Abstracts

Semiorthogonal decompositions for moduli of sheaves on curves

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(joint work with Swarnava Mukhopadhyay, Sergey Galkin)

Let C be a smooth projective curve over a field k. We will work over an algebraically closed field k of characteristic 0, suitable adaptations of the statements are expected to hold over more general fields. The curve C is often studied through the properties its various associated moduli spaces of sheaves. Because a sheaf on C splits as the direct sum of a vector bundle and a sheaf with finite support, these moduli spaces come in 2 (or 3) families:

1. the symmetric powers $\mathrm{Sym}^i C$, which are isomorphic to the Hilbert scheme of i points $\mathrm{Hilb}^i C$ on C;

2a. the Picard varieties $\mathrm{Pic}^d C$ parametrising line bundles of degree d;

2b. the moduli spaces $\mathrm{M}_C(r,d)$ of (semi)stable vector bundles of fixed rank $r \geq 2$ and degree d, and we will assume $\gcd(r,d) = 1$ to ensure that there are no strictly semistable bundles and that the moduli spaces are smooth.

The study of the moduli space $\mathrm{M}_C(r,d)$ is often reduced to that of $\mathrm{M}_C(r,\mathcal{L})$, which is the fiber of the determinant map $\det\colon \mathrm{M}_C(r,d) \to \mathrm{Pic}^d C$ over a line bundle $\mathcal{L}$ of degree d. The moduli space $\mathrm{M}_C(r,\mathcal{L})$ is a smooth projective Fano variety of dimension $(r^2-1)(g-1)$, with $\mathrm{Pic}\,\mathrm{M}_C(r,\mathcal{L}) \cong \mathbb{Z}\cdot\Theta$ where Θ is the ample generator, and index 2, i.e. we have that $\omega^\vee_{\mathrm{M}_C(r,\mathcal{L})} \cong \Theta^{\otimes 2}$.

We will explain how the derived categories of these moduli spaces of sheaves (are expected to) behave, depending on the parameters, by describing natural semiorthogonal decompositions. For background one is referred to Kuznetsov's ICM address [5]. These decompositions give rise to new connections between the different families of moduli spaces, and it is moreover possible to recover various known results as corollaries (not discussed here).

Under our assumptions on k we have that $\mathrm{Pic}^d C \cong \mathrm{Jac}\, C$, and it is well-known that the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{Jac}\, C)$ is indecomposable as $\mathrm{Jac}\, C$ is an abelian variety. On the other hand one expects interesting semiorthogonal decomposition for $\mathrm{M}_C(r,\mathcal{L})$, as one does for every Fano variety. And it is well-known that $\mathrm{Sym}^i C$ when $i \geq 2g-1$ is a projective bundle over $\mathrm{Pic}^i C$, so one again obtains a semiorthogonal decomposition. Hence one desires to understand decompositions of the derived categories

1. $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)$ for all $i \geq 1$,

2b'. $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L}))$ for $r \geq 2$ and $\mathcal{L}$ of degree d coprime to r

into indecomposable pieces. We will summarise the state of the art on decompositions for $\mathrm{Sym}^i C$ and $\mathrm{M}_C(r,\mathcal{L})$, and state some natural conjectures in this context.

Besides proving these conjectures in the stated settings it could be interesting to study these moduli problems in the context of noncommutative algebraic geometry for sheaves of hereditary orders on smooth projective curves, or equivalently, orbifold curves, where parabolic sheaves might play a role.

1. Symmetric powers

First we consider symmetric powers, as they will also make an appearance when describing $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L}))$. Recently, Toda showed the following semiorthogonal decomposition [9, corollary 5.11].

Theorem 1 (Toda)**.** *With C as above, there exists a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1+n} C) = \left\langle \overbrace{\mathbf{D}^{\mathrm{b}}(\mathrm{Jac}\, C), \dots, \mathbf{D}^{\mathrm{b}}(\mathrm{Jac}\, C)}^{n\ copies}, \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1-n} C) \right\rangle \tag{1}$$

for all $n \geq 0$.

When $n \geq g$ the Abel–Jacobi morphism $\mathrm{Sym}^{g-1+n} C \to \mathrm{Pic}^{g-1+n} C$ is a $\mathbb{P}^{n-1}$-bundle, and $\mathrm{Sym}^{g-1-n} C = \emptyset$, so the description reduces to Orlov's blow-up formula. For $n \leq g-1$ the semiorthogonal decomposition is obtained by studying wall-crossing for moduli spaces on Calabi–Yau 3-folds, and it would be interesting to give a more self-contained proof.

The following natural conjecture then provides the final ingredient for the study of semiorthogonal decompositions on symmetric powers.

