On the homotopy theory of differential graded categories

mémoire du Master 2 : Analyse, Arithmétique et Géométrie

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de heer Wadman in *De paradijsvogel*
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Introduction

> "Tiger got to hunt, bird got to fly;
> Man got to sit and wonder 'why, why, why?'
> Tiger got to sleep, bird got to land;
> Man got to tell himself he understand."
>
> *Cat's Cradle*
> KURT VONNEGUT

Comme le titre de ce mémoire suggère : le but est de discuter la théorie homotopique des catégories différentielles graduées. D'abord, on peut se demander pourquoi c'est intéressant d'étudier des catégories différentielles graduées, ce qui est addressé dans le chapitre 1. Le slogan est que les catégories différentielles graduées peuvent résoudre des problèmes inhérents des catégories dérivées. Ainsi on peut essayer de trouver des généralisations et interprétations plus naturelles des résultats classiques.

Afin d'étudier ce que l'on appelle la théorie homotopique d'une catégorie (et donc ici on s'intéresse à la catégorie des catégories différentielles graduées) on doit introduire une structure de catégorie de modèles de Quillen, ce qui est d'abord introduit en toute généralité dans le chapitre 2 et l'appendice A. Après la théorie générale, on peut tout spécialiser à la catégorie à laquelle nous nous intéressons, ce qui est fait dans le chapitre 3 et l'appendice B. Ces résultats sont dûs à Tabuada et Toën.

La partie importante (et technique) de ce mémoire est le chapitre 4. Ici on essaie d'expliquer les résultats principaux de l'article [Toe07] de Toën, dont nous nous sommes inspirés pour le titre de ce mémoire. On peut résumer ces résultats comme une description explicite de la structure simpliciale de la structure de catégorie de modèles, et l'existence d'une structure monoïdale fermée sur la catégorie homotopique au sens de Quillen.

Ces résultats nous permettent de discuter des applications dans la cohomologie de Hochschild et la géométrie algébrique, qui sont basées sur une forme de théorie de Morita dérivée, ce qu'on fait dans le chapitre 5.

Dans le chapitre 6 on fait un grand bond en avant, et on discute le champ dérivé de modules d'un carquois comme introduit par Toën-Vaquié. Il n'est pas possible de développer toute la théorie nécessaire ici, étant donne que l'on a besoin du formalisme de la géométrie algébrique dérivée au sens de Toën-Vezzosi. Basé sur un article recent de Keller-Scherotzke, on essaie de généraliser leur foncteur de stratification, qui permet d'étudier d'une façon explicite le champ dérivé de modules d'un carquois.

CHAPTER 1

Differential graded categories

"From now on, I'll describe the cities to you, in your journeys you will see if they exist."

Kublai Khan in *Invisible Cities*
ITALO CALVINO

Triangulated categories are the technical tool used for derived categories, and were originally introduced by Jean-Louis Verdier [Ver67]. Using this tool we can formulate many important and strong (geometric) results, e.g. Grothendieck duality. The idea is that derived categories are the correct object of study when one is looking at categories of sheaves and functors between them.

Unfortunately there are technical issues with triangulated categories. For instance the non-functoriality of cones: Verdier proved that if the cones are functorial in a (countably) (co)complete triangulated category that category is necessarily semisimple abelian [Ver67, proposition II.1.2.13]. Since not every triangulated category is semisimple abelian, it suffices to consider the derived category of abelian groups, we don't have functoriality in the interesting cases. One could say that triangulated categories forget too much of the structure we need. To solve these issues one is tempted to "remember higher homotopies", which is exactly what is forgotten when going from an abelian category to its derived category.

Two other problems with triangulated categories are worth mentioning.

(i) Let $\mathcal{D}_1 \to \mathcal{C}$ and $\mathcal{D}_2 \to \mathcal{C}$ be triangulated functors between triangulated categories. Then the fibered product $\mathcal{D}_1 \times_{\mathcal{C}} \mathcal{D}_2$ is not triangulated in general.

(ii) Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories. Then the category $\mathrm{Func}_{\mathrm{tr}}(\mathcal{C}, \mathcal{D})$ is not triangulated in general.

A possible approach to tackle this problem is the use of differential graded categories, which is the main object of study of this thesis. One wants to enrich the structure of a triangulated category, in order to keep more information [BK91]. In example 1.16 the way to recover the triangulated category is explained.

This chapter is greatly inspired by [Kel06; LNM2008; Toe07], and will provide a quick overview of the notions used in dg categories. In the remainder of this text we will fix a commutative ring k, and whenever we have the choice we will use cochain complexes (i.e. our differential has degree 1, or "goes to the right"). Also, from this point on we will always say dg category instead of differential graded category.

1.1 Preliminaries

To formally define dg categories it will suffice to say that:

Definition 1.1. A *dg category* is a category enriched over complexes of k-modules.

In order to give more flesh to this high-brow definition we will explain what this really means. For this we need several preliminary concepts.

Definition 1.2. A *k-category* $\mathcal{C}$ is a category in which the sets of homomorphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are k-modules[1], for all $X,Y \in \mathrm{Obj}(\mathcal{C})$. The composition of two morphisms is then described by the k-linear associative maps

$$(1.1)\quad -\circ-\colon \mathrm{Hom}_{\mathcal{C}}(Y,Z)\otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y)\to \mathrm{Hom}_{\mathcal{C}}(X,Z):(f,g)\mapsto fg.$$

of k-modules, for all $X,Y,Z\in\mathrm{Obj}(\mathcal{C})$, such that there is an unit element $\mathrm{e}_X\in\mathrm{Hom}_{\mathcal{C}}(X,X)$ which acts as the unit for multiplication. In other words it is a map $\mathrm{e}_X\colon k\to\mathrm{Hom}_{\mathcal{C}}(X,X)$. The associativity condition can be written more explicitly by requiring the commutativity of the diagram

$$(1.2)\quad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Z,T)\otimes_k \mathrm{Hom}_{\mathcal{C}}(Y,Z)\otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\mathrm{id}\otimes-\circ-"] \arrow[d, "-\circ-\otimes\mathrm{id}"'] & \mathrm{Hom}_{\mathcal{C}}(Y,T)\otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[d, "-\circ-"] \\ \mathrm{Hom}_{\mathcal{C}}(Z,T)\otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[r, "-\circ-"'] & \mathrm{Hom}_{\mathcal{C}}(X,T) \end{tikzcd}$$

while conditions on the unit morphism are described by the commutativity of the diagrams

$$(1.3)\quad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y)\cong k\otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\mathrm{e}_X\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Hom}_{\mathcal{C}}(X,X)\otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[d, "-\circ-"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,Y) \end{tikzcd}$$

and

$$(1.4)\quad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y)\cong \mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_k k \arrow[r, "\mathrm{id}\otimes\mathrm{e}_Y"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_k \mathrm{Hom}_{\mathcal{C}}(Y,Y) \arrow[d, "-\circ-"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,Y) \end{tikzcd}$$

Example 1.3. This definition generalizes the notion of k-algebra, which is nothing but a k-category with a single object. Thus in general k-categories can be considered as k-categories with multiple objects, just like groupoids can be considered as groups with multiple objects.

Definition 1.4. A *k-linear category* $\mathcal{C}$ is a k-category which is closed under finite direct sums.

Example 1.5. Let A be a k-algebra, the category A-Mod of right A-modules is a k-linear category with many objects: the Hom-sets all have the structure of a k-module.

We now fix some conventions of an algebraic nature. The notions of graded k-modules, morphisms of graded k-modules and shifts are considered to be known, all gradings are over $\mathbb{Z}$. The category of graded k-modules will be denoted k-grMod. In the remainder of the text we will always denote a grading using $\bullet$, i.e. whenever an object C is equipped with a grading it will be denoted $C^\bullet$. This rather pedantic rule will not be applied to morphisms equipped with a grading as it would make the notation too cumbersome.

Definition 1.6. Let $V^\bullet$ and $W^\bullet$ be two graded k-modules. The *tensor product of graded k-modules* is the graded k-module $(V^\bullet\otimes_k W^\bullet)^\bullet$ with components

$$(1.5)\quad (V^\bullet\otimes_k W^\bullet)^n := \bigoplus_{p+q=n} V^p\otimes_k W^q.$$

[1] One could say it is a category enriched over k-modules. But we were not going to resort to "abstract" notions at this point.

The *tensor product of morphisms* $f: V_1^\bullet \to V_2^\bullet$ and $g: W_1^\bullet \to W_2^\bullet$ of graded k-modules is defined using the Koszul sign rule, which states that for $v \in V_1^q$ and p the degree of g we have

$$(1.6)\quad (f \otimes g)(v \otimes w) := (-1)^{pq} f(v) \otimes g(w)$$

which is extended linearly to all of $(V_1^\bullet \otimes_k W_1^\bullet)^\bullet$, hence we get a map

$$(1.7)\quad (f \otimes g)\colon (V_1^\bullet \otimes_k W_1^\bullet)^\bullet \to (V_2^\bullet \otimes_k W_2^\bullet)^\bullet.$$

This allows us to define:

Definition 1.7. A *graded k-algebra* is a graded k-module $A^\bullet$ together with a multiplication map $A^\bullet \otimes_k A^\bullet \to A^\bullet$ of degree 0 which is required to be associative, and there must exist an element $1 \in A^0$ which acts as a unit for the multiplication.

Definition 1.8. A *dg k-module* is a graded k-module $V^\bullet$ which is equipped with a *differential*, i.e. a k-linear morphism $\mathrm{d}_{V^\bullet}\colon V^\bullet \to V^\bullet$ of degree 1 such that $\mathrm{d}_{V^\bullet}^{n+1} \circ \mathrm{d}_{V^\bullet}^n = 0$. So $V^\bullet$ can be considered as a cochain complex of k-modules.
The *shifted dg k-module* $V[1]^\bullet$ is the shift of the graded k-module together with the differential $-\mathrm{d}_{V^\bullet}$.
The *tensor product of dg k-modules* is the graded k-module $(V^\bullet \otimes_k W^\bullet)^\bullet$ equipped with the differential $\mathrm{d}_{V^\bullet} \otimes \mathrm{id}_{W^\bullet} + \mathrm{id}_{V^\bullet} \otimes \mathrm{d}_{W^\bullet}$. To see that this is a differential we take $v \in V^q$, $w \in W^\bullet$ arbitrary and observe that the differentials have degree 1 while the identities have degree 0, so we get

$$(1.8)\quad \begin{aligned} \mathrm{d}^2_{(V^\bullet \otimes_k W^\bullet)^\bullet}(v \otimes w) &= (\mathrm{d}_{V^\bullet} \otimes \mathrm{id}_{W^\bullet} + \mathrm{id}_{V^\bullet} \otimes \mathrm{d}_{W^\bullet})\left(\mathrm{d}_{V^\bullet}(v) \otimes w + (-1)^q v \otimes \mathrm{d}_{W^\bullet}(w)\right) \\ &= \mathrm{d}^2_{V^\bullet}(v) \otimes w + (-1)^{q+1}\mathrm{d}_{V^\bullet}(v) \otimes \mathrm{d}_{W^\bullet}(w) \\ &\quad + (-1)^q \mathrm{d}_{V^\bullet}(v) \otimes \mathrm{d}_{W^\bullet}(w) + (-1)^{q^2} v \otimes \mathrm{d}^2_{W^\bullet}(w) \\ &= 0. \end{aligned}$$

Having reviewed what dg k-algebras are we are ready to define our main object of study:

Definition 1.9. A *dg category* is a k-category $\mathcal{C}$ such that the Hom-sets are dg k-modules, the compositions

$$(1.9)\quad \mathrm{Hom}_{\mathcal{C}}(Y,Z)^\bullet \otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet \to \mathrm{Hom}_{\mathcal{C}}(X,Z)^\bullet$$

being morphisms of dg k-modules.

Hence we have recovered what we had originally taken as our definition in definition 1.1.

Example 1.10. Just like the case of example 1.3 a dg k-algebra $A^\bullet$ can be interpreted as a dg category with a single object. Such a dg algebra can be considered as a cochain complex and a k-algebra. Examples are Koszul complexes or tensor algebras. Remark that the Koszul sign rule implies that the multiplication in a dg k-algebra satisfies the *graded Leibniz rule*

$$(1.10)\quad \mathrm{d}_{A^\bullet}(ab) = \mathrm{d}_{A^\bullet}(a)b + (-1)^p a \mathrm{d}_{A^\bullet}(b)$$

for $a \in A^p$.
If we equip a k-algebra A with the trivial differential, i.e. $\mathrm{d}_A = 0$ we get an instance of a dg category.

Example 1.11. There are also less trivial examples, which generalize example 1.5. Let A again be a k-algebra, consider the category Ch(A-Mod) of complexes of (right) A-modules. Instead of taking the usual morphisms of cochain complexes we will introduce the category $\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})$ which has exactly the cochain complexes of A-modules as objects.
But for the morphisms we define the dg k-module $\mathrm{Hom}_{\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})}(M^\bullet, N^\bullet)^\bullet$ for cochain complexes $M^\bullet$ and $N^\bullet$ to have in its nth degree the morphisms of degree n, i.e. for each $p \in \mathbb{Z}$

the map $f^p\colon M^p \to N^{n+p}$ is a morphism of A-modules, composition being the composition of graded morphisms which clearly is compatible with this structure. The differential between these Hom-structures is defined by setting

$$\mathrm{d}(f) = \mathrm{d}_{N^\bullet} \circ f - (-1)^n f \circ \mathrm{d}_{M^\bullet} \tag{1.11}$$

for f a morphism of degree n, and this is where the original structure of cochain complexes is used. To check that this defines a differential, we see that

$$\begin{aligned} \mathrm{d}^2(f) &= \mathrm{d}_{N^\bullet}\left(\mathrm{d}_{N^\bullet} \circ f - (-1)^n f \circ \mathrm{d}_{M^\bullet}\right) - (-1)^{n+1}\left(\mathrm{d}_{N^\bullet} \circ f - (-1)^n f \circ \mathrm{d}_{M^\bullet}\right) \circ \mathrm{d}_{M^\bullet} \\ &= \mathrm{d}_{N^\bullet}^2 \circ f - (-1)^n \mathrm{d}_{N^\bullet} \circ f \circ \mathrm{d}_{M^\bullet} - (-1)^{n+1}\mathrm{d}_{N^\bullet} \circ f \circ \mathrm{d}_{M^\bullet} + (-1)^{2n+1} f \circ \mathrm{d}_{M^\bullet}^2 \\ &= 0. \end{aligned} \tag{1.12}$$

Given a dg category $\mathcal{C}$ we can construct four related categories, where the first is again a dg category, the others being "only" k-linear categories (one of which is graded).

Definition 1.12. Let $\mathcal{C}$ be a dg category. Its *opposite dg category* $\mathcal{C}^{\mathrm{op}}$ is the dg category such that $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Obj}(\mathcal{C})$. Its morphisms are defined as $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y)^\bullet := \mathrm{Hom}_{\mathcal{C}}(Y,X)^\bullet$, with composition being

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y)^p \otimes_k \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y,Z)^q \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Z)^{p+q} : f \otimes g \mapsto (-1)^{pq} gf \tag{1.13}$$

and then extended linearly.

In the definition of a dg category we have required that the sets of morphisms had a rather special structure. It is time we put this extra structure to use, and this will hint at a solution to the problem discussed in the introduction.

Definition 1.13. The *underlying category* $\mathrm{Z}^0(\mathcal{C})$ is the category with $\mathrm{Obj}(\mathrm{Z}^0(\mathcal{C})) := \mathrm{Obj}(\mathcal{C})$ but we take

$$\mathrm{Hom}_{\mathrm{Z}^0(\mathcal{C})}(X,Y) := \mathrm{Z}^0\left(\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet\right). \tag{1.14}$$

To be more precise, the morphisms in $\mathrm{Z}^0(\mathcal{C})$ are exactly those morphisms which live in the kernel of $\mathrm{d}\colon \mathrm{Hom}_{\mathcal{C}}(X,Y)^0 \to \mathrm{Hom}_{\mathcal{C}}(X,Y)^1$.

This definition ties in with the general theory of enriched categories, in which the notion of underlying category is defined for general enriched categories [LNM145, §1.3]. One could argue to use the term *cocycle category* in this case, as it more clearly represents the way we recover the underlying category, but we will not do this.

Definition 1.14. The *homotopy category* $\mathrm{H}^0(\mathcal{C})$ is the k-linear category with

$$\mathrm{Obj}(\mathrm{H}^0(\mathcal{C})) := \mathrm{Obj}(\mathcal{C}) \tag{1.15}$$

but we take

$$\mathrm{Hom}_{\mathrm{H}^0(\mathcal{C})}(X,Y) := \mathrm{H}^0\left(\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet\right). \tag{1.16}$$

Definition 1.15. The *graded homotopy category* $\mathrm{H}^\bullet(\mathcal{C})$ is the k-linear category with objects

$$\mathrm{Obj}(\mathrm{H}^\bullet(\mathcal{C})) := \mathrm{Obj}(\mathcal{C}) \tag{1.17}$$

but we take

$$\mathrm{Hom}_{\mathrm{H}^\bullet(\mathcal{C})}(X,Y)^\bullet := \mathrm{H}^\bullet\left(\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet\right). \tag{1.18}$$

Again, a different terminology could be used here, but "cohomology category" and "graded cohomology category" suggest a choice of differential which is not necessary for the development of the theory, and it is less standard.

Now that we have defined these related categories, we can see how they might provide a solution to the problem of triangulated categories by using this $\mathrm{Ch}(k\text{-Mod})$-enrichment.

Example 1.16. Continuing with the dg category $\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})$ from example 1.11 we can see that

$$\mathrm{Z}^0(\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})) = \mathrm{Ch}(A\text{-Mod}) \tag{1.19}$$

because a morphism $f: M^\bullet \to L^\bullet$ in $\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})$ (which is at this point not a map of cochain complexes) belongs to the kernel of the differential of $\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})$ if and only if $\mathrm{d}_{M^\bullet} \circ f - f \circ \mathrm{d}_{L^\bullet} = 0$ which is exactly the condition that squares commute in maps of cochain complexes. So we observe that the category $\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})$ is a $\mathrm{Ch}(k\text{-Mod})$-enrichment of $\mathrm{Ch}(A\text{-Mod})$.
Similarly we get that

$$\mathrm{H}^0(\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})) = \mathbf{K}(A\text{-Mod}) \tag{1.20}$$

where $\mathbf{K}(A\text{-Mod})$ is the *category of complexes up to homotopy*, as it occurs in the (classical) construction of the derived category of the category of chain complexes. So the higher homotopies are in a way contained in $\mathrm{Ch}_{\mathrm{dg}}(A\text{-Mod})$ and we will be able to use them.

Remark 1.17. Now might be a good time to remark that every time we have used modules over a ring we could have used any k-linear Grothendieck abelian category $\mathcal{A}$. There is nothing to change in the definitions related to A-Mod, in a completely similar way we obtain $\mathrm{Ch}_{\mathrm{dg}}(\mathcal{A})$. We will use this in §5.3 for categories of sheaves for a scheme X defined over k.

1.2 The category of small dg categories

Now we will introduce the category of all small dg categories, which we will denote $\mathrm{dg}\,\mathrm{Cat}_k$. This object will be of great importance.

Definition 1.18. A *dg functor* $F: \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are two (small) dg categories is a map $F: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ on the level of the objects such that for $X, Y \in \mathrm{Obj}(\mathcal{C})$ we have a morphism

$$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))^\bullet \tag{1.21}$$

of dg k-modules between the morphism spaces. These maps are required to be compatible with composition and units, which implies the commutativity of

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z)^\bullet \otimes_k \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet \arrow[r, "-\circ-"] \arrow[d, "F_{Y,Z}\otimes F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(X,Z)^\bullet \arrow[d, "F_{X,Z}"] \\
\mathrm{Hom}_{\mathcal{D}}(F(Y),F(Z))^\bullet \otimes_k \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))^\bullet \arrow[r, "-\circ-"'] & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Z))^\bullet
\end{tikzcd} \tag{1.22}$$

and

$$\begin{tikzcd}
k \arrow[r, "\mathrm{e}_X"] \arrow[dr, "\mathrm{e}_{F(X)}"'] & \mathrm{Hom}_{\mathcal{C}}(X,X)^\bullet \arrow[d, "F_{X,X}"] \\
& \mathrm{Hom}_{\mathcal{D}}(F(X),F(X))^\bullet
\end{tikzcd} \tag{1.23}$$

for $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$.
The *category of small dg categories* $\mathrm{dg}\,\mathrm{Cat}_k$ is the category whose objects are all small dg categories together with the dg functors as morphisms.

Remark 1.19. To prevent size issues we have defined $\mathrm{dg}\,\mathrm{Cat}_k$ to consist only of *small* dg categories, fixed to a given universe, similar to Cat being the category of all small categories [GTM5, §I.6]. This will always be the situation, but we will also need to consider bigger universes for certain constructions, especially in chapters 3 and 4. This will be given more attention in due time, most notably in remark 4.13. For now it suffices that we fix a universe relative to which we consider small objects.

Remark 1.20. The category $\mathrm{dg\,Cat}_k$ has the empty dg category as its initial object[2] The dg category with one object $*$, equipped with the zero endomorphism ring, i.e.

$$\mathrm{Hom}_{\mathrm{dg\,Cat}_k}(*,*)^\bullet = 0, \tag{1.24}$$

is the final object.

We can endow the category $\mathrm{dg\,Cat}_k$ with a tensor product and an internal Hom-functor, hence it will become a closed symmetric tensor category. This is nothing but using the enrichment over the symmetric monoidal category $\mathrm{Ch}(k\text{-Mod})$ [LNM145, §1.4].

Definition 1.21. The *tensor product* $\mathcal{C}\otimes\mathcal{D}$ of two dg categories $\mathcal{C}$ and $\mathcal{D}$ is defined by taking $\mathrm{Obj}(\mathcal{C}\otimes\mathcal{D}) := \mathrm{Obj}(\mathcal{C})\times\mathrm{Obj}(\mathcal{C})$ and setting

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}\left((X,Y),(X',Y')\right)^\bullet := \mathrm{Hom}_{\mathcal{C}}(X,X')^\bullet \otimes_k \mathrm{Hom}_{\mathcal{D}}(Y,Y')^\bullet. \tag{1.25}$$

Remark 1.22. The unit for the monoidal structure is the dg category k, where k by abuse of notation denotes both the dg category and the dg algebra with k concentrated in degree 0, using example 1.10.

To define the internal Hom-functor we need to explain how we will use the $\mathrm{Ch}(k\text{-Mod})$-enrichment. Again this is just a special case of the theory of enriched categories.

Definition 1.23. Let $\mathcal{C}$ and $\mathcal{D}$ be small dg categories. Let $F,G\colon\mathcal{C}\to\mathcal{D}$ be two dg functors. A *natural transformation of degree n* $\phi\colon F\Rightarrow G$ is a family of morphisms $(\phi_X)_{X\in\mathrm{Obj}(\mathcal{C})}$ such that $\phi_X\in\mathrm{Hom}_{\mathcal{D}}(F(X),G(X))^n$ for $X\in\mathrm{Obj}(\mathcal{C})$ satisfying $G(f)(\phi_X)=\phi_Y(F(f))$ for all $f\in\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $Y\in\mathrm{Obj}(\mathcal{C})$. In other words, if f is homogeneous of degree m, we have the commutativity of the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\phi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\phi_Y"'] & G(Y) \end{tikzcd} \tag{1.26}$$

up to the sign $(-1)^{nm}$.

