**How do semiorthogonal decompositions behave in families?**

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(joint work with Shinnosuke Okawa, Andrea T. Ricolfi)

In this talk I gave a brief history of semiorthogonal decompositions, explained how they can be studied using Fourier–Mukai transforms, and how they behave in families. What follows is an overview of the historical motivation with additional references, and a summary of the results on moduli spaces of semiorthogonal decompositions.

## 1. SEMIORTHOGONAL DECOMPOSITIONS

For an introduction to semiorthogonal decompositions, and many more examples, one is referred to [13]. They are introduced to understand the structure of the derived category of coherent sheaves of a smooth projective variety $X$, from now on denoted $\mathbf{D}^{\mathrm{b}}(X)$.

**A brief history of semiorthogonal decompositions: 3 examples**

In [2] Beĭlinson described the derived category of $\mathbb{P}^n$, starting the whole field.

**Example 0.1.** *Using the usual notation for exceptional collections we have that*

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle \tag{1}$$

*Here all the admissible subcategories are equivalent to* $\mathbf{D}^{\mathrm{b}}(k)$*.*

The next step came with the introduction of semiorthogonal decompositions by Bondal–Orlov in [6].

**Example 0.2.** *The first example is given by the (smooth) intersection of two even-dimensional quadrics* $X = Q_1 \cap Q_2$ *in* $\mathbb{P}^{2g+1}$*, for which we have*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{D}^{\mathrm{b}}(C), \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(2g-3) \rangle \tag{2}$$

*where* $C$ *is a hyperelliptic curve of genus* $g$*. As we will discuss in the next example,* $\mathbf{D}^{\mathrm{b}}(C)$ *cannot be decomposed further, so this is a semiorthogonal decomposition into "atomic" components.*

By now there exists a vast literature on exceptional collections and semiorthogonal decompositions. The final example we want to give discusses the *absence* of non-trivial semiorthogonal decompositions.

**Example 0.3.** *Let* $X$ *be either a Calabi–Yau variety, or a curve of genus* $g \geq 2$*. Then by Bridgeland* [7] *resp. Okawa* [14, Theorem 1.1] *we have that* $\mathbf{D}^{\mathrm{b}}(X)$ *is indecomposable.*

**Families of varieties** Once one understands a semiorthogonal decomposition of one fibre in a family, what can be said about semiorthogonal decompositions for other fibres?

An important guiding principle here is *Dubrovin's conjecture* [8]. It states that $\mathrm{BQH}^\bullet(X)$ is (generically) semisimple if and only if $\mathbf{D}^{\mathrm{b}}(X)$ admits a full exceptional collection. But $\mathrm{BQH}^\bullet(X)$ depends on the symplectic, not the complex structure of $X$. Hence conjecturally the existence of full exceptional collection is constant in nice families. A more general version of this conjecture for arbitrary semiorthogonal decompositions can be found in [17].

The behavior of exceptional collections in families is studied in [10] using Lieblich's deformation theory of perfect complexes. Full exceptional collections are shown to extend to étale neighbourhoods. In what follows we discuss how this deformation theory result generalises, by constructing an actual moduli space of semiorthogonal decompositions with arbitrary components.

## 2. The moduli space of semiorthogonal decompositions

The general procedure to make exceptional collections and semiorthogonal decompositions behave well in a family over a base $S$ is that of $S$-linearity [12]. The main result is then following [4, Theorem A].

**Theorem 1.** *Let $f\colon \mathcal{X} \to S$ be a smooth projective morphism, where $S$ is an excellent scheme. Then there exists an algebraic space $\mathrm{SOD}_f \to S$, such that*

(1) $\mathrm{SOD}_f \to S$ *is* étale*;*
(2) *there exists a functorial bijection between $\mathrm{SOD}_f(T \to S)$ and the set of $T$-linear semiorthogonal decompositions (of length 2) of* $\operatorname{Perf} \mathcal{X} \times_T S$.

The proof consists of checking Artin's axioms for étale algebraic spaces, in the form of Hall–Rydh [9, Theorem 11.3]. For this we use that $S$-linear semiorthogonal decompositions can be represented using (morphisms of) Fourier–Mukai kernels. The main technical ingredient is then a deformation theory of *morphisms* of complexes in a derived category (with a fixed lift of the codomain), generalising the deformation theory of complexes.