Conjecture 1.1. *With C as above, the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)$ is indecomposable for $i = 1, \dots, g-1$.*

If $i = 1$, then $\mathrm{Sym}^1 C = C$ and indecomposability was shown in [8]. For $i = 2, \dots, g-1$ the symmetric power is an i-dimensional variety of general type, where the study of the base locus of the complete linear system $|\omega_{\mathrm{Sym}^i C}|$ should shed light on the indecomposability. In the weeks after the workshop, during discussions with Francesco Bastianelli, Shinnosuke Okawa and Andrea Ricolfi we have obtained preliminary results in this direction.

2. Moduli of vector bundles

As $\mathrm{M}_C(r,\mathcal{L})$ is a fine moduli space we have a universal (Poincaré) bundle $\mathcal{W}$ on $C \times \mathrm{M}_C(r,\mathcal{L})$. We can use it to construct the Fourier–Mukai functor

$$\Phi_{\mathcal{W}} \colon \mathbf{D}^{\mathrm{b}}(C) \to \mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L})). \tag{2}$$

Recently the following result was shown by Fonarev–Kuznetsov [4] and independently by Narasimhan [7].

Theorem 2 (Fonarev–Kuznetsov, Narasimhan)**.** *With C as above, when $r = 2$ and $\deg \mathcal{L} = 1$, the functor $\Phi_{\mathcal{W}}$ is fully faithful.*

In the approach by Fonarev–Kuznetsov one needs to take C generic, as the fully faithfulness is based on an explicit check for hyperelliptic curves. The approach by Narasimhan uses the Hecke correspondence, and one needs to impose $g \geq 4$. In a joint work with Swarnava Mukhopadhyay [1] we have generalised the approach using the Hecke correspondence to arbitrary rank, and degree 1, subjct to a bound on the genus in terms of a function $g_0(r)$. The proof for other degrees will require a new ingredient.

Theorem 3 (Belmans–Mukhopadhyay)**.** *With C as above, when $r \geq 2$ and $\deg \mathcal{L} = 1$, the functor $\Phi_{\mathcal{W}}$ is fully faithful for $g \geq g_0(r)$.*

Having found a non-trivial component in the derived category, one would like to know how it relates to the exceptional line bundles $\Theta^\vee$ and $\mathcal{O}_{\mathrm{M}_C(r,\mathcal{L})}$. A special case of the following result when $r = 2$ and involving only a single copy of the curve was shown by Narasimhan.

Theorem 4 (Belmans–Mukhopadhyay)**.** *With C as above and $\mathcal{L}$ as above, we have that the sequence*

$$\Theta^\vee, \Phi_{\mathcal{W}}(\mathbf{D}^{\mathrm{b}}(C)) \otimes \Theta^\vee, \mathcal{O}_{\mathrm{M}_C(r,\mathcal{L})}, \Phi_{\mathcal{W}}(\mathbf{D}^{\mathrm{b}}(C) \tag{3}$$

is the start of a semiorthogonal decomposition for $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L}))$, for $g \geq g_0(r)$.

To describe the complement of these 4 admissible subcategories when $r = 2$ we propose the following conjecture. After the workshop, it was announced in [6] that Narasimhan independently proposed this conjecture.

Conjecture 4.1. *With C as above and $\mathcal{L}$ as above, the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}))$ has a semiorthogonal decomposition of the form*

$$\begin{aligned} \mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L})) = \big\langle\, & \mathbf{D}^{\mathrm{b}}(k), \mathbf{D}^{\mathrm{b}}(k), \\ & \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(C), \\ & \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^2 C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^2 C), \\ & \dots, \\ & \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-2} C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-2} C), \\ & \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1} C)\big\rangle \end{aligned} \tag{4}$$

where there are 2 copies of $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)$ for $i = 0, \dots, g-2$ and 1 copy of $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1} C)$.

For $g = 2$ we have that $\mathrm{M}_C(2,\mathcal{L})$ is the intersection of two quadrics in $\mathbb{P}^5$, and the proposed semiorthogonal decomposition is known by [3].

In a joint work in progress with Sergey Galkin and Swarnava Mukhopadhyay [2] we have collected some evidence for this conjecture, from two angles:

(1) an identity in the Bondal–Larsen–Lunts Grothendieck ring of categories;
(2) a (partial) description of the Landau–Ginzburg mirror, and a study of the eigenvalues of the endomorphism $\mathrm{c}_1(\mathrm{M}_C(2,\mathcal{L})) \star -$ on the quantum cohomology of $\mathrm{M}_C(2,\mathcal{L})$.

It is not clear at the moment what the precise statement for the analogous conjecture for $r \geq 3$ would have to be, but it will probably involve components like $\mathbf{D}^{\mathrm{b}}(C^i)$ and $\mathbf{D}^{\mathrm{b}}(C^i \times \operatorname{Sym}^j C)$. A better understanding of the Chow motive or the class in the Grothendieck ring of varieties would be helpful.

References

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