The *complex of graded morphisms* $\mathcal{H}\mathrm{om}(F,G)^\bullet$ for two dg functors $F,G\colon\mathcal{C}\to\mathcal{D}$ is the complex of graded morphisms (or rather natural transformations) such that $\mathcal{H}\mathrm{om}(F,G)^n$ consists of the natural transformations of degree n. The differential in this complex is given for each $X\in\mathrm{Obj}(\mathcal{C})$ by

$$\mathrm{d}^n_{\mathcal{H}\mathrm{om}(F,G)^\bullet}(\phi)(X) := \mathrm{d}^n_{\mathrm{Hom}_{\mathcal{D}}(F(X),G(X))^\bullet}(\phi_X) \tag{1.27}$$

which lands in $\mathrm{Hom}_{\mathcal{D}}(F(X),G(X))^{n+1}$. This means that $\mathrm{d}^n_{\mathcal{H}\mathrm{om}(F,G)^\bullet}(\phi)\in\mathcal{H}\mathrm{om}(F,G)^{n+1}$. To check that the sign convention is satisfied we apply d^{n+m} to the equality

$$G(f)\circ\phi_X = (-1)^{nm}\phi_Y\circ F(f) \tag{1.28}$$

obtained from (1.26), which lives in $\mathrm{Hom}_{\mathcal{D}}(F(X),G(Y))^\bullet$. We obtain

$$\begin{aligned} &\mathrm{d}^{n+m}_{\mathrm{Hom}_{\mathcal{D}}(F(X),G(Y))^\bullet}\left(G(f)\circ\phi_X\right) \\ &\quad= \mathrm{d}^m_{\mathrm{Hom}_{\mathcal{D}}(G(X),G(Y))^\bullet}\left(G(f)\right)\circ\phi_X + (-1)^m G(f)\circ\mathrm{d}^n_{\mathrm{Hom}_{\mathcal{D}}(F(X),G(X))^\bullet}(\phi_X) \\ &\quad= G\left(\mathrm{d}^m_{\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet}(f)\right)\circ\phi_X + (-1)^m G(f)\circ\mathrm{d}^n_{\mathcal{H}\mathrm{om}(F,G)^\bullet}(\phi)(X) \\ &\quad= (-1)^{n(m+1)}\phi_Y\circ F\left(\mathrm{d}^m_{\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet}(f)\right) + (-1)^m G(f)\circ\mathrm{d}^n_{\mathcal{H}\mathrm{om}(F,G)^\bullet}(\phi)(X) \end{aligned} \tag{1.29}$$

[2] In [Tab05a] the empty dg category is not considered to be a dg category, but one year later this concern has been dropped [Kel06]. The proof of [Tab05a] can be interpreted accordingly, as is required for lemma 3.6.

and

$$
\begin{aligned}
&(-1)^{nm}\mathrm{d}^{n+m}_{\mathrm{Hom}_{\mathcal{D}}(F(X),G(Y))^{\bullet}}(\phi_Y \circ F(f)) \\
(1.30)\qquad &= (-1)^{nm}\mathrm{d}^{n}_{\mathrm{Hom}_{\mathcal{D}}(F(Y),G(Y))^{\bullet}}(\phi_Y)\circ F(f) + (-1)^{n(m+1)}\phi_Y \circ \mathrm{d}^{n}_{\mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))^{\bullet}}(F(f)) \\
&= (-1)^{nm}\mathrm{d}^{n}_{\mathcal{H}\mathrm{om}(F,G)^{\bullet}}(\phi)(Y)\circ F(f) + (-1)^{n(m+1)}\phi_Y \circ F\left(\mathrm{d}^{m}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet}}(f)\right).
\end{aligned}
$$

By cancelling the corresponding terms and compensating for the sign, we obtain

$$(1.31)\quad G(f)\circ \mathrm{d}^{n}_{\mathcal{H}\mathrm{om}(F,G)^{\bullet}}(\phi)(X) = (-1)^{(n+1)m}\mathrm{d}^{n}_{\mathcal{H}\mathrm{om}(F,G)^{\bullet}}(\phi)(Y)\circ F(f)$$

which corresponds to the commutativity up to the sign $(-1)^{(n+1)m}$.

Example 1.24. Just like in example 1.16 we get that $\mathrm{Z}^0(\mathcal{H}\mathrm{om}(F,G)^{\bullet})$ describes the (classical) natural transformations $F \Rightarrow G$.

Definition 1.25. Let $\mathcal{C}$ and $\mathcal{D}$ be two dg categories. The *internal Hom* for $\mathcal{C}$ and $\mathcal{D}$ is the dg category $\mathcal{H}\mathrm{om}(\mathcal{C},\mathcal{D})$ which has the dg functors between $\mathcal{C}$ and $\mathcal{D}$ as objects and the complex of graded morphisms $\mathcal{H}\mathrm{om}(F,G)^{\bullet}$ between two dg functors $F, G\colon \mathcal{C}\to\mathcal{D}$ as morphism spaces.

If we take the dg category with the single object k as discussed in example 1.10 as the unit object, we have that the category $\mathrm{dg}\,\mathrm{Cat}_k$ is a symmetric tensor category, i.e. we have the adjunction

$$(1.32)\quad \mathrm{Hom}_{\mathrm{dg}\,\mathrm{Cat}_k}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \cong \mathrm{Hom}_{\mathrm{dg}\,\mathrm{Cat}_k}(\mathcal{A},\mathcal{H}\mathrm{om}(\mathcal{B},\mathcal{C}))$$

for $\mathcal{A},\mathcal{B},\mathcal{C}$ dg categories.

The pair $(\otimes,\mathcal{H}\mathrm{om})$ makes $\mathrm{dg}\,\mathrm{Cat}_k$ into a closed symmetric monoidal category. This structure will be important for the remainder of this work. We want our localisations as constructed in chapters 2 and 3 to be compatible with it, but this is the source of an important issue. An example of this problem is given in example 4.28, and a workaround is discussed §4.6.

One last property of the category $\mathrm{dg}\,\mathrm{Cat}_k$ is that its pullbacks and pushouts have a nice description. This is true in general but we state it nevertheless for future reference.

Lemma 1.26. Let $\mathcal{C}\to\mathcal{E}$ and $\mathcal{D}\to\mathcal{E}$ be two morphisms in $\mathrm{dg}\,\mathrm{Cat}_k$. Then the pullback (resp. pushout) is given by the pullbacks (resp. pushouts) of the object sets, together with pullbacks (resp. pushouts) of the cochain complexes as morphism spaces.

1.3 Differential graded modules

We have generalized (dg) algebras to (dg) algebras with multiple objects as discussed in examples 1.3, 1.5, 1.10 and 1.11. The same game can be played with dg modules, and this is crucial for the remainder of the thesis.

In this section we will introduce dg modules, but for the complete theory as we will need it we require model category structures on several of our objects. This will be explained in chapter 3.

The sometimes confusing notation found in the literature is hopefully improved by the puristic and verbose notation we will use.

Definition 1.27. Let $\mathcal{C}$ be a small dg category. We will define a *left dg $\mathcal{C}$-module* to be a dg functor $L\colon \mathcal{C}\to \mathrm{Ch}_{\mathrm{dg}}(k\text{-Mod})$ while a *right dg $\mathcal{C}$-module* is a dg functor $M\colon \mathcal{C}^{\mathrm{op}}\to\mathrm{Ch}_{\mathrm{dg}}(k\text{-Mod})$.

So a dg $\mathcal{C}$-module could also be defined as a "$\mathrm{Ch}(k\text{-Mod})$-enriched presheaf on the dg category $\mathcal{C}$". This terminology is not standard though, and we will not use it. But keeping this in mind can help in understanding the philosophy of certain statements and proofs.

As usual we can consider all dg modules and endow them with the structure of a category.

Definition 1.28. Let $\mathcal{C}$ be a small dg category. The *category of dg $\mathcal{C}$-modules* $\mathcal{C}\text{-dg}\,\mathrm{Mod}_k$ has all dg $\mathcal{C}$-modules as objects and morphisms of dg functors as morphism spaces.

It is an abelian category, where epi- resp. monomorphisms can be checked degreewise. These categories will be the main object of study, together with their generalizations in §3.2.

As evaluating a $\mathcal{C}$-dg module in an object X of $\mathcal{C}$ yields a complex in $\mathrm{Ch}(k\text{-Mod})$, it is natural to define the following.

Definition 1.29. Let $\mathcal{C}$ be a small dg category. Let M be a $\mathcal{C}$-dg module. The *cohomology* $\mathrm{H}^\bullet(M)$ of M is the functor

$$\mathrm{H}^\bullet(M)\colon \mathrm{H}^\bullet(\mathcal{C}) \to k\text{-gr}\,\mathrm{Mod} : X \mapsto \mathrm{H}^\bullet(M(X)^\bullet). \tag{1.33}$$

Let $f : M \to N$ be a morphism between two dg $\mathcal{C}$-modules. It is called a *quasi-isomorphism* if it induces an isomorphism on the level of the cohomology of M and N, i.e. we require

$$\mathrm{H}^\bullet(f)(X)\colon \mathrm{H}^\bullet(M)(X) \xrightarrow{\sim} \mathrm{H}^\bullet(N)(X) \tag{1.34}$$

for every $X \in \mathrm{Obj}(\mathcal{C})$.

These definitions will be used in chapter 3 after we have introduced the theory of model categories in chapter 2.

CHAPTER 2

Model categories

> "And yet I have constructed in my mind a model city from which all possible cities can be deduced. It contains everything corresponding to the norm. Since the cities that exist diverge in varying degree from the norm, I need only foresee in the exceptions and calculate the most probable combinations."
>
> Kublai Khan in *Invisible Cities*
> ITALO CALVINO

In 1967 Daniel Quillen introduced model categories as a tool to generalize homological algebra and lift the homotopy theory of topological spaces to a general categorical setting [LNM43]. By doing so he unified these two subjects, and provided new tools.

When inverting morphisms in a category one both gains and loses things. On one hand the category becomes more adapted to the problem at hand, with some of the objects being isomorphic. But the morphisms in this category have become harder to describe. By using the (strong) axioms for a model category, which are suggested by algebraic topology, one overcomes this difficulty and obtains again a nice description, as in theorem 2.3.

We will first introduce model categories and some of the specific choices we have made here. Then the central structural result is given in theorem 2.3, followed by a brief sketch of how derived functors act in homotopical algebra. The chapter ends by tying model categories and dg enrichments together by introducing monoidal model categories.

2.1 Preliminaries

The contents of this section (and by extension the rest of the chapter) are inspired by [LNM43; MSM63; D-S; MSM99]. For historical reasons the axioms are numbered more or less the way Quillen defined model categories [LNM43] but we immediately use the notion of a *closed* model category. So whenever the term model category is used, implicitly it will mean closed model category.

Definition 2.1. A *model category* $\mathcal{C}$ is a category in which we distinguish three classes of morphisms. These are

- the *weak equivalences*, denoted $\xrightarrow{\sim}$, and the collection of all weak equivalences is denoted $\mathcal{W}$;
- the *fibrations*, denoted $\twoheadrightarrow$;
- the *cofibrations*, denoted $\hookrightarrow$.

Whenever a morphism is both a (co)fibration and a weak equivalence we will call it a *acyclic (co)fibration*. The category $\mathcal{C}$ and these three classes must satisfy the following axioms.

(MC0) all limits and colimits exist in $\mathcal{C}$, i.e. it is bicomplete;

(MC1) each morphism f can be factored as $f = p \circ i$ where either

(i) i is a cofibration and p an acyclic fibration;

(ii) i is an acyclic cofibration and p a fibration.

(MC2) consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd} \tag{2.1}$$

where the arrow $h\colon B \to X$ (if it exists) is called a *lift*, we require that lifts exist if

(i) i is a cofibration and p an acyclic fibration;

(ii) i is an acyclic cofibration and p a fibration.

(MC3) the classes of fibrations and cofibrations contain the isomorphisms and are stable under composition;

(MC4) if f is a retract of g and g is either a fibration, cofibration or weak equivalence, so is f;

(MC5) if f and g are morphisms in $\mathcal{C}$ such that $g \circ f$ is defined, then if two of these morphisms are weak equivalences, so is the third, and the class of weak equivalences contains all isomorphisms.

Remark 2.2. Several things should be said about this definition:

(i) Instead of bicompleteness originally only finitely bicompleteness was asked. But in every case we will consider we have the stronger version, and it makes certain arguments (e.g. the small object argument as discussed in appendix A.1) easier.

(ii) Axiom (MC4) is not Quillen's original axiom, but rather a combination of his axioms (MC4) and (MC6) where the latter is the notion of a closed model category. The equivalence of our axiom (MC4) with Quillen's axioms is explained in [D-S, propositions 3.13 and 3.14].

(iii) The bicompleteness implies the existence of both an initial object $\emptyset$and a terminal object $*$. We will say an object C of a model category $\mathcal{C}$ is a *fibrant object* if the map $C \to *$ is a fibration and similarly we will say it is a *cofibrant object* if the map $\emptyset \to C$ is cofibrant.

(iv) Let C be an object in a model category $\mathcal{C}$. If we apply (MC1)(i) to the map $\emptyset \to C$ we get a factorisation $\emptyset \hookrightarrow \mathrm{Q}(C) \overset{\sim}{\twoheadrightarrow} C$ where $\mathrm{Q}\colon \mathcal{C} \to \mathcal{C}$ is an endomorphism of $\mathcal{C}$ that we will call the *cofibrant replacement*. Hence for every object we get a weakly equivalent cofibrant object.

Likewise using (MC1)(ii) we can factor $C \to *$ as $C \overset{\sim}{\hookrightarrow} \mathrm{R}(C) \twoheadrightarrow *$ and call $\mathrm{R}\colon \mathcal{C} \to \mathcal{C}$ the *fibrant replacement*. These will correspond to the injective and projective resolutions of homological algebra, as explained in remark A.10.

(v) As in [MSM63] we will require that the factorisations in (MC1) are functorial [MSM63, definition 1.1.1(2)]. With this extra condition on our model categories certain constructions are more natural.

One has to make sure that while proving the model category structure on some category $\mathcal{C}$ the presence of functorial factorisation is taken care of. But this will be the case in the situations of interest, by lemma A.5.

2.2 Homotopy categories

As indicated in the introduction to this chapter, model categories are a tool to construct localisations of categories. Using the extra structure provided by the model category structure we will be able to describe this localised category more explicitly than in the standard case. This is the main use of model categories, and the remainder of this thesis can be considered as an application of this idea in the case of (dg modules over) dg categories.

The localisation $\mathcal{C}[S^{-1}]$ of a category $\mathcal{C}$ with respect to a set of morphisms S is defined by a universal property of the localisation functor

$$\gamma\colon \mathcal{C} \to \mathcal{C}[S^{-1}]. \tag{2.2}$$

If we consider a model category $\mathcal{C}$ with $S = \mathcal{W}$ its collection of weak equivalences the *homotopy category* $\operatorname{Ho}\mathcal{C} = \mathcal{C}[\mathcal{W}^{-1}]$ is well-defined [MSM63, theorem 1.2.10] and satisfies this hypothesis.

But more importantly, one can construct an explicit "model" for this homotopy category, which is a category equivalent to $\operatorname{Ho}\mathcal{C}$, making calculations easier. To do so one introduces the subcategories $\mathcal{C}_{\mathrm{cof}}$, $\mathcal{C}_{\mathrm{fib}}$ and $\mathcal{C}_{\mathrm{cof,fib}}$, which are full subcategories of $\mathcal{C}$ on the cofibrant (resp. fibrant, resp. both fibrant and cofibrant) objects. Hence we restrict our point of view to objects which are "nice" in terms of homotopical algebra. This point is elaborated upon in appendix A.2. By the presence of the fibrant and cofibrant replacements functors this is a reasonable restriction to make.

Then one introduces the homotopy relation on a model category [MSM63, definition 1.2.4], and proves that it is an equivalence relation for $\mathcal{C}_{\mathrm{cof,fib}}$ [MSM63, corollary 1.2.7]. This yields the following theorem, which proves the existence of $\mathcal{C}[\mathcal{W}^{-1}]$ as a category and the equivalence with the explicit "model" we have constructed.

Theorem 2.3 (The homotopy category as a quotient)**.** Let $\mathcal{C}$ be a model category. Then

(i) we have the equivalence of categories $\mathcal{C}_{\mathrm{cof,fib}}/\!\sim\, \cong \operatorname{Ho}\mathcal{C}$;

(ii) we have natural isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(\mathrm{Q}\circ\mathrm{R}(X), \mathrm{Q}\circ\mathrm{R}(Y))/\!\sim\, &\cong \operatorname{Hom}_{\operatorname{Ho}\mathcal{C}}(\gamma(X), \gamma(Y)) \\ &\cong \operatorname{Hom}_{\mathcal{C}}(\mathrm{R}\circ\mathrm{Q}(X), \mathrm{R}\circ\mathrm{Q}(Y))/\!\sim\end{aligned} \tag{2.3}$$

for $X, Y \in \operatorname{Obj}(\mathcal{C})$;

(iii) if moreover X is cofibrant and Y fibrant the natural isomorphisms from (ii) reduce to

$$\operatorname{Hom}_{\mathcal{C}}(X, Y)/\!\sim\, \cong \operatorname{Hom}_{\operatorname{Ho}\mathcal{C}}(\gamma(X), \gamma(Y)); \tag{2.4}$$

(iv) if $f : X \to Y$ is a morphism in $\mathcal{C}$ which is mapped to an isomorphism in $\operatorname{Ho}\mathcal{C}$ by the localisation, then f is a weak equivalence in $\mathcal{C}$.

2.3 Quillen functors and adjunctions

The notion of functors and adjunctions has a special role in homotopy theory, just like it does in general category theory. The well-known adjunctions arising from forgetful functors and base change in algebra or algebraic geometry will arise in the context of this thesis, as explained in §3.2.

The functors in homotopy theory are not required to preserve all properties of a model category, i.e. to send (co)fibrations and weak equivalences to (co)fibrations and weak equivalence in the codomain. To get interesting results it already suffices to preserve "half" of the structure, hence we get the following definition.

Definition 2.4. A *left Quillen functor* $F\colon \mathcal{C} \to \mathcal{D}$ between model categories $\mathcal{C}$ and $\mathcal{D}$ is a left adjoint functor that preserves cofibrations and acyclic fibrations. Likewise, a *right Quillen functor* $G\colon \mathcal{D} \to \mathcal{C}$ is a right adjoint functor that preserves fibrations and acyclic cofibrations.

So Quillen functors always come in adjoint pairs, and such a pair (F, G, Φ) with Φ the natural isomorphism

$$\Phi\colon \operatorname{Hom}_{\mathcal{C}}(F(-), -) \stackrel{\simeq}{\Rightarrow} \operatorname{Hom}_{\mathcal{D}}(-, G(-)). \tag{2.5}$$

is called a *Quillen adjunction*.

Given the construction of a homotopy category in §2.2 we would like to know what happens with our newly introduced Quillen functors. We obtain a notion of derived functors, which generalizes the well-known concept of derived functors from homological algebra, given the model category structure on $\mathrm{Ch}(k\text{-Mod})$ and the interpretation of the (co)fibrant replacement, as in remark A.10.

Definition 2.5. Let (F, G, Φ) be a Quillen adjunction between model categories $\mathcal{C}$ and $\mathcal{D}$. The *total left derived functor* $\mathbf{L}F\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ is defined by

$$\begin{tikzcd} \operatorname{Ho}\mathcal{C} \arrow[r, "\operatorname{Ho}\mathrm{Q}"] \arrow[dr, "\mathbf{L}F"'] & \operatorname{Ho}\mathcal{C}_{\mathrm{cof}} \arrow[d, "\operatorname{Ho}F"] \\ & \operatorname{Ho}\mathcal{D}, \end{tikzcd} \tag{2.6}$$

while the *total right derived functor* $\mathbf{R}G\colon \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$ is the composition

$$\begin{tikzcd} \operatorname{Ho}\mathcal{D} \arrow[r, "\operatorname{Ho}\mathrm{R}"] \arrow[dr, "\mathbf{R}G"'] & \operatorname{Ho}\mathcal{D}_{\mathrm{fib}} \arrow[d, "\operatorname{Ho}G"] \\ & \operatorname{Ho}\mathcal{C}. \end{tikzcd} \tag{2.7}$$

Whenever we have a natural transformation $\tau\colon F \Rightarrow F'$ (resp. $\tau\colon G \Rightarrow G'$) we define the *total derived natural transformation* $\mathbf{L}\tau$ (resp. $\mathbf{R}\tau$) to be $\operatorname{Ho}\tau \circ \operatorname{Ho}\mathrm{Q}$ (resp. $\operatorname{Ho}\tau \circ \operatorname{Ho}\mathrm{R}$), i.e. its action on each object is given by $(\mathbf{L}\tau)_X = \tau_{\mathrm{Q}(X)}$ (resp. $(\mathbf{R}\tau)_X = \tau_{\mathrm{R}(X)}$).

Remark 2.6. At this moment it should be noted that we need the factorisations from axiom (MC1) to be functorial, hence the (co)fibrant replacement Q and R are actual endofunctors. By making the functorial factorisation a part of the definition we can define total derived functors using only the data available in $\mathcal{C}$. Otherwise we have to assume that we are given functorial factorisations from outside. In all the cases we consider we can equip $\mathcal{C}$ with functorial factorisations, which is a consequence of the small object argument, see appendix A.1.

We do not have functoriality for total derived functors: if we consider the functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ we see that $\mathbf{L}\,\mathrm{id}_{\mathcal{C}} = \operatorname{Ho}\mathrm{Q}$ by definition of the total left derived tensor product, while we would need $\mathrm{id}_{\operatorname{Ho}\mathcal{C}}$. So unless $\mathcal{C} = \operatorname{Ho}\mathcal{C}$ and the cofibrant replacement is the identity, we do not have functoriality. The properties we *do* have are given in the next proposition.

Proposition 2.7. Let $\mathcal{C}$ be a model category.

(i) We have a natural transformation $\alpha_{\mathcal{C}}\colon \mathbf{L}\,\mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathrm{Ho}\,\mathcal{C}}$ that is a natural isomorphism.

Now let $F\colon \mathcal{C} \to \mathcal{D}$ and $F'\colon \mathcal{D} \to \mathcal{E}$ be left Quillen functors between model categories. There exists a natural isomorphism $\mathrm{m}_{F',F}\colon \mathbf{L}F' \circ \mathbf{L}F \Rightarrow \mathbf{L}(F' \circ F)$. This isomorphism satisfies three properties.

(ii) Let $F\colon \mathcal{C} \to \mathcal{D}$, $F'\colon \mathcal{D} \to \mathcal{E}$ and $F''\colon \mathcal{E} \to \mathcal{F}$ be three left Quillen functors, then the *associativity coherence diagram*

$$\begin{tikzcd}
(\mathbf{L}F'' \circ \mathbf{L}F') \circ \mathbf{L}F \arrow[r, Rightarrow, "\mathrm{m}_{F'',F'} \circ \mathbf{L}F"] \arrow[d, equal] & \mathbf{L}(F'' \circ F') \circ \mathbf{L}F \arrow[r, Rightarrow, "\mathrm{m}_{(F'' \circ F'),F}"] & \mathbf{L}\left((F'' \circ F') \circ F\right) \arrow[d, equal] \\
\mathbf{L}F'' \circ (\mathbf{L}F' \circ \mathbf{L}F) \arrow[r, Rightarrow, "\mathbf{L}F'' \circ \mathrm{m}_{F',F}"] & \mathbf{L}F'' \circ \mathbf{L}(F' \circ F) \arrow[r, Rightarrow, "\mathrm{m}_{F'',F' \circ F}"] & \mathbf{L}\left(F'' \circ (F' \circ F)\right)
\end{tikzcd} \tag{2.8}$$

commutes.

(iii) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a left Quillen functor, then the *left unit coherence diagram*

$$\begin{tikzcd}
\mathbf{L}\,\mathrm{id}_{\mathcal{D}} \circ \mathbf{L}F \arrow[r, Rightarrow, "\mathrm{m}_{\mathrm{id}_{\mathcal{D}},F}"] \arrow[d, Rightarrow, "\alpha_{\mathcal{D}} \circ \mathrm{id}_{\mathbf{L}F}"'] & \mathbf{L}(\mathrm{id}_{\mathcal{D}} \circ F) \arrow[d, equal] \\
\mathrm{id}_{\mathrm{Ho}\,\mathcal{D}} \circ \mathbf{L}F \arrow[r, equal] & \mathbf{L}F
\end{tikzcd} \tag{2.9}$$

commutes.

(iv) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a left Quillen functor, then the *right unit coherence diagram*

$$\begin{tikzcd}
\mathbf{L}F \circ \mathbf{L}\,\mathrm{id}_{\mathcal{C}} \arrow[r, Rightarrow, "\mathrm{m}_{F,\mathrm{id}_{\mathcal{C}}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\mathbf{L}F} \circ \alpha_{\mathcal{C}}"'] & \mathbf{L}(F \circ \mathrm{id}_{\mathcal{C}}) \arrow[d, equal] \\
\mathbf{L}F \circ \mathrm{id}_{\mathrm{Ho}\,\mathcal{C}} \arrow[r, equal] & \mathbf{L}F
\end{tikzcd} \tag{2.10}$$

commutes.

