstract we work over an algebraically closed field k of characteristic 0, so that we avoid adequate moduli spaces and geometrically stable representations, but op. cit. is written in greater generality.

2. Projectivity for moduli spaces of quiver representations

Let us assume that Q is acyclic in what follows. Step (1) consists of writing the usual setup in a suitable functor-of-points language and quickly deducing the necessary properties. For step (2) one applies the Alper–Halpern-Leistner–Heinloth existence criterion for the moduli stack $\mathcal{M}_Q^{\theta\text{-ss}}(\mathbf{d})$, by explicitly checking Θ-reductivity, S-completeness, and the valuative criterion for universal closedness. The latter is done by giving a version of Langton's semistable reduction for quiver representations. The existence criterion then yields a good moduli space $\mathrm{M}_Q^{\theta\text{-ss}}(\mathbf{d})$, and shows it is a proper algebraic space.

In order to show projectivity we need a line bundle on $\mathrm{M}_Q^{\theta\text{-ss}}(\mathbf{d})$ for which we can prove ampleness. In the setting of vector bundles on a curve such line bundles are provided by descending *determinantal line bundles* from the moduli stack: considering the Fourier–Mukai functor given by the universal vector bundle, a vector bundle F on C gives a 2-term complex of vector bundles on the moduli stack. One can take the determinant of this complex to obtain a line bundle on the moduli stack, which only depends on the numerical invariants of F. If the rank and degree of F are chosen appropriately so that $\chi(C, E\otimes F)=0$ (where E has rank r and degree d) it is possible to descend this line bundle to the good moduli space, and moreover construct a section of this line bundle (which does depend on the isomorphism class of F).

A similar construction can be done for moduli of quiver representations using the universal representation. Suitably interpreted in concrete terms (which is how Schofield originally introduced them): if M is a $\mathbf{d}$-dimensional representation, and N is $\mathbf{e}$-dimensional, we define

$$d_N^M\colon \bigoplus_{i\in Q_0}\mathrm{Hom}_k(M_i,N_i)\to\bigoplus_{\alpha\in Q_1}\mathrm{Hom}_k(M_{t(\alpha)},N_{h(\alpha)}) \tag{1}$$

as

$$d_N^M\colon (\phi_i)_{i\in Q_0}\mapsto(\phi_{h(\alpha)}\circ M_\alpha - N_\alpha\circ\phi_{t(\alpha)})_{\alpha\in Q_1}. \tag{2}$$

If $\langle\mathbf{d},\mathbf{e}\rangle=0$, then d_N^M is in fact a square matrix, and following Schofield we define the *determinantal semi-invariant* $c(M,N):=\det d_N^M$. In what follows we will usually fix some N (or more precisely try to construct one with special properties) so that $c(\_, N)$ can be seen as a section of a determinantal line bundle on $\mathcal{M}_Q^{\theta\text{-ss}}(\mathbf{d})$. One important result on these determinantal semi-invariants is that by varying over all N of dimension vector $\mathbf{e}$ orthogonal to $\mathbf{d}$ they span the ring of semi-invariants (as a vector space), a result independently proven by Derksen–Weyman, Schofield–Van den Bergh and Domokos–Zubkov. The semi-invariant $c(\_, N)$ has weight $-\langle \_, \mathbf{e}\rangle$.

The next step is to produce enough determinantal semi-invariants to show that the determinantal line bundles (for an appropriate choice of multiple of the dimension vector $\mathbf{e}$) is basepoint-free. This can be done using the analogue of Faltings's characterization of semistable vector bundles, which says that a vector bundle is semistable if and only if there exists a vector bundle orthogonal in the sense from above for which $\mathrm{Hom}_C(F^\vee, E) = \mathrm{Ext}^1_C(F^\vee, E) = 0$. Such characterizations were known already to Schofield(–Van den Bergh) and Crawley-Boevey. But interestingly, from our proof we also obtain effective bounds on which power of this semiample determinantal line bundle becomes basepoint-free.

The final step, where truly new ingredients are needed, is to prove for an acyclic quiver Q that determinantal line bundle is *ample*, and not just semiample. This is done by performing a dimension count, which for curves is done in [5]. The semiample determinantal line bundle provides us with a morphism from a proper algebraic space to some $\mathbb{P}^N$, and by constructing suitable determinantal semi-invariants we can separate enough points to prove that this map is finite, thus the determinantal line bundle is ample. This part of the argument builds upon the (limited) compatibility between Auslander–Reiten functors and semistability.

References

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