The main important geometric feature of this moduli space (which is rather strange in other respects, see §3) is that it is étale over $S$. This is consistent with the suggestion of Dubrovin's conjecture.

**Application: indecomposability** We can use the moduli space of semiorthogonal decompositions to show that having an indecomposable derived category *specialises* in a family of smooth projective varieties. The details for this are contained in the joint work [1] with Francesco Bastianelli. We can obtain for example the following result.

**Theorem 2.** *Let $C$ be a smooth projective curve of genus $g \geq 2$. Let $n = 1, \dots, \lfloor \frac{g+3}{2} \rfloor - 1$. Then $\mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^n C)$ is indecomposable.*

Its proof is obtained by bootstrapping from the indecomposability result [11, Theorem 1.4], analysing the relationship between the gonality of a curve and the base locus of the canonical linear system (see also [5]). This theorem settles (the easier) half of [3, Conjecture 2], which suggests the indecomposability of $\mathbf{D}^{\mathrm{b}}(\operatorname{Sym}^n C)$ for $n$ up to $g-1$.

## 3. Examples, pathologies and amplifications

By describing $\mathrm{SOD}_f$ in 3 instances, we can see how this algebraic space has an interesting geometry, and what kind of variations we can moreover study.

**Example 2.1.** *Let $f\colon X \to \operatorname{Spec} k$ be an example from Example 0.3. Then $\mathrm{SOD}_f$ consists of two points, given by the trivial semiorthogonal decompositions $\langle \mathbf{D}^{\mathrm{b}}(X), \emptyset \rangle$ and $\langle \emptyset, \mathbf{D}^{\mathrm{b}}(X) \rangle$.*

To remedy this, one can study the open and closed algebraic subspace $\mathrm{ntSOD}_f \subset \mathrm{SOD}_f$, only parametrising non-trivial semiorthogonal decompositions.

More interestingly we can consider Beĭlinson's semiorthogonal decompositions from Example 0.1.

**Example 2.2.** *For $n = 1$ (folklore) and $n = 2$ [16] there exists a classification of semiorthogonal decompositions of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$. For $f\colon \mathbb{P}^1 \to \operatorname{Spec} k$ it shows that $\mathrm{ntSOD}_f = \bigcup_{i \in \mathbb{Z}} \operatorname{Spec} k$, indexing the decomposition $\langle \mathcal{O}_{\mathbb{P}^1}(i), \mathcal{O}_{\mathbb{P}^1}(i+1) \rangle$.*

This shows that $\mathrm{SOD}_f$ and $\mathrm{ntSOD}_f$ are usually *not quasicompact*. It also shows the necessity to extend the definition of the moduli space to incorporate semiorthogonal decompositions of length $\ell$. This can be done, and yields moduli spaces $\mathrm{SOD}_f^\ell$ and $\mathrm{ntSOD}_f^\ell$ with similar properties.

One can show that $\mathrm{SOD}_f^\ell$ and $\mathrm{ntSOD}_f^\ell$ admit (commuting) actions by the group $\mathrm{Auteq}(f)$ of $f$-linear autoequivalences and the braid group $\mathrm{Br}_\ell$ acting by mutations. The quotient by these groups might yield more tractable moduli spaces.

Finally, the most interesting behaviour is showcased by the following example.

**Example 2.3.** *Let $f\colon \mathcal{X} \to \mathbb{A}^1$ be the degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ into the second Hirzebruch surface $\mathbb{F}_2$ (at the point $0 \in \mathbb{A}^1$). By comparing the classification of exceptional objects for the quadric with the results of [15], one can construct distinct families of exceptional objects, which agree on $\mathbb{A}^1 \setminus \{0\}$, but give different exceptional objects in $\mathbf{D}^{\mathrm{b}}(\mathbb{F}_2)$.*

This shows that $\mathrm{SOD}_f$ can in general be *non-separated*. This is an important feature of the behavior of semiorthogonal decompositions in families, and it would be interesting to understand this in more instances.

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