It is possible that a Quillen adjunction (F, G, Φ) is not an equivalence of categories on the level of the original categories, but that the *derived adjunction* $\mathbf{L}(F, G, \Phi) := (\mathbf{L}F, \mathbf{R}G, \mathbf{R}\Phi)$ as obtained in [MSM63, lemma 1.3.10] is one. In that case we can define

Definition 2.8. Let (F, G, Φ) be a Quillen adjunction between model categories $\mathcal{C}$ and $\mathcal{D}$. We call it a *Quillen equivalence* if for X a cofibrant object of $\mathcal{C}$ and Y a fibrant object of $\mathcal{D}$ a map $f : F(X) \to Y$ is a weak equivalence in $\mathcal{D}$ if $\Phi(f)\colon X \to G(Y)$ is a weak equivalence in $\mathcal{C}$. In other words, weak equivalences are identified under the natural isomorphism Φ with the weak equivalences.

2.4 Ch(k-Mod)-model categories

We now wish to equip model categories with an extra structure, which arises from the theory of monoidal categories. This is nothing but the well-known generalization of rings and modules to the situation of model categories. Hence if $\mathcal{A}$ is a monoidal (or "ring-like") model category we wish to consider it as a coefficient ring, and define an $\mathcal{A}$-model category which consists of "modules" over this category. This ties in with the notion of dg modules over a dg category, as introduced in §1.3. Not all the notions of [MSM63, chapter 4] are reproduced explicitly.

Definition 2.9. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be model categories. Let $(\otimes, \mathrm{Hom}_{\mathrm{r}}, \mathrm{Hom}_{\ell}, \varphi_{\mathrm{r}}, \varphi_{\ell})$ be an adjunction of two variables. We say it is a *Quillen adjunction of two variables* if for every cofibration $f: U \hookrightarrow V$ in $\mathcal{C}$ and every cofibration $g: W \to X$ in $\mathcal{D}$ the *pushout product*

$$\begin{tikzcd}
U \otimes W \arrow[r] \arrow[d] & U \otimes X \arrow[d] \arrow[ddr] & \\
V \otimes W \arrow[r] \arrow[drr] & V \otimes W \sqcup_{U\otimes W} U \otimes X \arrow[dr, "f \square g"] & \\
& & V \otimes X
\end{tikzcd} \tag{2.11}$$

is again a cofibration in $\mathcal{E}$. If moreover f or g is acyclic then so is $f \square g$. The left adjoint is then called a *Quillen bifunctor*.

This allows us to define the compatibility of the monoidal and the model category structures.

Definition 2.10. Let $(\mathcal{A}, \otimes)$ be a monoidal category with a model category structure. We say $\mathcal{A}$ is a *monoidal model category* if $- \otimes -$ is a Quillen bifunctor and for all cofibrant $A \in \mathrm{Obj}(\mathcal{A})$ the maps $\mathrm{Q}(\mathbf{1}) \otimes A \to \mathbf{1} \otimes A$ and $A \otimes \mathrm{Q}(\mathbf{1}) \to A \otimes \mathbf{1}$ are weak equivalences, where $\mathbf{1}$ denotes the unit in the monoidal structure.

If the unit for the monoidal structure is cofibrant (which will be the case in this text) the second condition is vacuous. The condition on the pushout product in definition 2.9 on the other hand will turn out the be problematic, as discussed in example 4.28.

There is a general theory of $\mathcal{A}$-model categories [MSM63, chapter 4]. But we will only consider "the coefficient categories" $\mathrm{Ch}(k\text{-Mod})$ and sSet, where the latter is the category of simplicial sets.

Definition 2.11. Let $\mathcal{A}$ be either $\mathrm{Ch}(k\text{-Mod})$ or sSet together with their structures of a (symmetric) monoidal category given by the tensor product which we will denote $- \otimes -$ in both cases here. A *(left) $\mathcal{A}$-module* $\mathcal{M}$ is given by a functor

$$- \otimes - : \mathcal{A} \times \mathcal{M} \to \mathcal{M} \tag{2.12}$$

such that we have the natural isomorphisms $A \otimes (B \otimes M) \cong (A \otimes B) \otimes M$ and $\mathbf{1} \otimes M \cong M$ for $A, B \in \mathrm{Obj}(\mathcal{A})$ and $M \in \mathrm{Obj}(\mathcal{M})$.

Now we can state the extra conditions we need to impose to ensure compatibility with the model category structures.

Definition 2.12. Let $\mathcal{M}$ be both an $\mathcal{A}$-module and a model category. We say it is an $\mathcal{A}$)*-model category* if the scalar multiplication $- \otimes - : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ is a Quillen bifunctor and for all $M \in \mathrm{Obj}(\mathcal{M})$ cofibrant the map $\mathrm{Q}(\mathbf{1}) \otimes M \to \mathbf{1} \otimes M$ is a weak equivalence.

Example 2.13. If we take $\mathcal{A} = \mathrm{Ch}(k\text{-Mod})$ the "scalar multiplication" from (2.12) can be taken to be $- \otimes_k -$, i.e. the module structure is obtained from the monoidal structure on $\mathrm{Ch}(k\text{-Mod})$. In this case $\mathrm{Ch}(k\text{-Mod})$ is *symmetric monoidal model category* [MSM63, definition 4.2.6].

Example 2.14. If we on the other hand take $\mathcal{A} = \mathrm{sSet}$ the "scalar multiplication" from (2.12) corresponds to the tensor and cotensor enrichment using simplicial sets. In this case we will call a sSet-model category a *simplicial model category*.

Example 2.15. The main example in which we will be interested for the applications (see chapter 5) is the case of sheaves. If we take X a ringed space over k (resp. a scheme over k), and consider $\mathrm{Ch}(\mathcal{O}_X\text{-Mod})$ (resp. $\mathrm{Ch}(\mathrm{Qcoh}_X)$) the category of complexes of $\mathcal{O}_X$-modules (resp. the category of complexes of quasicoherent sheaves) we can equip it with a model category structure by applying the ideas of [Hov01] as discussed in appendix A. The particular case we use in §5.3 is the category of quasicoherent sheaves Qcoh_X on a quasi-compact and quasi-separated scheme X.

The Ch(k-Mod)-module structure is given by defining $M^\bullet \otimes \mathcal{F}^\bullet$ to be the sheafification of the presheaf

$$U \mapsto M^\bullet \otimes_k \mathcal{F}^\bullet(U) \tag{2.13}$$

where $U \subseteq X$ open, $M^\bullet \in \mathrm{Obj}(\mathrm{Ch}(k\text{-Mod}))$ and $\mathcal{F}^\bullet \in \mathrm{Ch}(\mathcal{O}_X\text{-Mod})$ (resp. $\mathrm{Ch}(\mathrm{Qcoh}_X)$).

The main reason for defining this is the fact that the structure is compatible with deriving model categories, i.e. when $\mathcal{C}$ is a Ch(k-Mod)-model category we get that Ho($\mathcal{C}$) is enriched over Ho(Ch(k-Mod)). The derived $\mathcal{H}$om of $\mathcal{C}$ is given by

$$\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{C}}(X,Y)^\bullet := \mathcal{H}\mathrm{om}_{\mathcal{C}}(\mathrm{Q}(X),\mathrm{R}(Y))^\bullet \tag{2.14}$$

for $X, Y \in \mathrm{Obj}(\mathcal{C})$, which is an object in Ho(Ch(k-Mod)). This implies that we can calculate the Homs in $\mathrm{Ho}\,\mathcal{C}$ for X and Y fibrant-cofibrant by

$$\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(X,Y) \cong \mathrm{H}^0(\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{C}}(X,Y)^\bullet) \tag{2.15}$$

which is a motivation for localising with respect to a model category structure.

CHAPTER 3

Model category structures and differential graded categories

"I have also thought of a model city from which I deduce all others. It is a city made only of exceptions, exclusions, incongruities, contradictions. If such a city is the most improbable, by reducing the number of abnormal elements, we increase the probability that the city really exists. So I have only to subtract exceptions from my model, and in whatever direction I proceed, I will arrive at one of the cities which, always as an exception, exist. But I cannot force my operation beyond a certain limit: I would achieve cities too probable to be real."

Marco Polo in *Invisible Cities*
ITALO CALVINO

Having defined dg categories and model categories in the previous chapters we can now put these notions to use. In this chapter we will discuss several ways in which model categories can be used in the context of dg categories. These will be

(i) a model category structure on $\mathrm{dg}\,\mathrm{Cat}_k$ itself, §3.1;

(ii) for every dg category $\mathcal{C}$ a model category structure on $\mathcal{C}$-$\mathrm{dg}\,\mathrm{Mod}_k$, §3.2;

(iii) generalisations of the categories $\mathcal{C}$-$\mathrm{dg}\,\mathrm{Mod}_k$ to dg module categories with values in arbitrary $\mathrm{Ch}(k\text{-Mod})$-model categories, §3.2.

The main sources of inspiration for this chapter are [Kel06; LNM2008; Toe07; TV07].

3.1 A model category structure on $\mathrm{dg}\,\mathrm{Cat}_k$

The morphisms we would like to invert in $\mathrm{dg}\,\mathrm{Cat}_k$ are the quasi-equivalences. These are a type of morphisms specific to dg categories and are the dg enriched analog of categorical equivalences. The invariants we are interested in in this thesis are compatible with this structure. It is possible to consider a different collection of weak equivalences, the Morita equivalences [Tab05b; Tab07a; Tab07b], see remark B.22. For certain applications these would be more appropriate, but for the ones discussed here quasi-equivalences will suffice.

Definition 3.1. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{dg}\,\mathrm{Cat}_k$, i.e. a dg functor between dg categories. It is said to be *quasi-fully faithful* if for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ the map

(3.1) $f_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet \to \mathrm{Hom}_{\mathcal{D}}(f(X), f(Y))^\bullet$

of cochain complexes is a quasi-isomorphism. It is said to be *quasi-essentially surjective* if the induced functor

$$\mathrm{H}^0(f)\colon \mathrm{H}^0(\mathcal{C}) \to \mathrm{H}^0(\mathcal{D}) \tag{3.2}$$

on the level of (k-linear) categories is essentially surjective.

These are versions of the two conditions necessary to define the equivalence of categories that are compatible with the enrichment, hence we can define the following.

Definition 3.2. Let $f:\mathcal{C}\to\mathcal{D}$ be a morphism in $\mathrm{dg}\,\mathrm{Cat}_k$. It is said to be a *quasi-equivalence* if it is both quasi-fully faithful and quasi-essentially surjective.

We will take the weak equivalences in our model category structure on $\mathrm{dg}\,\mathrm{Cat}_k$ to be the quasi-equivalences. Hence to be able to localize $\mathrm{dg}\,\mathrm{Cat}_k$ it suffices to specify the class of fibrations [MSM63, lemma 1.1.10]. To do so we apply the same philosophy as is used in defining the model category structure on Ch(k-Mod) in appendix A.2, i.e. we use a pointwise requirement. But we also ask for a certain lifting property to get compatibility with the homotopy categories.

Definition 3.3. Let $f:\mathcal{C}\to\mathcal{D}$ be a morphism in $\mathrm{dg}\,\mathrm{Cat}_k$. It is said to be a *quasi-fibration* if

(i) for all $X,Y\in\mathrm{Obj}(\mathcal{C})$ the map

$$f_{X,Y}\colon \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet \to \mathrm{Hom}_{\mathcal{D}}(f(X),f(Y))^\bullet \tag{3.3}$$

is a fibration in Ch(k-Mod) for the projective model category structure on Ch(k-Mod), i.e. is it an epimorphism in every degree (see also table A.1);

(ii) for all $X\in\mathrm{Obj}(\mathcal{C})=\mathrm{Obj}(\mathrm{H}^0(\mathcal{C}))$ and for all isomorphisms $v\colon \mathrm{H}^0(f)(X)\to Y'$ in $\mathrm{H}^0(\mathcal{D})$ we can lift v to an isomorphism $u\colon X\to Y$ in $\mathrm{H}^0(\mathcal{C})$ such that $\mathrm{H}^0(f)(u)=v$.

So if we take in addition to the quasi-equivalences as weak equivalences the quasi-fibrations as fibrations we can prove that these induce a model category structure on $\mathrm{dg}\,\mathrm{Cat}_k$ [Tab05a]. A detailed discussion of the proof can be found in appendix B. This structure is crucial for the remainder of the text.

In case a morphism in $\mathrm{dg}\,\mathrm{Cat}_k$ is not quasi-essentially surjective we can still be interested in its essential image, which we will generalize accordingly.

Definition 3.4. Let $f:\mathcal{C}\to\mathcal{D}$ be a morphism in $\mathrm{dg}\,\mathrm{Cat}_k$. Its *quasi-essential image* is the full dg subcategory on all objects $Y\in\mathrm{Obj}(\mathcal{D})$ such that $\mathrm{H}^0(Y)$ lies in the essential image of the functor $\mathrm{H}^0(f)$ between the (k-linear) categories $\mathrm{H}^0(\mathcal{C})$ and $\mathrm{H}^0(\mathcal{D})$.

The model category structure on $\mathrm{dg}\,\mathrm{Cat}_k$ satisfies some extra properties. These are given in the next lemmas. The first one is obvious but we will nevertheless give a full proof.

Lemma 3.5. Let $\mathcal{C}$ be a small dg category. Then $\mathcal{C}$ is a fibrant object in $\mathrm{dg}\,\mathrm{Cat}_k$.

Proof. The final object in $\mathrm{dg}\,\mathrm{Cat}_k$ is the dg category $\{*\}$ with a single object $*$, as explained in remark 1.20. The map $f:\mathcal{C}\to\{*\}$ is the obvious (and only) one sending every object to $*$ and every Hom-complex to the zero complex. For all $X,Y\in\mathrm{Obj}(\mathcal{C})$ we see that

$$f_{X,Y}\colon \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet \to \mathrm{Hom}_{\{*\}}(f(X),f(Y))^\bullet = \mathrm{Hom}_{\{*\}}(*,*)^\bullet = 0 \tag{3.4}$$

is an epimorphism in Ch(k-Mod).

For the second condition in definition 3.3, every isomorphism $v\colon \mathrm{H}^0(f)(X)=*\to *$ is actually the (zero) identity map id_*. So taking $u:=\mathrm{id}_{\mathcal{C}}$ we get $\mathrm{H}^0(f)(X)(u)=v$. So all the conditions to be a quasi-fibration are fulfilled. □

The next lemma is an important result on the cofibrant replacement functor of $\mathrm{dg}\,\mathrm{Cat}_k$. Together with lemma 3.7 it provides important insight in the cofibrant objects of $\mathrm{dg}\,\mathrm{Cat}_k$.

Lemma 3.6. We can choose the cofibrant replacement functor Q on $\mathrm{dg\,Cat}_k$ such that it is the identity on the objects, for all $\mathcal{C} \in \mathrm{Obj}(\mathrm{dg\,Cat}_k)$.

Proof. In the proof of the model category structure on $\mathrm{dg\,Cat}_k$ in appendix B the set of generating cofibrations consists of two types of morphisms. Because both are the identity on the objects every cofibration is the identity on the objects. Remark that we don't make the distinction between $\mathrm{dg\,Cat}_k$ and its pointed version $\mathrm{dg\,Cat}_{k,*}$. □

From now on we will fix this choice of cofibrant replacement functor. The next lemma relates the model category structures on $\mathrm{dg\,Cat}_k$ and $\mathrm{Ch}(k\text{-Mod})$, and will prove the necessary information to use lemma 3.14, as explained in remark 3.17.

Lemma 3.7. Let $\mathcal{C}$ be a cofibrant object in $\mathrm{dg\,Cat}_k$. For all $X, Y \in \mathrm{Obj}(\mathcal{C})$ we have that the morphism complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet$ is a cofibrant object in $\mathrm{Ch}(k\text{-Mod})$.

Proof. By the small object argument every cofibrant object can be written as the transfinite composition of pushouts of generating cofibrations. Moreover, the Hom's in a category always commute with filtered colimits, and the filtered colimit over cofibrations is again a cofibration. So to prove that $\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet$ is a cofibrant cochain complex it suffices to prove that this property is preserved under pushout along a generating cofibration. But by the description of the generating cofibrations in appendix B this is clear:

(i) pushout along Q does not change to cochain complexes;

(ii) pushout along $S(n)$ preserves the objects and is a pushout along a generating cofibration of the model category structure on $\mathrm{Ch}(k\text{-Mod})$, see appendix A.2.

□

Now we introduce a notion in the general context of model categories, which in lemma 3.9 is characterised in the case of the model category structure on $\mathrm{dg\,Cat}_k$. Another characterisation in the presence of a simplicial model category structure is discussed in lemma 4.6, after having introduced the notion of simplicial model categories, and imposing this structure on $\mathrm{dg\,Cat}_k$.

Definition 3.8. Let $\mathcal{M}$ be a model category. Let $f : X \to Y$ be a morphism in $\mathcal{M}$. We call f a *homotopy monomorphism* if the natural diagonal map $\Delta_X^{\mathrm{h}} : X \to X \times_Y^{\mathrm{h}} X$ is a weak equivalence.

In case of the model category structure on $\mathrm{dg\,Cat}_k$ we can give a useful characterisation of homotopy monomorphisms using the notion of quasi-fully faithfulness of dg functors.

Lemma 3.9. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism in $\mathrm{dg\,Cat}_k$. Then f is a homotopy monomorphism if and only if it is quasi-fully faithful.

Proof. We can take a fibrant replacement, so f is assumed to be a (quasi-)fibration in $\mathrm{dg\,Cat}_k$. Then the homotopy pullback becomes an ordinary pullback, and the morphism we should consider is

(3.5) $\Delta : \mathcal{C} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{C}$.

Assume that f is a homotopy monomorphism, i.e. hen (3.5) is a quasi-equivalence. By the quasi-fully faithfulness of (3.5) we get that the induced

(3.6) $\Delta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet \to \mathrm{Hom}_{\mathcal{C}\times_{\mathcal{D}}\mathcal{C}}(\Delta(X),\Delta(Y))^\bullet$

are quasi-isomorphisms, for all $X, Y \in \mathrm{Obj}(\mathcal{C})$. Using lemma 1.26 we get

(3.7) $\mathrm{Hom}_{\mathcal{C}\times_{\mathcal{D}}\mathcal{C}}(\Delta(X),\Delta(Y))^\bullet \cong \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet \times_{\mathrm{Hom}_{\mathcal{D}}(f(X),f(Y))^\bullet} \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet$

hence we can consider

(3.8) $\mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet} \to \mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet} \times_{\mathrm{Hom}_{\mathcal{D}}(f(X),f(Y))^{\bullet}} \mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet}$.

This implies that the morphisms (by abuse of notation)

(3.9) $\Delta_{X,Y}\colon \mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet} \to \mathrm{Hom}_{\mathcal{D}}(f(X),f(Y))^{\bullet}$

are homotopy monomorphisms in Ch(k-Mod). As the homotopy groups and the cohomology groups coincide for the model category structure on Ch(k-Mod) and this model category structure is moreover stable [MSM63, chapter 7] we can move the isomorphisms in $\pi_i \cong \mathrm{H}^i$ for $i \geq 1$ around to obtain the isomorphism in *every* degree. So it is actually a quasi-isomorphism, and f is quasi-fully faithful.

Now assume that f is quasi-fully faithful. Because f was assumed to be a (quasi-)fibration, the morphism

(3.10) $\mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet} \to \mathrm{Hom}_{\mathcal{D}}(f(X),f(Y))^{\bullet}$

is a fibration in Ch(k-Mod). Hence by quasi-fully faithfulness it is a acyclic fibration. So we get that the morphism in (3.8) is a quasi-isomorphism, because the projection is again a acyclic fibration as this is preserved under pullback. Then we apply the 3-out-of-2 axiom to $\Delta_{X,Y}$, the identity on $\mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet}$ and the projection. So the dg functor Δ is *quasi-fully faithful*.

To see that Δ is also quasi-essentially surjective, consider an object $T \in \mathrm{Obj}(\mathcal{C} \times_{\mathcal{D}} \mathcal{C})$. By lemma 1.26 this can be considered as a tuple $(X,Y) \in \mathcal{C} \times \mathcal{C}$ such that $f(X) = f(Y)$. So we can consider the identity morphism $f(X) \to f(Y)$ in $\mathrm{H}^0(\mathcal{D})$, and by the quasi-fully faithfulness this is lifted to an isomorphism $u\colon X \to Y$ in $\mathrm{H}^0(\mathcal{C})$. Because (3.10) is a fibration this isomorphism can be considered in $\mathrm{Z}^0(\mathrm{Hom}_{\mathcal{C}}(X,Y)^{\bullet})$, and we see that it is mapped to the identity by $\mathrm{H}^0(f)$. So the tuple (X,Y) can actually be taken to be $\Delta(X)$ when looking on the H^0-level, so Δ is also *quasi-essentially surjective*. Hence it is a quasi-equivalence. □

3.2 Differential graded modules with coefficients in a model category

We can generalize the notion of a dg $\mathcal{C}$-module introduced in §1.3, which has values in the model category Ch(k-Mod) to "dg $\mathcal{C}$-modules with coefficients in a Ch(k-Mod)-model category $\mathcal{M}$", a notion introduced in §2.4. Observe that we have the Quillen adjunction

(3.11) $f_!\colon \mathcal{C}\text{-dg}\,\mathrm{Mod}_k \rightleftarrows \mathcal{D}\text{-dg}\,\mathrm{Mod}_k : f^*$

for $f\colon \mathcal{C} \to \mathcal{D}$ a dg functor, analogous to the classical case, by using the composition and the forgetful functor. This Quillen adjunction will lift to the more general situation we will introduce, and if f is a quasi-equivalence of dg categories it will even be a Quillen equivalence, which is the important result of this section.

Definition 3.10. Let $\mathcal{C}$ be a dg category and $\mathcal{M}$ a cofibrantly generated Ch(k-Mod)-model category. A *dg $\mathcal{C}$-module with values in* $\mathcal{M}$ is a dg functor $\mathcal{C} \to \mathcal{M}$, where $\mathcal{M}$ is a dg category because its Hom-sets are by assumption enriched over Ch(k-Mod).

The *category of dg $\mathcal{C}$-modules with values in* $\mathcal{M}$ has as objects the dg functors $\mathcal{C} \to \mathcal{M}$ and its morphisms are given by the complexes of graded morphisms (or natural transformations). It will be denoted $\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})$.

Proposition 3.11 (Model category structure on $\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})$)**.** Under the assumptions of definition 3.10 the category $\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})$ has a cofibrantly generated model category structure.

Proof. This is a modification of [MSM99, theorems 11.6.1 and 11.7.3] and it will be discussed in some detail because it is an important construction in the theory of model categories. The notation is as explained in [MSM99, §11.5].

Let I and J denote the generating cofibrations and the generating acyclic cofibrations of $\mathcal{M}$. The set of generating cofibrations of $\mathcal{C}$-dg$\,\mathrm{Mod}_k(\mathcal{M})$ is $\mathbf{F}_I^{\mathcal{C}}$ where $\mathbf{F}_I^{\mathcal{C}}$ is the set of morphisms in $\mathcal{C}$-dg$\,\mathrm{Mod}_k(\mathcal{M})$ given by the *free I-cell at C*

$$\mathbf{F}^C_{M_1} \to \mathbf{F}^C_{M_2} \tag{3.12}$$

for all $C \in \mathrm{Obj}(\mathcal{C})$ and $M_1 \to M_2$ in I. The objects $\mathbf{F}^C_{M_i}$ are the *free diagrams on M_i generated by C* and are given by functors that for $D \in \mathrm{Obj}(\mathcal{C})$ give

$$\mathbf{F}^C_{M_i}(D) := \coprod_{\mathrm{Hom}_{\mathcal{C}}(C,D)^\bullet} M_i. \tag{3.13}$$

The morphism (3.12) is then described by

$$\coprod_{\mathrm{Hom}_{\mathcal{C}}(C,D)^\bullet} M_1 \;\to \coprod_{\mathrm{Hom}_{\mathcal{C}}(C,D)^\bullet} M_2. \tag{3.14}$$

Similarly the generating acyclic cofibrations are taken to be $\mathbf{F}_J^{\mathcal{C}}$. One can then prove that these induce a cofibrantly generated model category structure on $\mathcal{C}$-dg$\,\mathrm{Mod}_k(\mathcal{M})$, as is done in [MSM99, theorem 11.6.1].

This model category has moreover an induced Ch(k-Mod)-model structure. To prove this it suffices to copy the definitions and the proof of [MSM99, §11.7] for the case of cochain complexes instead of simplicial sets. □

Remark 3.12. If we take $\mathcal{M} = \mathrm{Ch}(k\text{-Mod})$, which is a cofibrantly generated Ch(k-Mod)-model category structure as explained in §2.4 we recover the original definition of a dg $\mathcal{C}$-module from §1.3, i.e. we can write

$$\mathcal{C}\text{-dg}\,\mathrm{Mod}_k = \mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathrm{Ch}(k\text{-Mod})), \tag{3.15}$$

and $\mathcal{C}$-dg$\,\mathrm{Mod}_k$ now comes equipped with a model category structure.

Remark 3.13. The Quillen adjunction (3.11) generalises to a Quillen adjunction

$$f_!\colon \mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}) \rightleftarrows \mathcal{D}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}) : f^* \tag{3.16}$$

for every $f : \mathcal{C} \to \mathcal{D}$ in dg$\,\mathrm{Cat}_k$. This can be derived into the adjunction

$$\mathbf{L} f_!\colon \mathrm{Ho}\left(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})\right) \rightleftarrows \mathrm{Ho}\left(\mathcal{D}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})\right) : f^*. \tag{3.17}$$

Lemma 3.14 (Quasi-equivalences induce Quillen equivalences)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be a quasi-equivalence of dg categories, and $\mathcal{M}$ a cofibrantly generated Ch(k-Mod)-model category. We assume that the domains and codomains of the set of generating cofibrations for $\mathcal{M}$ are cofibrant objects in $\mathcal{M}$. We also assume that either

(i) if $X \in \mathrm{Obj}(\mathcal{M})$ is cofibrant and $M^\bullet \to N^\bullet$ a quasi-isomorphism in Ch(k-Mod) then

$$M^\bullet \otimes X \to N^\bullet \otimes A \tag{3.18}$$

is a weak equivalence in $\mathcal{M}$;

(ii) the dg modules $\mathrm{Hom}_{\mathcal{C}}(C, C')^\bullet$ and $\mathrm{Hom}_{\mathcal{D}}(D, D')^\bullet$ are cofibrant objects in Ch(k-Mod) for all $C, C' \in \mathrm{Obj}(\mathcal{C})$ and $D, D' \in \mathrm{Obj}(\mathcal{D})$.

Then the Quillen adjunction $(f_!, f^*)$ for $\mathcal{C}$-dg$\,\mathrm{Mod}_k(\mathcal{M})$ and $\mathcal{D}$-dg$\,\mathrm{Mod}_k(\mathcal{M})$ induces a Quillen equivalence $(\mathbf{L} f_!, f^*)$ between Ho($\mathcal{C}$-dg$\,\mathrm{Mod}_k(\mathcal{M})$) and Ho($\mathcal{D}$-dg$\,\mathrm{Mod}_k(\mathcal{M})$).

Proof. Using [MSM63, corollary 1.3.16] it suffices to prove that

$$\mathbf{R}f^* = f^* \colon \operatorname{Ho}(\mathcal{D}\text{-dg}\operatorname{Mod}_k(\mathcal{M})) \to \operatorname{Ho}(\mathcal{C}\text{-dg}\operatorname{Mod}_k(\mathcal{M})) \tag{3.19}$$

reflects isomorphisms and that the adjunction morphism

$$\mathrm{id}_{\operatorname{Ho}(\mathcal{C}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))} \Rightarrow f^* \circ \mathbf{L}f_! \tag{3.20}$$

is an isomorphism.
For the first condition, remark that the weak equivalences in $\mathcal{D}$-dg $\operatorname{Mod}_k(\mathcal{M})$ are defined objectwise. So if F and G are weakly equivalent dg modules with coefficients in $\mathcal{D}$ their restrictions $f^*(F)$ and $f^*(G)$ evaluated in an object X of $\mathcal{C}$ yield weakly equivalent evaluations $F(f(X))$ and $G(f(X))$ in $\mathcal{M}$, hence $f^*(F)$ and $f^*(G)$ are weakly equivalent, hence the underived restriction functor f^* preserves weak equivalences. As f is quasi-essentially surjective, the derived restriction functor f^* reflects isomorphisms as $\operatorname{Ho}(f)$ is essentially surjective on the level of the homotopy categories.
For the second condition we can choose the set of generating cofibrations for $\mathcal{C}$-dg $\operatorname{Mod}_k(\mathcal{M})$ to be the morphisms $\mathrm{h}^X \otimes^{\mathbf{L}} M \to \mathrm{h}^X \otimes^{\mathbf{L}} N$ for $M \hookrightarrow N$ a generating cofibration in $\mathcal{M}$, where we have also used the fact that a contravariant functor is the colimit of representable functors. Moreover, given the assumptions on $\mathcal{M}$ we can write an object F in $\operatorname{Ho}(\mathcal{C}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))$ as the homotopy colimit of objects $\mathrm{h}^X \otimes^{\mathbf{L}} M$, where M is a cofibrant object in $\mathcal{M}$ and X an arbitrary object in $\mathcal{C}$.
Because $\mathbf{L}f_!$ is a left (Quillen) adjoint and f^* is both a left and right adjoint these functors commute with homotopy colimits. So it suffices to prove that the map

$$\mathrm{h}^X \otimes^{\mathbf{L}} M \to f^* \circ \mathbf{L}f_!(\mathrm{h}^X \otimes^{\mathbf{L}} M) \tag{3.21}$$

is an isomorphism in $\operatorname{Ho}(\mathcal{C}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))$. As we have the chain of isomorphisms

$$\begin{aligned}
&\operatorname{Hom}_{\operatorname{Ho}(\mathcal{D}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))}\left(\mathbf{L}f_!(\mathrm{h}^X \otimes^{\mathbf{L}} M), F\right) \\
&\quad\cong \operatorname{Hom}_{\operatorname{Ho}(\mathcal{C}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))}\left(\mathrm{h}^X \otimes^{\mathbf{L}} M, f^*(F)\right) && (3.17) \\
&\quad\cong \operatorname{Hom}_{\operatorname{Ho}(\mathcal{C}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))}\left(\mathrm{h}^X, \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{M}}(M, f^*(F))^\bullet\right) && \text{derived } (1.32) \\
&\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{M}}\left(M, f^*(F)(X)\right)^\bullet && \text{Yoneda} \\
&\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{M}}\left(M, F(f(X))\right)^\bullet && \text{definition of } f^* \\
&\quad\cong \operatorname{Hom}_{\operatorname{Ho}(\mathcal{D}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))}\left(\mathrm{h}^{f(X)}, \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{M}}(M, F)^\bullet\right) && \text{Yoneda} \\
&\quad\cong \operatorname{Hom}_{\operatorname{Ho}(\mathcal{D}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))}\left(\mathrm{h}^{f(X)} \otimes^{\mathbf{L}} M, F\right) && \text{derived } (1.32)
\end{aligned} \tag{3.22}$$

for all $F \in \operatorname{Obj}(\mathcal{D}\text{-dg}\operatorname{Mod}_k(\mathcal{M}))$ we can conclude that

$$\mathbf{L}f_!(\mathrm{h}^X \otimes^{\mathbf{L}} M) \cong \mathrm{h}^{f(X)} \otimes^{\mathbf{L}} M. \tag{3.23}$$

So we see that (3.21) is reduced to proving the isomorphism

$$\mathrm{h}^X \otimes^{\mathbf{L}} M \to f^* \circ \mathbf{L}f_!(\mathrm{h}^X \otimes^{\mathbf{L}} M) \cong f^*\left(\mathrm{h}^{f(X)} \otimes^{\mathbf{L}} M\right). \tag{3.24}$$

We evaluate this morphism in all $Y \in \operatorname{Obj}(\mathcal{C})$ and we obtain.

$$f_{X,Y} \otimes^{\mathbf{L}} \mathrm{id}_M \colon \operatorname{Hom}_{\mathcal{C}}(X,Y)^\bullet \otimes M \to \operatorname{Hom}_{\mathcal{D}}(f(X), f(Y))^\bullet \otimes M \tag{3.25}$$

Because f assumed to be a quasi-equivalence it is quasi-fully faithful, hence the cochain complexes in (3.25) are quasi-isomorphisms. So if condition (i) is satisfied the morphism in (3.25) is an isomorphism in the homotopy category. If condition (ii) is fulfilled we get the isomorphim by the definition of a Ch(k-Mod)-model category. □

In order to apply this result to the situation we will encounter later in proposition 4.15 and lemma 4.20 we need an intermediate result which is an important structural result of $\mathcal{C}$-dg $\mathrm{Mod}_k(\mathcal{M})$. Recall that for a Quillen functor we only required that it preserves "half" of the structure, i.e. cofibrations and acyclic fibrations or vice versa. But if we consider the evaluation functor ev_X for $X \in \mathrm{Obj}(\mathcal{C})$ defined by

$$\mathrm{ev}_X : \mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}) \to \mathcal{M} : F \mapsto F(X) \tag{3.26}$$

we get something stronger, given an extra condition on $\mathcal{C}$. This extra condition is further discussed in remark 3.17.

Lemma 3.15. Let $\mathcal{C}$ be a (small) dg category such that for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ we have that the morphism complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)^\bullet$ is cofibrant in $\mathrm{Ch}(k\text{-Mod})$. Let $\mathcal{M}$ be a cofibrantly generated $\mathrm{Ch}(k\text{-Mod})$-model category. Then for all $X \in \mathrm{Obj}(\mathcal{C})$ the evaluation functor ev_X as defined in (3.26) preserves weak equivalences, fibrations and cofibrations.

Proof. That ev_X preserves weak equivalences and fibrations is acyclic, as these were defined to be taken positionwise in §3.1.

To tackle the case of cofibrations we make two reductions. First of all ev_X commutes with colimits because it is a left adjoint to the constant diagram functor and left adjoints preserves colimits in general [GTM5, theorem V.4.1]. Hence by using the small object argument as explained in appendix A.1 we are reduced to taking F a generating cofibration. Then ev_X must send this generating cofibration to a cofibration in $\mathcal{M}$.

By the general theory of model categories the generating cofibrations can be taken of the form $\mathrm{h}^Z \otimes A \to \mathrm{h}^Z \otimes B$ where Z where Z runs over $\mathrm{Obj}(\mathcal{C})$ and $A \hookrightarrow B$ over the cofibrations of $\mathcal{M}$ [MSM99, theorem 11.6.1]. So the proof is reduced to checking that

$$\mathrm{ev}_X \left(\mathrm{h}^Z \otimes A \hookrightarrow \mathrm{h}^Z \otimes B\right) = \mathrm{Hom}_{\mathcal{C}}(Z,X)^\bullet \otimes A \to \mathrm{Hom}_{\mathcal{C}}(Z,X)^\bullet \otimes B \tag{3.27}$$

is a cofibration, but as all Hom-complexes were assumed to be cofibrant objects in $\mathrm{Ch}(k\text{-Mod})$ we have that this is again a cofibration. □

Remark 3.16. Without the condition in lemma 3.15 on the Hom-complexes of $\mathcal{C}$ the evaluation functor is still a Quillen functor, and it forms a (Quillen) adjoint pair with its right adjoint that sends an object $\mathcal{M}$ to the constant functor in $\mathcal{C}$-dg $\mathrm{Mod}_k(\mathcal{M})$, as discussed in [MSM99, theorem 11.6.8].

Remark 3.17. By the description of the model category structure on $\mathrm{dg}\,\mathrm{Cat}_k$ in §3.1 we see that this at first sight artifical condition on the Hom-complexes is satisfied if $\mathcal{C}$ is a cofibrant dg category, as proved in lemma 3.7.

In the case k is a field we this condition is always fulfilled because every module is projective (and even free), so using the characterisation of cofibrations from [MSM63, proposition 2.3.9] or appendix A.2 we obtain the result.

Remark 3.18. Let $\mathcal{C}$ be a dg category such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)^\bullet$ is a cofibrant cochain complex of k-modules for all $X, Y \in \mathrm{Obj}(\mathcal{C})$. Take $\mathcal{M}$ to be $\mathcal{C}$-dg Mod_k. Then we see that by lemma 3.15 and the fact that $\mathrm{Ch}(k\text{-Mod})$ satisfies condition (i) in lemma 3.14 we can always apply lemma 3.14.

3.3 Internal dg categories

For every $\mathrm{Ch}(k\text{-Mod})$-model category $\mathcal{M}$ as we have encountered them in §3.2 we can define its associated "internal dg category" (even without the assumption that it is cofibrantly generated). The idea behind structure is similar to the one in example 1.16, and provides a dg enrichment of the homotopy category $\mathrm{Ho}\,\mathcal{M}$, as is shown in proposition 3.20.

Definition 3.19. Let $\mathcal{M}$ be a $\mathrm{Ch}(k\text{-Mod})$-model category. Its *internal category* $\mathrm{Int}(\mathcal{M})$ is the dg category whose objects are the both fibrant and cofibrant objects of $\mathcal{M}$, i.e.

$$\mathrm{Obj}(\mathrm{Int}(\mathcal{M})) = \mathrm{Obj}(\mathcal{M}_{\mathrm{cof,fib}}). \tag{3.28}$$

Its cochain complexes of morphisms are taken from the $\mathrm{Ch}(k\text{-Mod})$-enrichment of $\mathcal{M}$, i.e. we set

$$\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X,Y)^\bullet := \mathrm{Hom}_{\mathcal{M}}(X,Y)^\bullet \tag{3.29}$$

for $X, Y \in \mathrm{Obj}(\mathrm{Int}(\mathcal{M}))$.

This dg category serves as an enrichment for $\mathrm{Ho}\,\mathcal{M}$, as one might already guess from its definition.

Proposition 3.20 (Internal dg category as enrichment of the homotopy category)**.** Let $\mathcal{M}$ be a $\mathrm{Ch}(k\text{-Mod})$-model category. We have the equivalence

$$\mathrm{Ho}(\mathcal{M}) \cong \mathrm{H}^0(\mathrm{Int}(\mathcal{M})). \tag{3.30}$$

Proof. By the very definition of $\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ its objects are the objects of $\mathcal{M}_{\mathrm{cof,fib}}$, and in theorem 2.3 we have seen that this category is equivalent to $\mathrm{Ho}\,\mathcal{M}$, so we can use this equivalence to define the functor $\mathrm{Ho}(\mathcal{M}) \to \mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ on the level of objects. Now the essential surjectivity as obtained in theorem 2.3 immediately gives the *essential surjectivity* in this situation.

On the morphisms this functor is defined by sending $f : X \to Y$ in $\mathrm{Ho}\,\mathcal{M}$ first to the corresponding morphism in $\mathcal{M}_{\mathrm{cof,fib}}$ and then by using the $\mathrm{Ch}(k\text{-Mod})$-enrichment we can interpret it as a morphism in $\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X,Y)^0$ the zeroth degree of the complex. So after taking the H^0 of this morphism we get a well-defined morphism in $\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ which is compatible with the map as we have defined it on the objects. To see that it is *fully faithful* we observe that

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))}(X,Y) \\ &\quad = \mathrm{H}^0(\mathbf{R}\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X,Y)^\bullet) && \text{definition} \\ &\quad \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ch}(k\text{-Mod}))}(k, \mathbf{R}\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X,Y)^\bullet) && \text{Yoneda with } k \text{ in degree } 0 \\ &\quad \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(k \otimes^{\mathbf{L}} X, Y) && \text{total derived adjunction} \\ &\quad \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X,Y) && k \text{ is unit for } - \otimes^{\mathbf{L}} - \end{aligned} \tag{3.31}$$

for X and Y by abuse of notation and theorem 2.3 both in $\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ and $\mathrm{Ho}(\mathcal{M})$. So we can conclude that $\mathrm{H}^0(\mathrm{Int}(\mathcal{M}))$ and $\mathrm{Ho}(\mathcal{M})$ are indeed equivalent categories. □

Now let $\mathcal{C}$ be a dg category and X an object in $\mathcal{C}$. By remark 3.18 the Yoneda embedding h^X as an object in $\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k$ yields a cofibrant object. This object is moreover fibrant because the terminal object in $\mathrm{Ch}(k\text{-Mod})$ is the zero complex, i.e. we get the same proof as for lemma 3.5. So the Yoneda embedding $\mathrm{h}^- : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\text{-dg}\,\mathrm{Mod}_k$ actually gives us

$$\mathrm{h}^- : \mathcal{C}^{\mathrm{op}} \to \mathrm{Int}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k) \tag{3.32}$$

Similarly we have the dual version

$$\mathrm{h}_- : \mathcal{C} \to \mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k). \tag{3.33}$$

These ($\mathrm{Ch}(k\text{-Mod})$-enriched) Yoneda embeddings yield the following definitions.

Definition 3.21. Let $\mathcal{C}$ be a dg category and F a dg $\mathcal{C}^{\mathrm{op}}$-module. We say F is *representable* if there exists an object X in $\mathcal{C}$ such that F is isomorphic in $\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}$ to h_X. We say F is *quasi-representable* if there exists an object X in $\mathcal{C}$ such that F is isomorphic to h_X in $\mathrm{Ho}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)$.

Dually, let $\mathcal{C}$ be a dg category and F a dg $\mathcal{C}$-module. We say F is *corepresentable* if there exists an object X in $\mathcal{C}$ such that F is isomorphic in $\mathcal{C}\text{-dg}\,\mathrm{Mod}$ to h^X. We say F is *quasi-corepresentable* if there exists an object X in $\mathcal{C}$ such that F is isomorphic to h^X in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k)$.

This ties in with the other adjectives we have prefixed with quasi- in §3.1. And because the Yoneda embedding is quasi-fully faithful we get that $\mathcal{C}$ and the full dg subcategory of $\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod})$ of right quasi-representable objects are quasi-equivalent.

3.4 Right quasi-representable dg modules

In §1.2 we mentioned that $\mathrm{dg}\,\mathrm{Cat}_k$ is equipped with a closed symmetric monoidal structure. So we have the adjunction formula

(3.34) $\mathrm{Hom}_{\mathrm{dg}\,\mathrm{Cat}_k}(\mathcal{C}\otimes\mathcal{D},\mathcal{E})\cong\mathrm{Hom}_{\mathrm{dg}\,\mathrm{Cat}_k}(\mathcal{C},\mathcal{H}\mathrm{om}(\mathcal{E},\mathcal{D}))$.

for $\mathcal{C},\mathcal{D},\mathcal{E}\in\mathrm{Obj}(\mathrm{dg}\,\mathrm{Cat}_k)$. Now let $\mathcal{M}$ be a cofibrantly generated $\mathrm{Ch}(k\text{-Mod})$-model category as introduced in §3.2. By using the Yoneda lemma we can prove that

(3.35) $(\mathcal{C}\otimes\mathcal{D})\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})\cong\mathcal{D}\text{-dg}\,\mathrm{Mod}_k\left(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})\right)$

which is compatible with the model category structures, as $\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})$ is again cofibrantly generated by proposition 3.11.
Unfortunately this does not define a Quillen adjunction on $\mathrm{dg}\,\mathrm{Cat}_k$, as will be explained in §4.6. But using the cofibrant replacement functor and lemma 3.6 we can still derive the tensor product on $\mathrm{dg}\,\mathrm{Cat}_k$ into a functor which is defined by

(3.36) $-\otimes^{\mathbf{L}}-:\ \mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)\times\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)\to\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k):(\mathcal{C},\mathcal{D})\to\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}:=\mathrm{Q}\,\mathcal{C}\otimes\mathcal{D}$.

This definition is originally on the level of $\mathrm{dg}\,\mathrm{Cat}_k$, but as it respects quasi-equivalences it descends to the homotopy category.
Now if we consider $(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k$ we can define for every $X\in\mathrm{Obj}(\mathcal{C})$ a morphism of dg categories $\mathcal{D}^{\mathrm{op}}\to\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}$ which on the level of objects is defined as $Y\mapsto(X,Y)$ and therefore on the level of the morphisms as

(3.37) $\mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(Y,Z)^\bullet\mapsto\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}}((X,Y),(X,Z))^\bullet\overset{(1.25)}{=}\mathrm{Hom}_{\mathcal{C}}(X,X)^\bullet\otimes_k\mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(Y,Z)^\bullet$

for $Y,Z\in\mathrm{Obj}(\mathcal{D}^{\mathrm{op}})$. Because the cofibrant replacement functor of $\mathrm{dg}\,\mathrm{Cat}_k$ can be taken to be the identity on the objects as explained in lemma 3.6 this naturally defines

(3.38) $\mathrm{i}_X:\mathcal{D}^{\mathrm{op}}\to\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}}=\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}}$.

And this morphism yields the following important definition.

Definition 3.22. Let $\mathcal{C}$ and $\mathcal{D}$ be dg categories. A dg $(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})$-module F is *right quasi-representable* if for all $X\in\mathrm{Obj}(\mathcal{C})$ the induced dg $\mathcal{D}^{\mathrm{op}}$-module $\mathrm{i}_X(F)$ is quasi-representable in $\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k$.

Studying these right quasi-representable objects is the main goal of [Toe07] and hence of this thesis. They will provide the generalization of Morita theory as explained in the introduction.

CHAPTER 4

Simplicial properties

> This must mean that the one-of-each rule applies only to forms of life, such as little pards, and not to objects, such as bandannas.
>
> *Gravity's Rainbow*
> THOMAS PYNCHON

We now come to the main part of this thesis. We will prove that the model category $\mathrm{dg}\,\mathrm{Cat}_k$ comes equipped with an Map-object which is weakly equivalent to a nerve construction using dg bimodules. The full statement can be found in theorem 4.14. The proof of this theorem will require most of this chapter, and is rather technical. It can be found in §4.3.

This result shows that the homotopy theory of dg categories is a rich subject, which allows for many constructions. The main construction is the existence of an internal Hom-object in $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$, which is not a result of the monoidal and the model category structure (as these are incompatible). Using this result it is possible to introduce a so called derived Morita theory, which will be done in §5.1.

4.1 Simplicial structure of mapping spaces of $\mathrm{dg}\,\mathrm{Cat}_k$

For the proof of the main result in §4.3 we will need to construct a cosimplicial resolution functor, sending objects in $\mathrm{dg}\,\mathrm{Cat}_k$ to cosimplicial objects in $\mathrm{dg}\,\mathrm{Cat}_k^{\Delta}$. This requires the notion of a Reedy model category structure on $\mathrm{dg}\,\mathrm{Cat}_k^{\Delta}$ and the construction of something that serves as a generalization of the cofibrant replacement functor. We will not need the full theory of Reedy (model) categories, so we will just start with the general definition and some remarks on the abstract case, but we will quickly specialize to the (Reedy) category Δ, which is explained in example 4.3.

Definition 4.1. Let $\mathcal{C}$ be a small category. It is a *Reedy category* if it is equipped with two subcategories $\mathcal{C}^{\rightarrow}$ and $\mathcal{C}^{\leftarrow}$ called the *direct* and *inverse* subcategory such that

$$\mathrm{Obj}(\mathcal{C}) = \mathrm{Obj}(\mathcal{C}^{\rightarrow}) = \mathrm{Obj}(\mathcal{C}^{\leftarrow}) \tag{4.1}$$

and a non-negative number for each object called the *degree*, such that

(i) every non-identity morphism in $\mathcal{C}^{\rightarrow}$ increases the degree;

(ii) every non-identity morphism in $\mathcal{C}^{\leftarrow}$ decreases the degree;

(iii) every morphism f in $\mathcal{C}$ can be factored as $g = g^{\rightarrow} \circ g^{\leftarrow}$ such that $g^{\rightarrow}$ is a morphism in $\mathcal{C}^{\rightarrow}$ and $g^{\leftarrow}$ is a morphism in $\mathcal{C}^{\leftarrow}$.

Remark 4.2. This definition can be generalized such that the degree is an arbitrary ordinal [MSM63] but we will not need this situation as we will use Δ, in which case integers suffice.

There are two ways of putting a model category structure on $\mathrm{Func}(\mathcal{C},\mathcal{M})$ where $\mathcal{M}$ is a model category. The first is to assume some properties on $\mathcal{M}$, in which case we can get an injective or projective structure. One defines the weak equivalences and either fibrations or cofibrations to be taken positionwise. If $\mathcal{M}$ satisfies some conditions this induces a model category structure. The other possibility is to assume $\mathcal{C}$ satisfies an extra hypothesis, while $\mathcal{M}$ can be arbitrary. This hypothesis will be the Reedy structure. When both a injective or projective and Reedy model category structure exist on $\mathrm{Func}(\mathcal{C},\mathcal{M})$ there will be a relationship between these structures, but in general they will not be the same.

Example 4.3. The *cosimplicial indexing category* Δ has as its objects the ordered sets $[n]$, where $[n] = (0,\dots,n)$ for each $n \in \mathbb{N}$ and the weakly monotone functions as morphisms, i.e.

$$\mathrm{Hom}_\Delta([m],[n]) := \{\sigma\colon [m] \to [n] \mid \forall 0 \le i \le j \le n \colon \sigma(i) \le \sigma(j)\}\,. \tag{4.2}$$

If we assign degree n to the object $[n]$, and we take $\Delta^{\rightarrow}$ (resp. $\Delta^{\leftarrow}$) to be the subcategories of injective (resp. surjective maps) we get by the factorisation lemma [CSAM29, lemma 8.1.2] that Δ satisfies the conditions for a Reedy category. Moreover, it is exactly this property that more general Reedy categories are modelled after.

Before we can define the Reedy model category structure on $\mathrm{Func}(\mathcal{C},\mathcal{M})$ we need some constructions. Recall that the *category* $(C \downarrow \mathcal{C})$ *of objects in* $\mathcal{C}$ *under* C for $C \in \mathrm{Obj}(\mathcal{C})$ has as its objects the morphisms $C \to X$ in $\mathcal{C}$ and as its morphisms maps $f : X \to Y$ of objects under C inducing the commutative triangle

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd} \tag{4.3}$$

with composition being defined on the lower level. Similarly we have the *category* $(\mathcal{C} \uparrow C)$ *of objects in* $\mathcal{C}$ *over* C with objects the maps $X \to C$ in $\mathcal{C}$ and as its morphisms the maps $f : X \to Y$ of objects over C inducing the commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & C & \end{tikzcd} \tag{4.4}$$

with composition being defined on the upper level.
Given an under category $(C \downarrow \mathcal{C}^{\rightarrow})$ we define the *latching category* $\partial(C \downarrow \mathcal{C}^{\rightarrow})$ to be the full subcategory of $(C \downarrow \mathcal{C}^{\rightarrow})$ that contains all objects, i.e. all morphisms $D \to C$ in $\mathcal{C}$ except for $\mathrm{id}_{\mathcal{C}}$. Likewise we define the *matching category* $\partial(C \downarrow \mathcal{C}^{\leftarrow})$ to be the full subcategory of $(C \downarrow \mathcal{C}^{\leftarrow})$ that contains all objects except $\mathrm{id}_{\mathcal{C}}$.

Definition 4.4. Let $\mathcal{M}$ be a bicomplete category. Let $\mathcal{C}$ be a Reedy category. Let C be an object of $\mathcal{C}$.

(i) The *latching space functor* $\mathrm{L}_C\colon \mathrm{Func}(\mathcal{C},\mathcal{M}) \to \mathcal{M}$ is defined to be the composite of

$$\mathrm{L}_C\colon \mathrm{Func}(\mathcal{C},\mathcal{M}) \overset{\mathrm{res}}{\to} \mathrm{Func}(\partial(C \downarrow \mathcal{C}^{\rightarrow}),\mathcal{M}) \overset{\mathrm{colim}}{\to} \mathcal{M} \tag{4.5}$$

where res is the obvious restriction functor.

(ii) The *matching space functor* $\mathrm{M}_C\colon \mathrm{Func}(\mathcal{C},\mathcal{M}) \to \mathcal{M}$ is defined to be the composite of

$$\mathrm{M}_C\colon \mathrm{Func}(\mathcal{C},\mathcal{M}) \overset{\mathrm{res}}{\to} \mathrm{Func}(\partial(C \downarrow \mathcal{C}^{\leftarrow}),\mathcal{M}) \overset{\lim}{\to} \mathcal{M} \tag{4.6}$$

where res is the obvious restriction functor.

By the universal properties of the constructions involved we get natural maps $\mathrm{L}_C X \to X(C)$ resp. $X(C) \to \mathrm{M}_C(X)$ which we will call *latching* resp. *matching map* at C, for $X : \mathcal{C} \to \mathcal{M}$ an object in $\mathrm{Func}(\mathcal{C}, \mathcal{M})$.

We are now ready to define the Reedy model category structure on $\mathrm{Func}(\mathcal{C}, \mathcal{M})$. If $\mathcal{M}$ is cofibrantly generated, which is the condition for the existence of the projective model category structure, then the weak equivalences will coincide, but in the Reedy model category structure there will be more cofibrations.

Theorem 4.5. Let $\mathcal{M}$ be a model category. Let $\mathcal{C}$ be a Reedy category. Let $f : X \to Y$ be a map in $\mathrm{Func}(\mathcal{C}, \mathcal{M})$. If we say f is

(i) a *Reedy weak equivalence* if and only if for all $C \in \mathrm{Obj}(\mathcal{C})$ the map

$$f_C : X(C) \to Y(C) \tag{4.7}$$

is a weak equivalence in $\mathcal{M}$;

(ii) a *Reedy cofibration* if and only if for every $C \in \mathrm{Obj}(\mathcal{C})$ the map

$$X(C) \sqcup_{\mathrm{L}_C X} \mathrm{L}_C Y \to Y(C) \tag{4.8}$$

is a cofibration in $\mathcal{M}$;

(iii) a *Reedy fibration* if and only if for every $C \in \mathrm{Obj}(\mathcal{C})$ the map

$$X(C) \to Y(C) \times_{\mathrm{M}_C Y} \mathrm{M}_C X \tag{4.9}$$

is a fibration in $\mathcal{M}$;

then this induces a model category structure on $\mathrm{Func}(\mathcal{C}, \mathcal{M})$ which we will call the *Reedy model category structure*.

We will also need simplicial model categories, as introduced in example 2.14. In §4.5 we will prove that the category $\mathrm{dg\,Cat}_k$ comes equipped with this type of model category structure. But for now we will restrict ourselves to the definition and a characterisation of homotopy monomorphisms in this context.

Recall that in lemma 3.9 we have characterised the homotopy monomorphisms in $\mathrm{dg\,Cat}_k$. But there is another characterisation using simplicial model categories [Rez10, proposition 7.3].

Lemma 4.6. Let $\mathcal{M}$ be a simplicial model category. Let $f : X \to Y$ be a morphism in $\mathcal{M}$. Then f is a homotopy monomorphism if and only if for all $Z \in \mathrm{Obj}(\mathcal{M})$ the induced map

$$f_* \colon \mathrm{Map}_{\mathcal{M}}(Z, X)^* \to \mathrm{Map}_{\mathcal{M}}(Z, Y)^* \tag{4.10}$$

yields an injection on π_0 the connected components, and an isomorphism on all higher homotopy groups.

Remark 4.7. If we combine this with lemma 3.9 we see that for $f : \mathcal{C} \to \mathcal{D}$ a quasi-fully faithful morphism in $\mathrm{dg\,Cat}_k$ the map

$$f_* \colon \mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{B}, \mathcal{C})^* \to \mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{B}, \mathcal{D})^* \tag{4.11}$$

for all $\mathcal{B} \in \mathrm{Obj}(\mathrm{dg\,Cat}_k)$ is an injection on the level of π_0 and an isomorphism for all π_i. By using the enriched philosophy this implies that

$$\mathrm{Ho}(f) \colon \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{B}, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{B}, \mathcal{D}) \tag{4.12}$$

is an injection.

4.2 The fundamental weak equivalence: introduction

We can now prove the main technical result of [Toe07], which is given in theorem 4.14 and its interpretation in theorem 4.25. In order to prove this we need several lemmas and intermediate results. The proof itself is not changed, but the structure of the exposition is hopefully enhanced.

Before we start the proof we need to construct the morphisms which are used in the theorem and its proof. It is at this point that we will use the simplicial, or Dwyer-Kan, localisation. And we will need to construct a (rather involved) subcategory. The reason for this construction is that $\mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^*$ only considers representable objects, so if we want to relate arbitrary dg modules to the mapping spaces we need to restrict ourselves.

Definition 4.8. Let $[n]$ be an object of Δ, then we define using the cosimplicial resolution functor, the category

$$(\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}. \tag{4.13}$$

This is the subcategory of $(\Gamma^n(\mathcal{C})\otimes\mathcal{D})$-dg Mod_k where the objects are the right quasi-representable objects F such that for all $X\in\mathrm{Obj}(\Gamma^n(\mathcal{C}))$ the object $F(X,-)$ in $\mathcal{D}^{\mathrm{op}}$-dg Mod_k is cofibrant. For the morphisms in this category we only take the weak equivalences, hence it is not a full subcategory. In this thesis we will stick to this (potentially cumbersome) notation because it reflects the (involved) definition of this subcategory.

Remark 4.9. In the proof of theorem 4.25 we will show that the condition that $F(X,-)$ is a cofibrant object in $\mathcal{D}^{\mathrm{op}}$-dg Mod_k for all $X\in\mathrm{Obj}(\Gamma^n(\mathcal{C}))$ is only technical and can be dropped to reach the string of weak equivalences in (4.93).

For any category $\mathcal{A}$ we can construct its *nerve* $\mathrm{N}(\mathcal{A})^*$ [GTM5, §XII.2]. In our situation we then get that a morphism $[n]\to[m]$ in Δ (i.e. an order-preserving morphism $[m]\to[n]$) yields a morphism of simplicial sets

$$\mathrm{N}\left((\Gamma^m(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^*\to\mathrm{N}\left((\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^*. \tag{4.14}$$

Here we have used the pullback as introduced in §3.2, the interpretation of dg bimodules and dg modules with values in a category as discussed in (3.35), and the functoriality of the nerve construction. By combining all this we get a bisimplicial set defined by

$$\mathrm{N}\left((\Gamma^*(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^*:\begin{array}{l}\Delta^{\mathrm{op}}\to\mathrm{sSet}\\ {[n]}\mapsto\mathrm{N}\left((\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^*.\end{array} \tag{4.15}$$

The relation between the mapping space and this nerve construction is the main result, as given in theorem 4.14.

We will now describe the construction of the morphism that is used. By the definition of the nerve we know that 0-simplices in this case correspond to objects in the original category. Hence we can define a morphism of sets

$$\mathrm{Hom}_{\mathrm{dg\,Cat}_k}(\Gamma^n(\mathcal{C}),\mathcal{D})\to\mathrm{N}\left((\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^* \tag{4.16}$$

by sending a dg functor $F:\Gamma^n(\mathcal{C})\to\mathcal{D}$ to the $(\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})$-dg module $\phi^n_{\mathcal{D}}(F)$ defined by

$$\phi^n_{\mathcal{D}}(F)(X,Y)^\bullet=\mathrm{Hom}_{\mathcal{D}}(Y,F(X))^\bullet \tag{4.17}$$

for $X\in\mathrm{Obj}(\Gamma^n(\mathcal{C}))$ and $Y\in\mathrm{Obj}(\mathcal{D})$. This map lands in the correct subcategory, because we observe that

$$\phi^n_{\mathcal{D}}(F)(X,-)=\mathrm{h}_{F(X)} \tag{4.18}$$

is representable as a $\mathcal{D}^{\mathrm{op}}$-dg module hence right quasi-representable and cofibrant.

If we consider $\mathrm{Hom}_{\mathrm{dg\,Cat}_k}(\Gamma^*(\mathcal{C},\mathcal{D})$ as the constant simplicial set we get by the adjunction between the forgetful functor and the constant simplicial set a morphism

$$(4.19)\quad \phi^n_{\mathcal{D}}\colon \mathrm{Hom}_{\mathrm{dg\,Cat}_k}(\Gamma^n(\mathcal{C}),\mathcal{D}) \to \mathrm{N}\left((\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}}\right)^*$$

in sSet. Because $[n]\to[m]$ yields the corresponding commutative square by (4.16), and by the construction of the cosimplicial resolution functor we get functoriality in n. This corresponds to a morphism of bisimplicial sets

$$(4.20)\quad \phi^*_{\mathcal{D}}\colon \mathrm{Hom}_{\mathrm{dg\,Cat}_k}(\Gamma^*(\mathcal{C}),\mathcal{D}) \to \mathrm{N}\left((\Gamma^*(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}}\right)^*.$$

Finally, for each bisimplicial sets we can consider its diagonal [MSM99, definition 15.11.3].

Definition 4.10. The preceding discussion defines a morphism

$$(4.21)\quad \phi_{\mathcal{D}}\colon \mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^* \to \mathrm{diag}\left(\mathrm{N}\left((\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}}\right)^*\right).$$

We need a second morphism in the proof of theorem 4.14, for which we prove the essential property in proposition 4.15.

Definition 4.11. Using the preceding constructions we define

$$(4.22)\quad \psi_{\mathcal{D}}\colon \mathrm{N}\left((\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}}\right)^* \to \mathrm{diag}\left(\mathrm{N}\left((\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}}\right)^*\right)$$

which is obtained by the isomorphism $\Gamma^0(\mathcal{C})\cong\mathrm{Q}(\mathcal{C})$, the map $\Gamma^n\to\Gamma^0$, its induced pullback morphism and functoriality of the constructions that are involved.

Remark 4.12. We observe that the diagram

$$(4.23)\quad \begin{array}{ccc} \mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^* & \xrightarrow{\phi_{\mathcal{D}}} & \mathrm{diag}\left(\mathrm{N}\left((\Gamma^*(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}}\right)^*\right) \\ & & \uparrow{\scriptstyle\psi_{\mathcal{D}}} \\ & & \mathrm{N}\left((\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}}\right)^* \end{array}$$

is functorial in the first variable as all the constructions applied are functorial. The functoriality in the second variable on the other hand unfortunately does not hold, it is only a lax functor. In the proof of proposition 4.24 this is discussed.

Before we start, now is a good time to address the issue of universes, as indicated in remark 1.19. The original reference for this notion is [$\mathrm{SGA4}_1$, appendice, exposé I].

Remark 4.13. Whenever we used the notion of a *small* dg category, this was with respect to a universe $\mathbb{U}$. For this universe $\mathbb{U}$ moreover require that $\mathbb{N}\in\mathbb{U}$, i.e. the axiom of infinity holds. To prevent the use of the silly terminology "big" and "very big", especially in the statement of lemma 5.10 we can choose an inclusion of universes $\mathbb{U}\in\mathbb{V}\in\mathbb{W}$.

There is an important compatibility between $\mathrm{dg\,Cat}_k$ (without an explicit reference to $\mathbb{U}$) and $\mathrm{dg\,Cat}_{k,\mathbb{V}}$, the category of dg categories which are small with respect to $\mathbb{V}$. As the model category structure on $\mathrm{dg\,Cat}_k$ is cofibrantly generated, and the set of generating (acyclic) cofibrations is specified without reference to the universe (see [Tab05a] or appendix B) we can choose the same generating (acyclic) cofibrations for the model category $\mathrm{dg\,Cat}_{k,\mathbb{V}}$. Hence the inclusion

$$(4.24)\quad \mathrm{dg\,Cat}_k \hookrightarrow \mathrm{dg\,Cat}_{k,\mathbb{V}}$$

yields a *fully faithful* inclusion

$$(4.25)\quad \mathrm{Ho}(\mathrm{dg\,Cat}_k) \hookrightarrow \mathrm{Ho}(\mathrm{dg\,Cat}_{k,\mathbb{V}}).$$

and changing universes is harmless.

4.3 The fundamental weak equivalence: proof

Theorem 4.14 (Fundamental weak equivalence)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be dg categories. The canonical morphism

$$\mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^* \to \mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\right)^* \tag{4.26}$$

in sSet is a weak equivalence.

To show this we will first prove proposition 4.15, which will serve as an intermediate weak equivalence as depicted in (4.23). Tackling the other intermediate weak equivalence, which is proposition 4.24, will require more effort. The main tool will be lemma 4.23, of which conditions (iii) and (iv) are non-trivial. First we will prove in lemma 4.20 that the map induced on the connected components is an isomorphism, which is condition (iii). Then we will reduce our situation to the situation as depicted in lemma 4.23, and prove the second non-trivial condition, i.e. condition (iv), in lemma 4.22.

Proposition 4.15. The morphism

$$\psi_{\mathcal{D}}\colon \mathrm{N}\left((\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\right)^* \to \mathrm{diag}\left(\mathrm{N}\left((\Gamma^*(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\right)^*\right) \tag{4.27}$$

in sSet is a weak equivalence.

Proof. By the properties of the cosimplicial resolution functor the morphism

$$\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}} \to \mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}} \tag{4.28}$$

is a weak equivalence in $\mathrm{dg\,Cat}_k$ (i.e. a quasi-equivalence of dg categories) as these are defined pointwise in the Reedy model category structure [MSM99, proposition 16.1.3]. So if we apply lemma 3.14 the restriction functor

$$(\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k \to (\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k \tag{4.29}$$

is part of a Quillen equivalence, as condition (i) of lemma 3.14 is satisfied, which is explained in remark 3.17.

If an object of $(\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k$ is right quasi-representable then its image is right quasi-representable too, hence this map is compatible with the restriction to the non-full subcategories on both sides.

To conclude we observe that the nerve construction sends Quillen equivalences to weak equivalences in sSet. □

Now we will consider $\pi_0(\phi_{\mathcal{D}})$, the map induced by functoriality on the connected components of the mapping space and the diagonal. In lemma 4.20 this will turn out to be an isomorphism in our situation. To prove this we need two results which hold in general, and which will also be useful in the following sections. The first is a relation between dg functors and isomorphism classes of certain objects, which ties in with the philosophy of representability as explained in the introduction.

For a category $\mathcal{C}$ we will denote the set of isomorphism classes of $\mathcal{C}$ by $\mathrm{Isom}(\mathcal{C})$. Now we can state and prove the first general lemma.

Lemma 4.16. Let $\mathcal{M}$ be a cofibrantly generated $\mathrm{Ch}(k\text{-Mod})$-model category. Let $\mathcal{C}$ be a dg category. We have a natural morphism

$$\mathrm{Hom}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathrm{Int}(\mathcal{M})) \to \mathrm{Isom}\left(\mathrm{Ho}\left(\mathcal{C}\text{-}\mathrm{dg\,Mod}_k(\mathcal{M})\right)\right) \tag{4.30}$$

which is surjective.

Proof. The morphism is defined by sending a dg functor $\mathcal{C} \to \mathrm{Int}(\mathcal{M})$ to the object representing this functor, and then we take its isomorphism class.

By the properties of the homotopy category as discussed in §2.2 we can take an object F in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$ to be both fibrant and cofibrant in $\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})$. This object is a $\mathrm{Ch}(k\text{-Mod})$-enriched functor. Because this functor is both fibrant and cofibrant we get by lemma 3.15 that $F(X)$ is a both fibrant and cofibrant object in $\mathcal{M}$ for all $X \in \mathrm{Obj}(\mathcal{C})$. Hence we can interpret it as a dg functor

$$F\colon \mathcal{C} \to \mathrm{Int}(\mathcal{M}) \tag{4.31}$$

that is mapped to the representing object in $\mathrm{Isom}(\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})))$, so we obtain surjectivity. □

Remark 4.17. Under some extra conditions this map is an isomorphism, as discussed in [LNM2008, proposition 1]. For this we have to assume that for every cofibrant object X of $\mathcal{M}$ and every quasi-isomorphism $M^\bullet \to N^\bullet$ in $\mathrm{Ch}(k\text{-Mod})$ the morphism $M^\bullet \otimes X \to N^\bullet \otimes X$ is a quasi-equivalence, we obtain the injectivity. Remark that this condition is familiar from lemma 3.14(i).

The second general lemma is a lifting property: using the surjection obtained in lemma 4.16 we will prove that dg functors which have isomorphic objects in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$ are actually homotopic morphisms in $\mathrm{dg}\,\mathrm{Cat}_k$.

Lemma 4.18. Let $\mathcal{C}$ be a dg category. Let $\mathcal{M}$ be a cofibrantly generated $\mathrm{Ch}(k\text{-Mod})$-model category. Consider two dg functors $F, G\colon \mathcal{C} \to \mathrm{Int}(\mathcal{M})$, i.e. morphisms in $\mathrm{dg}\,\mathrm{Cat}_k$, such that their representing objects in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$ are isomorphic. Then F and G are homotopic maps in $\mathrm{dg}\,\mathrm{Cat}_k$.

Proof. By the general theory of model categories an isomorphism in the homotopy category of a model category can be written as the composition of acyclic fibrations and acyclic cofibrations, see theorem 2.3. Hence in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$ we can write an isomorphism between functors such that each evaluation in an object of $\mathcal{C}$ is both fibrant and cofibrant in $\mathcal{M}$ as a composition of acyclic (co)fibrations of functors such that each evaluation in an object of $\mathcal{C}$ is both fibrant and cofibrant. If we can prove that an acyclic (co)fibration yields homotopic maps we can then compose these homotopies, because by lemma 4.16 the map (4.30) is surjective hence we can lift our weak equivalence to a morphism in $\mathrm{dg}\,\mathrm{Cat}_k$.

Let $\alpha\colon F \to G$ be a cofibration in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$. We can interpret any morphism as a diagram using the functor category $\mathrm{Func}(\mathbf{2}, \mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$ where $\mathbf{2}$ is the category on two objects $\{0,1\}$ with a single morphism $0 \to 1$ between them. It is clear that α can be considered as an object in this category.

We play the same game with $\mathcal{M}$, and obtain $\mathrm{Func}(\mathbf{2}, \mathcal{M})$ which we can equip with the projective model category structure by the assumption on $\mathcal{M}$. Hence fibrations and weak equivalences are taken positionwise.

Because the evaluation functor preserves all the structure of a model category by lemma 3.15 we observe that for all $X \in \mathrm{Obj}(\mathcal{C})$ the image of the cofibration α is again a cofibration $\alpha_X : F(X) \to G(X)$ in $\mathcal{M}$. So by the model category structure induced on $\mathrm{Func}(\mathbf{2}, \mathcal{M})$ this is both a fibrant (by the positionwise definition) and cofibrant object (because we consider a finite cotower diagram) in $\mathrm{Func}(\mathbf{2}, \mathcal{M})$. So the morphism (or more appropriately: natural transformation) $\alpha\colon F \Rightarrow G$ lifts to a morphism

$$\overline{\alpha}\colon \mathcal{C} \to \mathrm{Int}(\mathrm{Func}(\mathbf{2}, \mathcal{M})). \tag{4.32}$$

We can embed $\mathrm{Int}(\mathcal{M})$ in $\mathrm{Int}(\mathrm{Func}(\mathbf{2}, \mathcal{M}))$ by sending a both fibrant and cofibrant object to its identity morphism. This is a morphism in $\mathrm{dg}\,\mathrm{Cat}_k$. It is quasi-fully faithful, i.e.

$$\mathrm{Hom}_{\mathrm{Int}(\mathcal{M})}(X, Y)^\bullet \to \mathrm{Hom}_{\mathrm{Int}(\mathrm{Func}(\mathbf{2}, \mathcal{M}))}(\mathrm{id}_X, \mathrm{id}_Y)^\bullet \tag{4.33}$$

is a quasi-isomorphism, because the morphisms in $\mathrm{Int}(\mathrm{Func}(\mathbf{2}, \mathcal{M}))$ correspond to commutative squares and in case of the identity morphisms at X and Y we obtain

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd} \tag{4.34}$$

so necessarily $f = g$ and we even get an isomorphism of chain complexes.

Define $\mathcal{C}' \subseteq \mathrm{Int}(\mathrm{Func}(\mathbf{2}, \mathcal{M}))$ to be the quasi-essential image of the embedding. We can interpret it as the full dg subcategory of $\mathrm{Int}(\mathrm{Func}(\mathbf{2}, \mathcal{M}))$ that consists of the weak equivalences in $\mathcal{M}$. Now we can construct our desired homotopy.

Evaluating an element of $\mathrm{Func}(\mathbf{2}, \mathcal{M})$ in 0 and 1 yields two functorial projections

$$p_0, p_1 \colon \mathcal{C}' \subseteq \mathrm{Int}(\mathrm{Func}(\mathbf{2}, \mathcal{M})) \to \mathrm{Int}(\mathcal{M}). \tag{4.35}$$

The embedding we constructed before yields a section for these projections. We obtain the commutative diagram

$$\begin{tikzcd} & \mathrm{Int}(\mathcal{M}) \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "\alpha"] \arrow[dr, "G"'] & \mathcal{C}' \arrow[u, "p_0"'] \arrow[d, "p_1"] \\ & \mathrm{Int}(\mathcal{M}) \end{tikzcd} \tag{4.36}$$

in $\mathrm{dg\,Cat}_k$ which serves as a homotopy between F and G.

Let $\alpha\colon F \to G$ be a fibration in $\mathrm{Ho}(\mathcal{C}\text{-}\mathrm{dg\,Mod}_k(\mathcal{M}))$. We repeat the proof for the case of a cofibration, now using the injective model structure on $\mathrm{Func}(\mathbf{2}, \mathcal{M})$, i.e. the weak equivalences are again the positionwise weak equivalences, while now the cofibration are taken positionwise.

□

Remark 4.19. We are allowed to put the injective model structure on this functor category even though $\mathcal{M}$ is not combinatorial, because the category $\mathbf{2}$ is "very small" [D-S, §10.13]. For a general functor category $\mathrm{Func}(I, \mathcal{M})$ we would need a combinatorial model category [HTT, §A.2.6] for the injective structure to exist. The same applies for the projective model structure, where in general we need $\mathcal{M}$ to be cofibrantly generated. But in the very small case this assumption is not required.

With these results in hand we can prove technical condition (iii) from lemma 4.23.

Lemma 4.20. Let $\mathcal{C}$ and $\mathcal{D}$ be dg categories. Then the map on the connected components

$$\pi_0(\phi_{\mathcal{D}})\colon \pi_0(\mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C}, \mathcal{D})^*) \to \pi_0\left(\mathrm{diag}\left(\mathrm{N}\left(\Gamma^*(\mathcal{C}) \otimes \mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^*\right)\right) \tag{4.37}$$

is an isomorphism.

Proof. First we make some reductions. By taking a cofibrant replacement $\mathrm{Q}(\mathcal{C})$ of the dg category $\mathcal{C}$ and setting $\mathcal{C} := \mathrm{Q}(\mathcal{C})$, which is allowed by remark 3.17 where $f\colon \mathrm{Q}(\mathcal{C}) \to \mathcal{C}$ is the cofibrant replacement we see that

$$\pi_0(\mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C}, \mathcal{D})^*) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{C}, \mathcal{D}) \tag{4.38}$$

so we can look at the domain of $\pi_0(\phi_{\mathcal{D}})$ in a more familiar way whenever necessary. For the codomain of $\pi_0(\phi_{\mathcal{D}})$ we observe that

$$\pi_0\left(\operatorname{diag}\left(\mathrm{N}\left(\left(\Gamma^*(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}}\right)\text{-dg}\,\mathrm{Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\right)^*\right)\right)\cong\pi_0\left(\mathrm{N}\left(\left(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}\right)\text{-dg}\,\mathrm{Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\right)^*\right)\tag{4.39}$$

because taking the connected components of the diagonal of a bisimplicial set reduces to the connected components of $\Gamma^0(\mathcal{C})$ but we have assumed $\mathrm{Q}(\mathcal{C})=\mathcal{C}$. Then we observe that

$$\pi_0\left(\mathrm{N}\left(\left(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}\right)\text{-dg}\,\mathrm{Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\right)^*\right)\cong\operatorname{Isom}\left(\operatorname{Ho}\left(\left(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}\right)\text{-dg}\,\mathrm{Mod}_{k}^{\mathrm{rqr}}\right)\right)\tag{4.40}$$

because we have restricted ourselves to the non-full subcategory of weak equivalences. Hence we can replace 4.37 using compositions of the isomorphisms, and a little abuse of notation, by

$$\pi_0(\phi)\colon \operatorname{Hom}_{\operatorname{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}(\mathcal{C},\mathcal{D})\to\operatorname{Isom}\left(\operatorname{Ho}\left(\left(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}\right)\text{-dg}\,\mathrm{Mod}_{k}^{\mathrm{rqr}}\right)\right).\tag{4.41}$$

We first prove that it is *surjective*. Take F an object of $\operatorname{Ho}\left((\mathcal{C}\otimes\mathcal{D})\text{-dg}\,\mathrm{Mod}_k^{\mathrm{rqr}}\right)$. We would like to find a morphism $f:\mathcal{C}\to\mathcal{D}$ in $\mathrm{dg}\,\mathrm{Cat}_k$ that is represented up to isomorphism by this dg module, i.e. such that $\pi_0(\phi_{\mathcal{D}})(f)\cong F$ in the homotopy category.
By applying lemma 4.16 to $\mathcal{M}=\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k$ we can find a morphism of dg categories

$$g\colon\mathcal{C}\to\operatorname{Int}\left(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)^{\mathrm{qr}}\tag{4.42}$$

that is mapped to F. Remark that by the condition on F the codomain of f actually is the full dg subcategory of quasi-representable object. We can also consider the Yoneda embedding on $\mathcal{D}$, and that way we obtain the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "g"] & \operatorname{Int}\left(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)^{\mathrm{qr}} \\ & \mathcal{D} \arrow[u, "\mathrm{h}_{-}"'] \end{tikzcd}\tag{4.43}$$

Because we embed $\mathcal{D}$ in the subcategory of weak equivalences of quasi-representable dg $\mathcal{D}^{\mathrm{op}}$-modules this is a quasi-equivalence. We asumed $\mathcal{C}$ to be cofibrant, so we can apply [MSM63, proposition 1.2.5(iv)] to find a morphism $f:\mathcal{C}\to\mathcal{D}$ in $\mathrm{dg}\,\mathrm{Cat}_k$. This morphism moreover yields a homotopy

$$\begin{tikzcd} \mathcal{C} \arrow[d, "i_0"] \arrow[dr, "g"] & \\ \mathcal{C}' \arrow[r, "H", "\sim"'] & \operatorname{Int}\left(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)^{\mathrm{qr}} \\ \mathcal{C} \arrow[u, "i_1"] \arrow[ur, "\phi_{\mathcal{D}}(f)"'] & \end{tikzcd}\tag{4.44}$$

in $\mathrm{dg}\,\mathrm{Cat}_k$ by the same proposition. The object $\mathcal{C}'$ is the cylinder object for $\mathcal{C}$, and it can be assumed to be cofibrant. Now let $p\colon\mathcal{C}'\to\mathcal{C}$ be the natural projection, i.e. $p\circ i_0=p\circ i_1=\mathrm{id}_{\mathcal{C}}$. By applying lemma 3.14 and remark 3.17 to $\mathcal{M}=\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k$ we get a string of equivalences of categories

$$i_0^*\cong i_1^*\cong(p^*)^{-1}\colon\operatorname{Ho}\left(\left(\mathcal{C}'\otimes\mathcal{D}^{\mathrm{op}}\right)\text{-dg}\,\mathrm{Mod}_k\right)\to\operatorname{Ho}\left(\left(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}\right)\text{-dg}\,\mathrm{Mod}_k\right)\tag{4.45}$$

by using the Quillen adjoint pairs $(i_{0,!}, i_0^*)$, $(i_{1,!}, i_1^*)$ and $(p_!, p^*)$. The maps i_0 and i_1 are quasi-equivalences by [MSM63, definition 1.2.4], and hence p is a quasi-equivalence by the 3-out-of-2 property. Now because the morphism H in (4.44) is a quasi-equivalence in $\mathrm{dg\,Cat}_k$ we get the sequence of weak equivalences in $\mathrm{dg\,Cat}_k$

$$F \simeq i_0^*(H) \simeq i_1^*(H) \simeq \phi_{\mathcal{D}}(f). \tag{4.46}$$

We obtain that the corresponding $(\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}})$-dg modules F and $\phi_{\mathcal{D}}(f)$ are isomorphic in $\mathrm{Ho}((\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k^{\mathrm{r\,qr}})$. By passing to the isomorphism classes this is a true equality in $\mathrm{Isom}(\mathrm{Ho}((\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k^{\mathrm{r\,qr}}))$, hence we get surjectivity of $\phi_{\mathcal{D}}(f)$ in (4.41).
We now prove that it is *injective*. Let $f, g\colon \mathcal{C} \to \mathcal{D}$ be two morphisms of dg categories, such that $\phi_{\mathcal{D}}(f) \cong \phi_{\mathcal{D}}(g)$ in $\mathrm{Ho}((\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k)$. Consider the commutative diagram obtained by using the dg Yoneda embedding as introduced in §3.3

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f{,}g"] \arrow[dr, dashed, "f'{,}g'"'] & \mathcal{D} \arrow[d, "\mathrm{h}^-"] \\ & \mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k) \end{tikzcd} \tag{4.47}$$

to define $f', g'\colon \mathcal{C} \to \mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k)$ as the compositions. Using the results from the same section the dg Yoneda embedding is quasi-fully faithful, so if f' and g' are homotopic in $\mathrm{dg\,Cat}_k$ then f and g are actually equal in $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$.
By remark 4.13, if $\mathbb{U} \in \mathbb{V}$ is the inclusion of universes, the inclusion of the homotopy category of all dg categories which are small with respect to the universe $\mathbb{U}$ in the homotopy category of all dg categories which are small with respect to the universe $\mathbb{V}$ is quasi-fully faithful. So we want to show that f' and g' are homotopic in the bigger category. But as $\mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k$ is the prototype of a cofibrantly generated $\mathrm{Ch}(k\text{-Mod})$-model category we can apply lemma 4.18. □

Remark 4.21. Before stating lemma 4.23 we will translate our situation to the one which is used there. The idea is to use the functors ϕ and ψ as depicted in the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^* \arrow[r, "\phi_{\mathcal{D}}"] & \mathrm{diag}\left(\mathrm{N}\left(\Gamma^*(\mathcal{C}) \otimes \mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^*\right) \\ & \mathrm{N}\left(\mathrm{Q}(\mathcal{C}) \otimes \mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^* \arrow[u, "\psi_{\mathcal{D}}"'] \end{tikzcd} \tag{4.48}$$

and translate this to the weakly equivalent diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^* \arrow[r, "\phi'_{\mathcal{D}}"] \arrow[dr, "k=(\psi'_{\mathcal{D}})^{-1}\circ\phi'_{\mathcal{D}}"'] & \mathrm{diag}\left(\mathrm{N}\left(\Gamma^*(\mathcal{C}) \otimes \mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof,str}}\right)^*\right) \\ & \mathrm{N}\left(\mathrm{Q}(\mathcal{C}) \otimes \mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof,str}}\right)^*. \arrow[u, "\psi'_{\mathcal{D}}"'] \end{tikzcd} \tag{4.49}$$

To obtain this diagram one has to apply the strictification procedure, as there is no strict functoriality in the second argument in this case (see also remark 4.12). This procedure is originally discussed in [SGA1, exposé VI], but the version that is applied in this situation is found in [HAG-II, appendix B]. Hence we obtain true functors with values in simplicial sets.

As $\psi_{\mathcal{D}}$ is a weak equivalence its strictification $\psi'_{\mathcal{D}}$ is one too, so in $\mathrm{Ho}(\mathrm{Func}(\mathrm{dg\,Cat}_k, \mathrm{sSet}))$ this is invertible. So it is indeed possible to define the map k as done in (4.49). Now we can prove that the map k satisfies condition (iv) of lemma 4.23.

Lemma 4.22. Let $\mathcal{C}$ be a cofibrant object in $\mathrm{dg\,Cat}_k$. Let $p\colon \mathcal{D}_1 \twoheadrightarrow \mathcal{D}_3$ and $q\colon \mathcal{D}_2 \to \mathcal{D}_3$ be morphisms of dg categories, where we assume that p is a fibration. Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{D} := \mathcal{D}_1 \times_{\mathcal{D}_3} \mathcal{D}_2 \arrow[r, "u"] \arrow[d, "v"] & \mathcal{D}_1 \arrow[d, two heads, "p"] \\ \mathcal{D}_2 \arrow[r, "q"'] & \mathcal{D}_3 \end{tikzcd} \tag{4.50}$$

in $\mathrm{dg\,Cat}_k$. Then applying $\mathrm{N}((\mathcal{C}\otimes -^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{r\,qr,cof,str}})^*$ yields the homotopy cartesian diagram

$$\begin{tikzcd} \mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{r\,qr,cof,str}}\right)^* \arrow[r] \arrow[d] & \mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}_1^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{r\,qr,cof,str}}\right)^* \arrow[d] \\ \mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}_2^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{r\,qr,cof,str}}\right)^* \arrow[r] & \mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}_3^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{r\,qr,cof,str}}\right)^* \end{tikzcd} \tag{4.51}$$

in sSet.

Proof. We first prove that the canonical morphism (4.52) is a *homotopy monomorphism*, i.e. it induces an injective morphism on the connected components, and an isomorphism for all higher homotopy groups by lemma 4.6.

To prove that (4.52) is a homotopy monomorphism it suffices to prove that the diagram of path spaces (4.53) is a homotopy pullback diagram for all $F, G \in (\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{r\,qr,cof,str}}$, where we denote $w = p\circ u = q\circ v$. Remark that this is an abuse of notation: we have suppressed $\mathrm{id}_{\mathcal{C}}$ from the diagram

$$\begin{tikzcd} \mathcal{C}\otimes\mathcal{D} \arrow[r, "\mathrm{id}_{\mathcal{C}}\otimes u"] \arrow[d, "v"'] & \mathcal{C}\otimes\mathcal{D}_1 \arrow[d, two heads, "\mathrm{id}_{\mathcal{C}}\otimes p"] \\ \mathcal{C}\otimes\mathcal{D}_2 \arrow[r, "\mathrm{id}_{\mathcal{C}}\otimes q"'] & \mathcal{C}\otimes\mathcal{D}_3. \end{tikzcd} \tag{4.54}$$

To make this reduction recall that taking loop spaces shifts the homotopy groups to the left, i.e. if (4.53) is a homotopy pullback we have the isomorphism of all π_i for the loop spaces, hence for all π_{i+1} in the original situation. In case of the connected components we see that the space is either empty or non-empty depending on the connected components the images of F and G lie in, hence we can also detect the connected components of the original situation using (4.53).

By [HAG-II, proposition A.0.3] and the ensuing discussion which in turn refers to [HAG-I, lemma 4.2.2] we obtain that (4.53) is equivalent to

$$\begin{tikzcd} \mathrm{Map}^{\mathrm{eq}}(F,G)^* \arrow[r] \arrow[d] & \mathrm{Map}^{\mathrm{eq}}(u_!(F),u_!(G))^* \arrow[d] \\ \mathrm{Map}^{\mathrm{eq}}(v_!(F),v_!(G))^* \arrow[r] & \mathrm{Map}^{\mathrm{eq}}(w_!(F),w_!(G))^* \end{tikzcd} \tag{4.55}$$

$$\mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^{*} \to \mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^{*} \times^{\mathrm{h}}_{\mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^{*}} \mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^{*} \tag{4.52}$$

$$\begin{array}{ccc}
\Omega_{F,G}\left(\mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^{*}\right)^{*} & \longrightarrow & \Omega_{u_!(F),u_!(G)}\left(\mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^{*}\right)^{*} \\
\downarrow & & \downarrow \\
\Omega_{v_!(F),v_!(G)}\left(\mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^{*}\right)^{*} & \longrightarrow & \Omega_{w_!(F),w_!(G)}\left(\mathrm{N}\left((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^{*}\right)^{*}
\end{array} \tag{4.53}$$

where $\mathrm{Map}^{\mathrm{eq}}$ is the subsimplicial set of weak equivalences in $\mathrm{Map}_{\mathcal{M}}$ where $\mathcal{M}$ is a simplicial model category.

By the adjunction from §3.2 we can rewrite the diagram as

$$\begin{tikzcd} \mathrm{Map}^{\mathrm{eq}}(F,G)^* \arrow[r] \arrow[d] & \mathrm{Map}^{\mathrm{eq}}(F,u^*\circ u_!(G))^* \arrow[d] \\ \mathrm{Map}^{\mathrm{eq}}(F,v^*\circ v_!(G))^* \arrow[r] & \mathrm{Map}^{\mathrm{eq}}(F,w^*\circ w_!(G))^*. \end{tikzcd} \tag{4.56}$$

Using the Yoneda lemma it now suffices to prove that the natural morphism

$$G \to u^*\circ u_!(G) \times^{\mathrm{h}}_{w^*\circ w_!(G)} v^*\circ v_!(G) \tag{4.57}$$

is a weak equivalence in $(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})$-$\mathrm{dg\,Mod}_k$. As we have restricted ourselves to the subcategory of right quasi-representable dg modules we can take $C\in\mathrm{Obj}(\mathcal{C})$ and check wether

$$G(C,-) \to \Big(u^*\circ u_!(G) \times^{\mathrm{h}}_{w^*\circ w_!(G)} v^*\circ v_!(G)\Big)(C,-) \tag{4.58}$$

is a weak equivalence in $\mathcal{D}^{\mathrm{op}}$-$\mathrm{dg\,Mod}_k$. Hence $\mathcal{C}$ no longer plays a role, and we can assume $\mathcal{C}=k$. Then we can find $D\in\mathrm{Obj}(\mathcal{D})$ such that $G\cong \mathrm{h}_D$, and we can write (4.58) as

$$G(-) \to \Big(u^*\circ u_!(G) \times^{\mathrm{h}}_{w^*\circ w_!(G)} v^*\circ v_!(G)\Big)(-). \tag{4.59}$$

By the interpretation of the adjunction from §3.2 we obtain for each $D'\in\mathrm{Obj}(\mathcal{D})$ the natural isomorphisms

$$\begin{cases} u^*\circ u_!(G)(D')^\bullet \cong \mathrm{Hom}_{\mathcal{D}_1}\big(u(D'),u(D)\big)^\bullet \\ v^*\circ v_!(G)(D')^\bullet \cong \mathrm{Hom}_{\mathcal{D}_2}\big(v(D'),v(D)\big)^\bullet \\ w^*\circ w_!(G)(D')^\bullet \cong \mathrm{Hom}_{\mathcal{D}_3}\big(w(D'),w(D)\big)^\bullet. \end{cases} \tag{4.60}$$

Hence (4.59) can be written as

$$\mathrm{Hom}_{\mathcal{D}}(D',D)^\bullet \to \mathrm{Hom}_{\mathcal{D}_1}(u(D'),u(D))^\bullet \times^{\mathrm{h}}_{\mathrm{Hom}_{\mathcal{D}_3}(w(D'),w(D))^\bullet} \mathrm{Hom}_{\mathcal{D}_2}(v(D'),v(D))^\bullet. \tag{4.61}$$

But as p is assumed to be a fibration in $\mathrm{dg\,Cat}_k$ this is a quasi-isomorphism of cochain complexes, as the pullback on the left is weakly equivalent (i.e. quasi-isomorphic) to the homotopy pullback [HTT, proposition A.2.4.4]. Hence (4.57) is a weak equivalence, so (4.56), and therefore (4.55) and (4.53) are homotopy pullbacks. As it suffices that the diagram of path spaces is a homotopy pullback to prove that (4.52) is a homotopy monomorphism, we have obtained the desired result.

To prove that we also have a *surjection on the level of the connected components* a construction is used. One first constructs an explicit model $\mathcal{N}$ for the codomain of (4.52) such that there is a natural isomorphism after considering the connected components. Then one proves the surjectivity for this model.

Let $\mathcal{N}$ be the category whose objects are diagrams

$$\begin{tikzcd} & p_!(F_1) \arrow[d, "a"] \\ q_!(F_2) \arrow[r, "b"'] & F_3 \end{tikzcd} \tag{4.62}$$

where $F_i \in \mathrm{Obj}((\mathcal{C}\otimes\mathcal{D}_i)\text{-dg}\,\mathrm{Mod}^{\mathrm{rqr,cof}}_{k,\mathrm{weq}})$, so a and b are morphisms in $(\mathcal{C}\otimes\mathcal{D}_3)\text{-dg}\,\mathrm{Mod}^{\mathrm{rqr,cof}}_{k,\mathrm{weq}}$. The morphisms in this category are the morphisms of diagrams. If we take the nerve of this category we get a good model for the homotopy fiber product on the level of the connected components.

There is moreover a natural map

$$
(4.63)\quad (\mathcal{C}\otimes\mathcal{D})\text{-dg}\,\mathrm{Mod}^{\mathrm{rqr,cof}}_{k,\mathrm{weq}} \to \mathcal{N} : F \mapsto \begin{tikzcd} & u_!(F) \arrow[d, "a"] \\ v_!(F) \arrow[r, "b"'] & w_!(F) \end{tikzcd}
$$

where a and b correspond to the isomorphisms $p_! \circ u_! \cong w_!$ and $q_! \circ v_! \cong w_!$. This induces a morphism

$$
(4.64)\quad \pi_0 \circ \mathrm{N}\left((\mathcal{C}\otimes\mathcal{D})\text{-dg}\,\mathrm{Mod}^{\mathrm{rqr,cof}}_{k,\mathrm{weq}}\right) \to \pi_0 \circ \mathrm{N}(\mathcal{N})
$$

and we would like to prove its surjectivity.

Consider an object of $\mathcal{N}$. Define

$$
(4.65)\quad F := u^*(F_1) \times^{\mathrm{h}}_{w^*(F_3)} v^*(F_2)
$$

in $\mathrm{Ho}((\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k)$. By the adjunction $(\mathbf{L}f_!, f^*)$ we obtain morphisms

$$
(4.66)\quad \mathbf{L}u_!(F) \to F, \qquad \mathbf{L}v_!(F) \to F_2, \qquad \mathbf{L}w_!(F) \to F_3
$$

in $\mathrm{Ho}((\mathcal{C}\otimes\mathcal{D}_i)\text{-dg}\,\mathrm{Mod}_k)$. As the role of $\mathcal{C}$ is negligible we can assume $\mathcal{C} = k$, the unit dg category. Then by the restrictions we have imposed on the categories we can write $F_i = \mathrm{h}_{D_i}$ for $D_i \in \mathrm{Obj}(\mathcal{D}_i)$. We can lift the weak equivalence

$$
(4.67)\quad a : p_!(\mathrm{h}_{D_1}) = \mathrm{h}_{p(D_1)} \xrightarrow{\sim} \mathrm{h}_{D_3}
$$

to a weak equivalence

$$
(4.68)\quad \mathrm{h}_{D_1} \xrightarrow{\sim} \mathrm{h}_{D_1'}
$$

in $\mathcal{D}_1^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k$ because p is assumed to be a fibration. This allows us to set $D_1 := D_1'$, hence $p(D_1) = D_3$ and $a = \mathrm{id}$. Similarly we can lift the weak equivalence

$$
(4.69)\quad b : q_!(\mathrm{h}_{D_2}) = \mathrm{h}_{q(D_2)} \xrightarrow{\sim} \mathrm{h}_{D_3} = \mathrm{h}_{p(D_1)}
$$

to a weak equivalence

$$
(4.70)\quad \mathrm{h}_{D_1''} \to \mathrm{h}_{D_1}
$$

in $(\mathcal{D}_1^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k$. Now setting $D_1 := D_1''$ yields $q(D_2) = p(D_1) = D_3$ and $a = b = \mathrm{id}$. Hence F is right quasi-representable by the triple (D_1, D_2, D_3) and we have an isomorphism in (4.66). This concludes the proof of the surjectivity. □

Remark that the condition on $\mathcal{C}$ is only to avoid notational issues, we can (and will) always consider a cofibrant replacement. Finally we can state our main technical lemma.

Lemma 4.23. Let $m : F \to G$ be a morphism in $\mathrm{Func}(\mathrm{dg}\,\mathrm{Cat}_k, \mathrm{sSet})$, i.e. a natural transformation between functors $F, G : \mathrm{dg}\,\mathrm{Cat}_k \to \mathrm{sSet}$. Let the following conditions be satisfied:

(i) the functors F and G send quasi-equivalences in $\mathrm{dg}\,\mathrm{Cat}_k$ to weak equivalences in sSet;

(ii) F and G map the terminal object in $\mathrm{dg\,Cat}_k$ (i.e. the category with one object and zero endomorphism ring) to the terminal object in sSet (i.e. the one-point simplicial set);

(iii) for all $\mathcal{C} \in \mathrm{dg\,Cat}_k$ is the associated morphism $m_{\mathcal{C}} : \pi_0(F(\mathcal{C})) \to \pi_0(G(\mathcal{C}))$ an isomorphism;

(iv) let $p : \mathcal{D}_1 \twoheadrightarrow \mathcal{D}_3$ and $\mathcal{D}_2 \to \mathcal{D}_3$ be morphisms of dg categories, assume p to be a fibration and consider the cartesian diagram

$$\begin{tikzcd} \mathcal{C}_1 \times_{\mathcal{D}_3} \mathcal{D}_2 \arrow[r] \arrow[d] & \mathcal{D}_1 \arrow[d, two heads, "p"] \\ \mathcal{D}_2 \arrow[r] & \mathcal{D}_3, \end{tikzcd} \tag{4.71}$$

then both induced commutative diagrams

$$\begin{tikzcd} F(\mathcal{C}_1 \times_{\mathcal{D}_3} \mathcal{D}_2) \arrow[r] \arrow[d] & F(\mathcal{D}_1) \arrow[d] \\ F(\mathcal{D}_2) \arrow[r] & F(\mathcal{D}_3) \end{tikzcd} \qquad \begin{tikzcd} G(\mathcal{C}_1 \times_{\mathcal{D}_3} \mathcal{D}_2) \arrow[r] \arrow[d] & G(\mathcal{D}_1) \arrow[d] \\ G(\mathcal{D}_2) \arrow[r] & G(\mathcal{D}_3) \end{tikzcd} \tag{4.72}$$

are homotopy cartesian diagrams in $\mathrm{dg\,Cat}_k$.

Then the natural transformation m induces a weak equivalence $k_{\mathcal{C}} : F(\mathcal{C}) \xrightarrow{\sim} G(\mathcal{C})$ in sSet.

Proof. Using condition (i) we obtain an Ho(sSet)-enrichment on the functors $\mathrm{Ho}\,F$ and $\mathrm{Ho}\,G$. Now let K be an object in Ho(sSet) and $\mathcal{C}$ a dg category. Then we get natural morphisms

$$F(\mathcal{C}^{\mathbf{R}K}) \to \mathrm{Map}_{\mathrm{sSet}}(K, F(\mathcal{C}))^* \tag{4.73}$$

and

$$G(\mathcal{C}^{\mathbf{R}K}) \to \mathrm{Map}_{\mathrm{sSet}}(K, G(\mathcal{C}))^* \tag{4.74}$$

in Ho(sSet). Now let K be a finite simplicial set, i.e. there are only finitely many non-degenerate simplices. Then it can be constructed using finitely many pushouts and coproducts, as for instance in [PiM174, proposition I.2.3]. Hence the object $\mathcal{C}^{\mathbf{R}K}$ can be constructed (functorially) using finitely many homotopy products and homotopy fibre products. So using (ii) and (iv) we see that (4.73) and (4.74) are actually isomorphisms.
If we now take the π_0 of the simplicial sets in (4.73) and (4.74) we obtain by (iii) the isomorphisms

$$\pi_0\left(F(\mathcal{C}^{\mathbf{R}K})\right) \xrightarrow{\cong} \pi_0\left(\mathrm{Map}_{\mathrm{sSet}}(K, F(\mathcal{C}))^*\right) \tag{4.75}$$

and

$$\pi_0\left(G(\mathcal{C}^{\mathbf{R}K})\right) \xrightarrow{\cong} \pi_0\left(\mathrm{Map}_{\mathrm{sSet}}(K, G(\mathcal{C}))^*\right) \tag{4.76}$$

in Set. By chaining these using (iii) we get the isomorphism

$$\pi_0\left(\mathrm{Map}_{\mathrm{sSet}}(K, F(\mathcal{C}))^*\right) \cong \pi_0\left(\mathrm{Map}_{\mathrm{sSet}}(K, G(\mathcal{C}))^*\right). \tag{4.77}$$

Now we can get the desired weak equivalence by observing that all the homotopy groups of $F(\mathcal{C})$ and $G(\mathcal{C})$ can be obtained by evaluating (4.77) in the standard simplicial sets Δ^k. $\square$

Recall that we have proved the weak equivalence of $\psi_{\mathcal{D}}$ in proposition 4.24. Then we can prove the following.

Proposition 4.24. The morphism

$$(4.78)\quad \phi_{\mathcal{D}}\colon \operatorname{Map}_{\operatorname{dg Cat}_k}(\mathcal{C},\mathcal{D})^* \to \operatorname{diag}\left(\mathrm{N}\left(((\Gamma^*(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\operatorname{dg Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\right)^*\right)$$

in sSet is a weak equivalence.

Proof. In §4.2 we have indicated the functoriality of the diagram (4.23) with respect to the first variable. By the functoriality of $\operatorname{Hom}_{\operatorname{dg Cat}_k}(-,-)$ in both variables we observe that the partial map

$$(4.79)\quad \operatorname{Map}_{\operatorname{dg Cat}_k}(\mathcal{C},-)^*\colon \operatorname{dg Cat}_k \to \mathrm{sSet} : \mathcal{D}\mapsto \operatorname{Hom}_{\operatorname{dg Cat}_k}(\Gamma^*(\mathcal{C}),\mathcal{D})$$

for $\mathcal{C}\in\operatorname{Obj}(\operatorname{dg Cat}_k)$ is indeed a true functor.

In the case of the functor

$$(4.80)\quad (\mathcal{C}\otimes -^{\mathrm{op}})\text{-}\operatorname{dg Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}\colon \operatorname{dg Cat}_k \to \mathrm{Cat} : \mathcal{D}\mapsto (\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-}\operatorname{dg Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}$$

there is unfortunately no functoriality. To see this, let $f:\mathcal{D}\to\mathcal{E}$ be a morphism in $\operatorname{dg Cat}_k$. The lower shriek

$$(4.81)\quad f_!\colon \mathcal{D}^{\mathrm{op}}\text{-}\operatorname{dg Mod}_k \to \mathcal{E}^{\mathrm{op}}\text{-}\operatorname{dg Mod}_k$$

can be extended to

$$(4.82)\quad \mathrm{id}_{\mathcal{C}}\otimes f_!\colon (\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-}\operatorname{dg Mod}_k \to (\mathcal{C}\otimes\mathcal{E}^{\mathrm{op}})\text{-}\operatorname{dg Mod}_k$$

or by (3.35) we can reinterpret this as

$$(4.83)\quad f_!\text{-}\operatorname{dg Mod}_k(\mathcal{C})\colon \mathcal{D}^{\mathrm{op}}\text{-}\operatorname{dg Mod}_k(\mathcal{C}) \to \mathcal{E}^{\mathrm{op}}\text{-}\operatorname{dg Mod}_k(\mathcal{C}).$$

This functor can be restricted to the category $(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-}\operatorname{dg Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}}$: an object F in this category yields a partial functor $F(X,-)$ for $X\in\operatorname{Obj}(\mathcal{C})$ that is by assumption cofibrant in $\mathcal{D}^{\mathrm{op}}\text{-}\operatorname{dg Mod}_k$. Because $f_!$ is a left Quillen functor it preserves cofibrations and acyclic cofibrations, so we can indeed restrict the codomain.

But for morphisms $f:\mathcal{D}\to\mathcal{E}$ and $g:\mathcal{E}\to\mathcal{F}$ in $\operatorname{dg Cat}_k$ the lower shriek only satisfies

$$(4.84)\quad (g\circ f)_! \cong g_!\circ f_!$$

and in general this is not the identity. The reason for this is that dg categories are a generalization of modules over a ring, and lower shriek is the analogue of the tensor product as base change. Because we cannot get functoriality in this baby case (but we do get pseudofunctoriality by the associativity of the tensor product) we cannot get functoriality in our general situation. However, we can apply the strictification procedure as indicated in remark 4.21, so we get a natural equivalence

$$(4.85)\quad (\mathcal{C}\otimes -^{\mathrm{op}})\text{-}\operatorname{dg Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof}} \Rightarrow (\mathcal{C}\otimes -^{\mathrm{op}})\text{-}\operatorname{dg Mod}_{k,\mathrm{w\,eq}}^{\mathrm{r\,qr,cof,str}}$$

and our pseudofunctor is lifted to a true presheaf of categories on $\operatorname{dg Cat}_k$.

We want to lift this strictification procedure to the diagram (4.23). To do so, observe that $\operatorname{Hom}_{\operatorname{dg Cat}_k}(-,-)$ is functorial in both variables, and that a dg functor in $\operatorname{Hom}_{\operatorname{dg Cat}_k}(\mathcal{C},\mathcal{D})$

can be considered as an object in $(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}}$. So we get natural transformations

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{dg\,Cat}_k}(\mathcal{C},-) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (\mathcal{C}\otimes -^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof}}_{k,\mathrm{w\,eq}} \arrow[d, Rightarrow] \\
& (\mathcal{C}\otimes -^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}} \arrow[d, Rightarrow, "\mathrm{N}"] \\
& \mathrm{N}\left((\mathcal{C}\otimes -^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^* .
\end{tikzcd} \tag{4.86}$$

Hence the composition is a functor $\mathrm{dg\,Cat}_k \to \mathrm{sSet}$. After applying the cosimplicial resolution functor in the first variable we can lift (4.23) using (4.86) to

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},-)^* \arrow[r, Rightarrow, "\phi^{\mathrm{str}}"] & \mathrm{diag}\left(\mathrm{N}\left((\mathcal{C}\otimes -^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^*\right) \\
& \mathrm{N}\left((\mathcal{C}\otimes -^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^* \arrow[u, Rightarrow, "\psi^{\mathrm{str}}"']
\end{tikzcd} \tag{4.87}$$

in $\mathrm{Func}(\mathrm{dg\,Cat}_k,\mathrm{sSet})$. After evaluating in $\mathcal{D}$ we get

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^* \arrow[r, "\phi^{\mathrm{str}}_{\mathcal{D}}"] & \mathrm{diag}\left(\mathrm{N}\left((\mathcal{C}\otimes \mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^*\right) \\
& \mathrm{N}\left((\mathcal{C}\otimes \mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^* \arrow[u, "\psi^{\mathrm{str}}_{\mathcal{D}}"']
\end{tikzcd} \tag{4.88}$$

which is a diagram in sSet, weakly equivalent to the one in (4.23). So in order to prove the weak equivalence of $\phi_{\mathcal{D}}$ as is the goal in this proposition it suffices to prove this for $\phi^{\mathrm{str}}_{\mathcal{D}}$. Now using that

(i) for weak equivalences we have the 3-out-of-2 property;

(ii) the map $\psi_{\mathcal{D}}$ is a weak equivalence by proposition 4.15, and therefore so is $\psi^{\mathrm{str}}_{\mathcal{D}}$;

(iii) every isomorphism in $\mathrm{Ho}(\mathrm{Func}(\mathrm{dg\,Cat}_k,\mathrm{sSet}))$ arises from a weak equivalence in the model category $\mathrm{Func}(\mathrm{dg\,Cat}_k,\mathrm{sSet})$ [MSM99, theorem 8.3.10];

we can consider the map

$$m := (\psi^{\mathrm{str}}_{\mathcal{D}})^{-1}\circ\phi_{\mathcal{D}}\colon \mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},-)^* \to \mathrm{N}\left((\mathcal{C}\otimes -^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}^{\mathrm{r\,qr,cof,str}}_{k,\mathrm{w\,eq}}\right)^* \tag{4.89}$$

in $\mathrm{Ho}(\mathrm{Func}(\mathrm{dg\,Cat}_k,\mathrm{sSet}))$. This notation is already reminiscent of lemma 4.23. If we can check that the conditions for this lemma are satisfied the proof is finished. For the functor $\mathrm{Map}_{\mathrm{dg\,Cat}_k}$ this is standard [MSM63, §5.4]. For the second functor one applies remark 3.18, lemma 4.20 and lemma 4.22. □

The proof of the fundamental weak equivalence is now reduced to piecing together the previous results and constructions.

Proof of theorem 4.14. Apply propositions 4.15 and 4.24 to obtain the result. □

4.4 The fundamental weak equivalence: corollaries

We can now discuss some immediate corollaries of the fundamental weak equivalence in theorem 4.14. The first corollary is actually an important result hence we will call it a theorem. Recall from §4.2 that we defined the rather involved object

$$(\Gamma^n(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}. \tag{4.90}$$

But it turns out that, as suggested in remark 4.9, we can simplify this object. So whereas the fundamental weak equivalence is mostly a technical result, we can give a nice interpretation in concrete terms. One might call this the fundamental bijection.

Theorem 4.25 (Fundamental bijection)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be small dg categories. We have a functorial bijection

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{C},\mathcal{D})\cong\mathrm{Isom}\left(\mathrm{Ho}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k^{\mathrm{rqr}}\right)\right). \tag{4.91}$$

Proof. The idea will be to add one more weak equivalence to the diagram in (4.23) which results in (4.93). We have the natural inclusion of the subcategory with the technical condition in the bigger category where we do not impose the cofibrant condition:

$$i\colon(\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\to(\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr}}. \tag{4.92}$$

By applying the cofibrant replacement functor on an object in the codomain of the inclusion we get an object in the domain, where we have used lemma 3.6 and using (3.35). So after applying the nerve construction we get the weak equivalences

$$\begin{tikzcd}
 & \mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C},\mathcal{D})^* \arrow[r, "\phi_{\mathcal{D}}"] & \mathrm{diag}\left(\mathrm{N}\left((\Gamma^*(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\right)^*\right) \\
\mathrm{N}\left(\mathrm{Q}(\mathcal{C}\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr}}\right)^* & & \mathrm{N}\left((\mathrm{Q}(\mathcal{C})\otimes\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathrm{weq}}^{\mathrm{rqr,cof}}\right)^*. \arrow[ll, "\mathrm{N}(i)"'] \arrow[u, "\psi_{\mathcal{D}}"']
\end{tikzcd} \tag{4.93}$$

Using the interpretation as explained in the proof of lemma 4.20 we can drop the enrichments and this yields the result in (4.91). ☐

4.5 Simplicial model category structure on $\mathrm{dg\,Cat}_k$

In the preceding we have used the nerve construction, which is a way of turning a category into a simplicial set. We can do the opposite, given a simplicial set associate a category to it.

Definition 4.26. Let K be a simplicial set. Its *simplex category* $(X\downarrow\Delta)$ is the category of objects under X.

It has a concrete interpretation: the objects of $(X\downarrow\Delta)$ are natural transformations $\Delta^n\to K$, hence can be considered as pairs $(n,x)\in\Delta\times K_n$. Its morphisms are natural transformations $\Delta^n\Rightarrow\Delta^m$ that fit in a triangle like (4.3) coming from a map $[n]\to[m]$, so we can interpret a morphism $([n],x)\to([m],y)$ as a morphism of simplicial sets $f:[n]\to[m]$ such that $f^*(y)=x$.

Using this construction we can deduce the simplicial structure that is present on $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$. In the previous paragraphs we have obtained the existence of a Map-object, and this structure is moreover part of a tensor-cotensor structure on $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$. The following theorem, which can be obtained by repeating the arguments in the previous paragraphs, describes the result from [Toe07, §5].

Theorem 4.27 (Simplicial structure on $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be small dg categories. Let K be a simplicial set. Then we have a functorial injective map

(4.94) $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left(K \otimes^{\mathbf{L}}_s \mathcal{C}, \mathcal{D}\right) \to \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left({}^k\langle (K \downarrow \Delta)\rangle \otimes^{\mathbf{L}} \mathcal{C}, \mathcal{D}\right).$

The image of this map consists of those $f\colon {}^k\langle (K \downarrow \Delta)\rangle \otimes^{\mathbf{L}} \mathcal{C} \to \mathcal{D}$ such that for all $X \in \mathrm{Obj}(\mathcal{C})$ the partial functor

(4.95) $f(-,X)\colon {}^k\langle (K \downarrow \Delta)\rangle = \mathrm{H}^0\left({}^k\langle (K \downarrow \Delta)\rangle\right) \to \mathrm{H}^0(\mathcal{D})$

sends all the morphisms in the category ${}^k\langle (K \downarrow \Delta)\rangle$ to isomorphisms in $\mathcal{D}$.

4.6 Existence of internal Hom's in $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$

Recall from §1.2 that we have both a symmetric monoidal structure on the category of differential graded categories, and from §3.1 that we have a model category structure on $\mathrm{dg\,Cat}_k$. But we can observe that it is not a symmetric monoidal model category [MSM63, definition 4.2.18]. The reason for this is that the tensor product of two cofibrant objects is not necessarily cofibrant, which is nevertheless one of the conditions [MSM63, definitions 1.3.1, 4.2.1 and 4.2.6].

Example 4.28. It is enough to restrict oneself to algebras considered as dg categories with a single object. In general a cofibrant dg algebra is built using retract of (noncommutative) free dg algebras. The differential is moreover chosed to be compatible with the order of the variables. But the tensor product of two free algebras in one variable yields

(4.96) $k\langle X\rangle \otimes_k k\langle Y\rangle \cong k[X,Y].$

This (commutative) polynomial algebra is no longer free, as a noncommutative free algebra in more than 1 variable has trivial center k, while $\mathrm{Z}(k[X,Y]) = k[X,Y]$. Moreover, taking retracts will not amend the situation by inspecting the centers in the retraction diagram.

We also constructed an internal Hom for $\mathrm{dg\,Cat}_k$ in definition 1.25, which we denoted by $\mathcal{H}\mathrm{om}$. This provided a *closed* monoidal structure on $\mathrm{dg\,Cat}_k$. But by the previously discussed incompatibility of the monoidal and model structures on $\mathrm{dg\,Cat}_k$ this internal Hom does not descend to $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$. The problem is that the $\mathcal{H}\mathrm{om}$ of both fibrant and cofibrant objects in $\mathrm{dg\,Cat}_k$ is not invariant under quasi-equivalences, so after localising we do not get a well-defined object. If we want to obtain a derived interal Hom for $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$ we will therefore have to construct one explicitly and prove that it satisfies the axioms.

Lemma 4.29. Let $\mathcal{M}$ be a cofibrantly generated $\mathrm{Ch}(k\text{-Mod})$-model category. Assume furthermore that the domains and codomains of the set of generating cofibrations are cofibrant objects in $\mathcal{M}$.

Let $\mathcal{M}_0$ be a full subcategory of $\mathcal{M}$ closed under its weak equivalences. Denote $\mathrm{Int}(\mathcal{M}_0)$ the full dg subcategory of $\mathrm{Int}(\mathcal{M})$ of both fibrant and cofibrant objects which belong to $\mathcal{M}_0$.

Let $\mathcal{C}$ be a cofibrant dg category. We will denote $\mathrm{Ho}(\mathcal{C}\text{-dg\,Mod}_k(\mathcal{M}_0))$ the full subcategory of $\mathrm{Ho}(\mathcal{C}\text{-dg\,Mod}_k(\mathcal{M}))$ consisting of the objects $F \in \mathrm{Ho}(\mathcal{C}\text{-dg\,Mod}_k(\mathcal{M}))$ such that $F(X)$ actually evaluates to an object in $\mathcal{M}_0$ for each $X \in \mathrm{Obj}(\mathcal{C})$. Then the natural morphism

(4.97) $\phi\colon \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{C}, \mathrm{Int}(\mathcal{M}_0)) \to \mathrm{Isom}(\mathrm{Ho}(\mathcal{C}\text{-dg\,Mod}_k(\mathcal{M}_0)))$

is an isomorphism.

Proof. As before the map ϕ in (4.97) sends a dg functor $\mathcal{C} \to \mathrm{Int}(\mathcal{M}_0)$ to the representing object, see lemma 4.16 for another instance of this theme. To check that we actually can pass to the set of isomorphism classes we observe that by applying lemma 3.14 to a quasi-equivalence $f\colon \mathcal{C} \xrightarrow{\sim} \mathrm{Int}(\mathcal{M}_0)$ we obtain a Quillen equivalence $(f_!, f^*)$ between the representing objects. So after taking the homotopy category of $\mathrm{dg}\,\mathrm{Cat}_k$ we get isomorphic objects in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$ and therefore as well in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}_0))$ by the definition of this category. So the map is well-defined.

We can now check that it is actually an isomorphism. First of all, it is *surjective* by applying lemma 4.16 and observing that the conditions for defining $\mathrm{Int}(\mathcal{M}_0)$ and $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}_0))$ are compatible.

To see that it is *injective*, let $f,g\colon \mathcal{C}\to\mathrm{Int}(\mathcal{M}_0)$ be two morphisms in $\mathrm{dg}\,\mathrm{Cat}_k$ such that their representing objects are isomorphic in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}_0))$. Then these objects are isomorphic in $\mathrm{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$ too. Denote $i\colon \mathrm{Int}(\mathcal{M}_0)\to\mathrm{Int}(\mathcal{M})$ the inclusion of dg categories. By applying lemma 4.18 to the compositions

$$i\circ f, i\circ g\colon \mathcal{C}\to\mathcal{M} \tag{4.98}$$

we get a homotopy in $\mathrm{dg}\,\mathrm{Cat}_k$. So we can consider its associated diagram

$$\begin{tikzcd} \mathcal{C} \arrow[d] \arrow[dr, "i\circ f"] & \\ \mathcal{C}' \arrow[r, "H"] & \mathrm{Int}(\mathcal{M}). \\ \mathcal{C} \arrow[u] \arrow[ur, "i\circ g"'] & \end{tikzcd} \tag{4.99}$$

By the assumptions $\mathcal{M}_0$ is closed in $\mathcal{M}$ under weak equivalences, so we can factor the (left) homotopy $\mathcal{C}'\to\mathrm{Int}(\mathcal{M})$ through $\mathrm{Int}(\mathcal{M}_0)$ as the evaluations all land in the correct subcategory. So using the details from [D-S, §5] we see that we can change (4.99) to the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[d] \arrow[dr, "f"'] \arrow[drr, "i\circ f"] & & \\ \mathcal{C}' \arrow[r, "H'"] \arrow[rr, bend right, "H"'] & \mathrm{Int}(\mathcal{M}_0) \arrow[r, "i"] & \mathrm{Int}(\mathcal{M}) \\ \mathcal{C} \arrow[u] \arrow[ur, "g"] \arrow[urr, "i\circ g"'] & & \end{tikzcd} \tag{4.100}$$

where H' is a (left) homotopy between f and g, hence these morphisms are identified in the homotopy category $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$. □

Now we are ready to state the main theorem of this section. The situation is reminiscent of the construction of the exceptional inverse image functor in algebraic geometry, which is denoted $f^!$ or $\mathbf{R}f^!$. But neither notation is good because this functor lives in a derived categories context, so $f^!$ does not suggest the correct context, while $\mathbf{R}f^!$ suggests that it is the derived functor of some other functor, but that is not the case. The "derived" internal Hom in $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$ is by the discussion in the introduction of this section *not* the derived version of the internal Hom on $\mathrm{dg}\,\mathrm{Cat}_k$, so one could argue about the notation $\mathbf{R}\mathcal{H}\mathrm{om}$. We will nevertheless use it.

Theorem 4.30 (Internal Hom for $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$)**.** It is possible to equip the monoidal category $(\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k, -\otimes^{\mathbf{L}}-))$ with an internal Hom-object which we will denote $\mathbf{R}\mathcal{H}\mathrm{om}$, hence it is a closed monoidal category. This internal Hom is moreover characterised by

$$\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C},\mathcal{D})\cong\mathrm{Int}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k^{\mathrm{rqr}}\right) \tag{4.101}$$

for $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k))$.

Proof. Using the proposed characterisation we can check that

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{E}\otimes^{\mathbf{L}}\mathcal{C},\mathcal{D}) \\
&\quad\cong \mathrm{Isom}\left(\mathrm{Ho}\left(\left((\mathcal{E}\otimes^{\mathbf{L}}\mathcal{C})\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}}\right)\text{-dg}\,\mathrm{Mod}_k^{\mathrm{rqr}}\right)\right) && (4.41)\\
&\quad\cong \mathrm{Isom}\left(\mathrm{Ho}\left(\left(\mathcal{E}\otimes^{\mathbf{L}}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\right)\text{-dg}\,\mathrm{Mod}_k^{\mathrm{rqr}}\right)\right) && \text{associativity } -\otimes^{\mathbf{L}}-\\
&\quad\cong \mathrm{Isom}\left(\mathrm{Ho}\left(\mathcal{E}\text{-dg}\,\mathrm{Mod}_k((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k^{\mathrm{rqr}})\right)\right) && (3.35)\\
&\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left(\mathcal{E},\mathrm{Int}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k^{\mathrm{rqr}}\right)\right) && \text{lemma 4.29}
\end{aligned}\tag{4.102}$$

where lemma 4.29 is applied with

$$\begin{aligned}
\mathcal{M} &:= (\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k,\\
\mathcal{M}_0 &:= (\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k^{\mathrm{rqr}}.
\end{aligned}\tag{4.103}$$

So we can conclude there exists an object that serves as the internal Hom for $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$. □

Remark 4.31. The closedness of the monoidal structure yields the *derived dg tensor-Hom adjunction*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D},\mathcal{E})\cong\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{C},\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{D},\mathcal{E}))\tag{4.104}$$

for all $\mathcal{C},\mathcal{D}$ and $\mathcal{E}$ dg categories. This adjunction is generalized in corollary 4.33.

In §4.5 we have introduced the simplicial model category structure on $\mathrm{dg\,Cat}_k$. By deriving this structure we get a simplicial structure on $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$ which is compatible with the derived tensor product and internal Hom.

Corollary 4.32. Let $\mathcal{C}$ and $\mathcal{D}$ be two dg categories. Let K be a simplicial set. Then we have the functorial isomorphism

$$K\otimes_s^{\mathbf{L}}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D})\cong(K\otimes_s^{\mathbf{L}}\mathcal{C})\otimes^{\mathbf{L}}\mathcal{D}\tag{4.105}$$

in $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$.

The internal Hom-functor as we have obtained it in theorem 4.30 is moreover yields an enriched adjointness.

Corollary 4.33. Let $\mathcal{C},\mathcal{D},\mathcal{E}$ be dg categories. Then we have the functorial isomorphism

$$\mathrm{Map}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D},\mathcal{E})^*\cong\mathrm{Map}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{C},\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{D},\mathcal{E}))^*\tag{4.106}$$

in Ho(sSet).

Proof. Let K be a simplicial set. Then we have the chain of isomorphisms

$$\begin{aligned}
&\pi_0\left(\mathrm{Hom}_{\mathrm{Ho(sSet)}}\left(K,\mathrm{Map}_{\mathrm{dg\,Cat}_k}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D},\mathcal{E})^*\right)\right)\\
&\quad\cong\pi_0\left(\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left(K\otimes_s^{\mathbf{L}}(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}),\mathcal{E}\right)\right) && \text{[MSM99, definition 9.1.6]}\\
&\quad\cong\pi_0\left(\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left((K\otimes_s^{\mathbf{L}}\mathcal{C})\otimes^{\mathbf{L}}\mathcal{D},\mathcal{E}\right)\right) && \text{corollary 4.32}\\
&\quad\cong\pi_0\left(\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left(K\otimes_s^{\mathbf{L}}\mathcal{C},\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{D},\mathcal{E})\right)\right) && \text{theorem 4.30}\\
&\quad\cong\pi_0\left(\mathrm{Hom}_{\mathrm{Ho(sSet)}}\left(K,\mathrm{Map}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}(\mathcal{C},\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{D},\mathcal{E}))^*\right)\right) && \text{[MSM99, definition 9.1.6]}
\end{aligned}\tag{4.107}$$

which yields the adjointness property in (4.106). □

The following corollary is immediate by the adjointness from corollary 4.33, and $\mathrm{Map}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}$ commuting with all homotopy colimits.

Corollary 4.34. Let $\mathcal{C}$ be a dg category. Then the functor

$$- \otimes^{\mathbf{L}} \mathcal{C} \colon \mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k) \to \mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k) \tag{4.108}$$

commutes with homotopy colimits.

As a final corollary we find that internal Hom "commutes with", or preserves quasi-fully faithful morphisms in $\mathrm{dg}\,\mathrm{Cat}_k$.

Corollary 4.35. Let $f : \mathcal{C} \to \mathcal{D}$ be a quasi-fully faithful morphism in $\mathrm{dg}\,\mathrm{Cat}_k$. Let $\mathcal{E}$ be a dg category. We have that

$$\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{E}, -)(f) \colon \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{C}) \to \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{D}) \tag{4.109}$$

is again quasi-fully faithful.

Proof. We have characterized quasi-fully faithful morphisms as homotopy monomorphisms in lemma 3.9. So we would like to prove that the endofunctor $\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{E}, -)$ preserves homotopy monomorphisms. But using the adjointness from corollary 4.33 this follows by observing that $\mathcal{E}$ is in the first position, and hence it is reduced to the preservation properties of the Map functor. $\square$

CHAPTER 5

Applications

"You can't make it with geometry and geometrical systems of thinking. It's all *this*!"

Dean Moriarty in *On the Road*
JACK KEROUAC

In [Toe07] three applications of theorem 4.14 are discussed. These are:

(i) the relation between Hochschild cohomology and $\mathbf{R}\mathcal{H}\mathrm{om}$, its Morita invariance, the link with the geometric realisation and the derived Picard group [Toe07, §8.1];

(ii) the existence of a good theory for localisations and quotients of dg categories [Toe07, §8.2];

(iii) (continuous) functors between the derived categories of quasicoherent sheaves on a sufficiently nice scheme are represented by an element of the derived category of quasicoherent sheaves on their product [Toe07, §8.3]

To obtain these results one first introduces derived Morita theory (or Morita theory for dg categories). This is done in §5.1.

In this text the first and last application will be discussed. The first one is a generalisation of Hochschild cohomology to dg categories, and the fact that this Hochschild cohomology can be expressed in terms of the internal $\mathbf{R}\mathcal{H}\mathrm{om}$.

Concerning the last result, recall from [Orl97] that triangulated equivalences between the derived categories of quasicoherent sheaves of *varieties* are representable in this way using so-called Fourier-Mukai transforms. If the schemes are quasicompact and separated over a ring k we have to consider the continuous functors to get representability, as discussed in theorem 5.19 [Toe07, theorem 8.9]. If one moreover assumes that the schemes are smooth and proper over a ring k one can drop the continuity condition if one restricts himself to the subcategory of perfect complexes [Toe07, theorem 8.15]. Using the formalism of derived algebraic geometry this theorem has been generalised in [BFN10] to the more general context of (derived) stacks.

5.1 Derived Morita theory

In this section we prove the main result from [Toe07, §7], which gives an incarnation of "derived Morita theory". Recall from the introduction that standard Morita theory for rings gives an equivalence between the functors A-Mod $\to$ B-Mod that preserve colimits and the category $(A^{\mathrm{op}} \otimes_k B)$-Mod, where A and B are associative algebras over a field k.

The correspondence gives us that every cocontinuous functor $F\colon A\text{-Mod} \to B\text{-Mod}$ can be represented by a bimodule P in $(A^{\mathrm{op}} \otimes_k B)$-Mod in the sense that $F(M) = M \otimes_A P$ for M in A-Mod.

One could try to play the same game for associative dg algebras $A^\bullet$ and $B^\bullet$, and their derived (in the classical sense) categories of dg modules $\mathbf{D}(A^\bullet\text{-dg}\,\mathrm{Mod}_k)$ and $\mathbf{D}(B^\bullet\text{-dg}\,\mathrm{Mod}_k)$, where we have used the notation from definition 1.28 and the interpretation of a dg algebra as a dg category from example 1.10. In a similar way as for the original case we obtain that the functor

$$\mathbf{D}(A^\bullet\text{-dg}\,\mathrm{Mod}_k) \to \mathbf{D}(B^\bullet\text{-dg}\,\mathrm{Mod}) : M^\bullet \mapsto M^\bullet \otimes^{\mathbf{L}}_{A^\bullet} P^\bullet \tag{5.1}$$

for $P^\bullet$ in $(A^{\bullet,\mathrm{op}} \otimes_k B^\bullet)\text{-dg}\,\mathrm{Mod}_k$ yields a triangulated functor. Unfortunately there exist triangulated functors that cannot be represented in this way [DS09], see also [DS04, remarks 2.5 and 6.8] for more context.

So if one wants to extend Morita theory to dg algebras, and more generally dg categories we have to change our approach. The theory as developed up to now will provide the solution we are looking for. The result we obtain is called derived Morita theory, as in [Toe07], but in fact "dg Morita theory" would be more appropriate, as there is a Morita theory for derived categories [Ric89]. Nevertheless we will use the original terminology. The actual result is given in theorem 5.12. It forms the cornerstone of all further applications, two of which will be discussed in sections 5.2 and 5.3.

First we have to set the stage with a few definitions and preliminary lemmas. The first lemma gives a relation between the internal dg category for the category of dg $\mathcal{C}$-modules (where $\mathcal{C}$ is as usual a small dg category) and the internal Hom for $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$ as constructed in §4.6.

Lemma 5.1. Let $\mathcal{C}$ be a small dg category. We have the isomorphism

$$\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k) \cong \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C}^{\mathrm{op}}, \mathrm{Int}(\mathrm{Ch}(k\text{-Mod}))) \tag{5.2}$$

in $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$.

Proof. Let $\mathcal{D}$ be a small dg category. We have the string of isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{D}, \mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right) \\
&\quad\cong \mathrm{Isom}\left(\mathrm{Ho}\left(\mathcal{D}\text{-dg}\,\mathrm{Mod}_k\left(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)\right)\right) && \text{lemma 4.29} \\
&\quad\cong \mathrm{Isom}\left(\mathrm{Ho}\left((\mathcal{D} \otimes^{\mathbf{L}} \mathcal{C}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k\right)\right) && (3.35) \\
&\quad\cong \mathrm{Isom}\left(\mathrm{Ho}\left((\mathcal{D} \otimes^{\mathbf{L}} \mathcal{C}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k\left(\mathrm{Ch}(k\text{-Mod})\right)\right)\right) && \text{remark 3.12} \\
&\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{D} \otimes^{\mathbf{L}} \mathcal{C}^{\mathrm{op}}, \mathrm{Int}(\mathrm{Ch}(k\text{-Mod}))\right) && \text{lemma 4.29} \\
&\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{D}, \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C}^{\mathrm{op}}, \mathrm{Int}(\mathrm{Ch}(k\text{-Mod})))\right) && \text{adjunction (4.104)}
\end{aligned} \tag{5.3}$$

where we applied lemma 4.29 with $\mathcal{M} = \mathcal{M}_0$ and the appropriate categories of dg modules. □

Remark 5.2. This lemma explains the terminology "internal dg category", as introduced in §3.3: to construct the internal dg category of a category of dg $\mathcal{C}$-modules we just have to consider the derived Hom of our coefficient category into the internal dg category of cochain complexes.

Remark 5.3. We also observe that this is related to the isomorphism

$$\mathrm{Int}(\mathrm{Ch}_{\mathrm{dg}}(k\text{-Mod})) \cong \mathrm{Int}(k\text{-dg}\,\mathrm{Mod}_k) \tag{5.4}$$

where k denotes the unit dg category for the monoidal structure on $\mathrm{dg}\,\mathrm{Cat}_k$, as discussed in remark 1.22.

Remark 5.4. We can rephrase the condition that a morphism $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{dg\,Cat}_k$ is continuous by requiring that for all all small families $(X_i)_{i\in I}$ of objects in $\mathcal{C}$ the natural morphism

$$\bigoplus_{i\in I}^{\mathbf{L}} F(X_i) \to F\left(\bigoplus_{i\in I} X_i\right) \tag{5.5}$$

is an isomorphism in $\mathrm{Ho}(\mathcal{D})$.

We now introduce some notation.

Definition 5.5. Let $\mathcal{C}$ and $\mathcal{D}$ be small dg categories. We define the *dg category of Morita morphisms* to be

$$\mathrm{Morita}_k(\mathcal{C},\mathcal{D}) \coloneqq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathrm{cont}}\left(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k), \mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right). \tag{5.6}$$

In the proof of proposition 5.11 we will need two (easy) isomorphisms. These straight-forward calculations are lemmas 5.6 and 5.7.

Lemma 5.6. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be small dg categories. Then we have a natural isomorphism

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{E}, \mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right)\\ &\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{C}, \mathrm{Int}\left((\mathcal{E}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k\right)\right).\end{aligned} \tag{5.7}$$

Proof. We have the string of isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{E}, \mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right)\\ &\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{E}, \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{D}^{\mathrm{op}}, \mathrm{Int}(k\text{-dg}\,\mathrm{Mod}_k)\right)\right) && \text{lemma 5.1}\\ &\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{E})\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}}, \mathrm{Int}(k\text{-dg}\,\mathrm{Mod}_k)\right) && \text{adjunction (4.104)}\\ &\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{C}\otimes^{\mathbf{L}}(\mathcal{E}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}}), \mathrm{Int}(k\text{-dg}\,\mathrm{Mod}_k)\right) && \text{associativity of } -\otimes^{\mathbf{L}}-\\ &\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{C}, \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{E}^{\mathrm{op}}\otimes^{\mathbf{L}}\mathcal{D}, \mathrm{Int}(k\text{-dg}\,\mathrm{Mod}_k)\right)\right) && \text{adjunction (4.104)}\\ &\quad\cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{C}, \mathrm{Int}\left((\mathcal{E}^{\mathrm{op}}\otimes^{\mathbf{L}}\mathcal{D})\text{-dg}\,\mathrm{Mod}_k\right)\right) && \text{lemma 5.1}\end{aligned} \tag{5.8}$$

which proves the lemma. □

A similar result can be proved where we replace the occurrences of $\mathcal{C}$ by its internal dg category of dg $\mathcal{C}$-modules and we restrict ourselves to continuous morphisms.

Lemma 5.7. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be small dg categories. Then we have a natural isomorphism

$$\begin{aligned}&\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\otimes^{\mathbf{L}}\mathcal{E}, \mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right)\\ &\quad\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathrm{Int}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k), \mathrm{Int}\left((\mathcal{E}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k\right)\right).\end{aligned} \tag{5.9}$$

Proof. This proof is a formal adaptation of the proof of lemma 5.6, with the addition of the cont index at every point. These should be interpreted in terms of the partial functors that are obtained in each step of the proof. □

Lemma 5.8. Let $\mathcal{C}$ be a small dg category. Let $\mathcal{M}$ be a combinatorial $\mathrm{Ch}(k\text{-Mod})$-model category. Let

$$F\colon\ \mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k) \to \mathcal{M} \tag{5.10}$$

be a continuous morphism. Let $X\colon I \to \mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k$ be a small diagram such that $X(i)$ is a cofibrant object in $\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k$ for all $i \in \mathrm{Obj}(I)$. Then we have that F commutes with homotopy colimits, i.e. the natural morphism

$$\operatorname*{hocolim}_{i\in I} F(X(i)) \to F\left(\operatorname*{hocolim}_{i\in I} X(i)\right) \tag{5.11}$$

is an isomorphism in $\mathrm{Ho}(\mathcal{M})$.

Proof. By the general theory of model categories every homotopy colimit is the composition of homotopy pushouts and homotopy direct sums. So if we prove commutativity with a continuous morphism for these two types of limits we can compose the obtained isomorphisms for an arbitrary homotopy colimit.

Because F was taken to be continuous it commutes with *homotopy direct sums*, as explained in remark 5.4.

For the commutativity of a continuous functor with an *homotopy pushout* in $\mathrm{Ho}(\mathcal{M})$ we can replace F by a fibrant and cofibrant object. Then using the model category structure on $\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod})\text{-dg}\,\mathrm{Mod}(\mathcal{M})$ which exists by the requirement of a combinatorial structure on $\mathcal{M}$ we see that all evaluations of F actually land in $\mathrm{Int}(\mathcal{M})$, so we can write every F as

$$F\colon \mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k) \to \mathrm{Int}(\mathcal{M}). \tag{5.12}$$

We now apply the dual Yoneda embedding to the domain and codomain of F^{op} which is the objectwise opposite of F. We obtain a diagram

$$\begin{tikzcd}
\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)^{\mathrm{op}} \arrow[r, "F^{\mathrm{op}}"] \arrow[d, "\mathrm{h}^-"'] & \mathrm{Int}(\mathcal{M})^{\mathrm{op}} \arrow[d, "\mathrm{h}^-"] \\
\mathrm{Int}\left((\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k\right) \arrow[r, "F_!^{\mathrm{op}}"'] & \mathrm{Int}\left((\mathrm{Int}(\mathcal{M})^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k\right).
\end{tikzcd} \tag{5.13}$$

The situation is now sufficiently flexible to prove the result. As $F_!^{\mathrm{op}}$ is a left Quillen functor it commutes with homotopy pushouts up to weak equivalences. Using [MSM63, theorem 7.1.11] and the fact that $\mathrm{Int}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k)^{\mathrm{op}}$ and $\mathrm{Int}(\mathcal{M})\text{-dg}\,\mathrm{Mod}_k$ are stable model categories we see that $F_!^{\mathrm{op}}$ also commutes with homotopy pullbacks, up to weak equivalences.

As we consider the dual Yoneda embedding we observe moreover that homotopy pushouts are sent to homotopy pullbacks. And a commutative square is a homotopy pushout diagram in $\mathrm{Int}(\mathcal{M})$ if and only if its Yoneda embedding in $\mathrm{Int}(\mathcal{M})\text{-dg}\,\mathrm{Mod}_k$ is a homotopy pullback diagram. So chasing a homotopy pushout around we see that F preserves them. □

To apply this result in the proof of lemma 5.10 we need the compatibility of a continuous morphism with the monoidal structure.

Lemma 5.9. Let $\mathcal{C}$ be a small dg category. Let $\mathcal{M}$ be a combinatorial $\mathrm{Ch}(k\text{-Mod})$-model category. Let

$$F\colon \mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k) \to \mathcal{M} \tag{5.14}$$

be a continuous morphism. Let $X^\bullet$ be a cochain complex of k-modules and take $M \in \mathrm{Obj}(\mathcal{M})$. Then the morphism

$$X^\bullet \otimes^{\mathbf{L}} F(M) \to F(X^\bullet \otimes^{\mathbf{L}} M) \tag{5.15}$$

is an isomorphism.

Proof. We can construct $X^\bullet$ from the trivial complex having k in degree zero, by repeated applications of homotopy colimits and loop objects. Using lemma 5.8 we have that F already commutes with homotopy colimits.

To prove that F also commutes with loop objects, it suffices to observe that the loop and suspension functor are inverses in the homotopy category. But the suspension functor is a homotopy pushout, so F commutes with suspension, hence F commutes with the loop functor in the homotopy category. $\square$

We will denote using $h := \mathrm{h}^-$ the Yoneda embedding

(5.16) $h\colon \mathcal{C} \to \mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)$.

Lemma 5.10. Let $\mathcal{C}$ be a small dg category. Let $\mathcal{M}$ be a combinatorial $\mathrm{Ch}(k\text{-Mod})$-model category, where the class of generating cofibrations is small with respect to a universe $\mathbb{V}$, and the category itself small with respect to a universe $\mathbb{W}$, as in remark 4.13.

Assume that the domains and codomains of the generating cofibrations are cofibrant objects in $\mathcal{M}$. Assume moreover that for every cofibrant object $M \in \mathrm{Obj}(\mathcal{M})$ and for every quasi-isomorphism $X^\bullet \to Y^\bullet$ in $\mathrm{Ch}(k\text{-Mod})$ the morphism $X^\bullet \otimes M \to Y^\bullet \otimes M$ is a weak equivalence in $\mathcal{M}$.

Then the Quillen adjunction

(5.17) $h_!\colon \mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}) \rightleftarrows \mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}) : h^*$

(i) yields a fully faithful functor

(5.18) $\mathbf{L}h_!\colon \mathrm{Ho}\left(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})\right) \to \mathrm{Ho}\left(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})\right)$

(ii) and its essential image describes all the dg $\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-}\mathrm{Mod}_k)$-modules with values in $\mathcal{M}$ that represent the continuous morphisms $\mathcal{C} \to \mathcal{M}$ in $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$.

Proof. By the general properties of Quillen adjoint functors we have that $\mathbf{L}h_!$ and h^* commute with homotopy colimits [MSM99, theorem 19.4.5]. So as in the proof of lemma 3.14 the situation is reduced to the case where we need to show that the adjunction morphism

(5.19) $M \otimes^{\mathbf{L}} \mathrm{h}^C \to h^* \circ \mathbf{L}h_! \left(M \otimes^{\mathbf{L}} \mathrm{h}^C\right)$

for $M \in \mathrm{Obj}(\mathcal{M})$, $C \in \mathrm{Obj}(\mathcal{C})$ is a weak equivalence in $\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M})$. To check this we take $D \in \mathrm{Obj}(\mathcal{C})$ and observe that

$$\begin{aligned} h^* \circ \mathbf{L}h_!(M \otimes^{\mathbf{L}} \mathrm{h}^C)(D) &\cong h^*(M \otimes^{\mathbf{L}} \mathrm{h}^{h(C)})(D) \\ &= M \otimes^{\mathbf{L}} \mathrm{Hom}_{\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)}(h(C), h(D))^\bullet \\ &\cong M \otimes^{\mathbf{L}} \mathrm{Hom}_{\mathcal{C}}(C, D)^\bullet \\ &\cong (M \otimes^{\mathbf{L}} \mathrm{h}^C)(D) \end{aligned} \tag{5.20}$$

where we were able to apply the condition on the tensor product because the Yoneda embedding is fully faithful.

If we restrict to the essential image we get an equivalence of categories, hence the other induced adjunction morphism

(5.21) $\mathbf{L}h_! \circ h^*(F) \to F$

should be an isomorphism in $\mathrm{Ho}(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\text{-dg}\,\mathrm{Mod}(\mathcal{M}))$, if F is an object in this homotopy category representing a continuous morphism. This will follow if h^* is conservative, i.e. reflects isomorphisms, because $\mathbf{L}h_!$ is fully faithful, hence in the subcategory of continuous

functors we can consider all the evaluations to obtain the isomorphism of the functors in the homotopy category.
For the second part, consider $f : F \to G$ where

(5.22) $F, G\colon \operatorname{Int}\left(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right) \to \mathcal{M}$

are dg functors such that $h^*(f)\colon h^*(F) \to h^*(G)$ is an isomorphism in $\operatorname{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$. To check this we will evaluate the morphism, or more accurately natural transformation, f in an arbitrary $C \in \operatorname{Obj}(\operatorname{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k))$. Then C can be written as the homotopy colimit of objects $X_i^\bullet \otimes \mathrm{h}^{C_i}$ for $X_i^\bullet \in \operatorname{Obj}(\operatorname{Ch}(k\text{-Mod}))$ and $C_i \in \operatorname{Obj}(\mathcal{C})$. So we have

(5.23) $C = \operatorname*{hocolim}_{i\in I} X_i^\bullet \otimes \mathrm{h}^{C_i}.$

So we have the commutative diagram

(5.24)
$$\begin{array}{ccc}
\operatorname{hocolim}_{i\in I} F\left(X_i^\bullet \otimes^{\mathbf{L}} \mathrm{h}_{C_i}\right) & \longrightarrow & \operatorname{hocolim}_{i\in I} G\left(X_i^\bullet \otimes^{\mathbf{L}} \mathrm{h}_{C_i}\right) \\
\downarrow & & \downarrow \\
F\left(\operatorname{hocolim}_{i\in I} X_i^\bullet \otimes^{\mathbf{L}} \mathrm{h}_{C_i}\right) & \xrightarrow[f_C]{} & G\left(\operatorname{hocolim}_{i\in I} X_i^\bullet \otimes^{\mathbf{L}} \mathrm{h}_{C_i}\right)
\end{array}$$

where the vertical arrows are isomorphisms by lemma 5.9. Using lemma 5.8 for the top horizontal arrow we observe that for all $i \in I$ the morphism

(5.25) $F(X_i^\bullet \otimes^{\mathbf{L}} \mathrm{h}_{C_i}) \cong X_i^\bullet \otimes^{\mathbf{L}} F(\mathrm{h}_{C_i}) \to G(X_i^\bullet \otimes^{\mathbf{L}} \mathrm{h}_{C_i}) \cong X_i^\bullet \otimes^{\mathbf{L}} G(\mathrm{h}_{C_i})$

can be written as $\mathrm{id}_{X_i^\bullet} \otimes^{\mathbf{L}} h^*(f)$. But as we have $h^*(f)$ assumed to be an isomorphism in the homotopy category $\operatorname{Ho}(\mathcal{C}\text{-dg}\,\mathrm{Mod}_k(\mathcal{M}))$ the top horizontal arrow in (5.24) is an isomorphism. This implies that $f_C\colon F(C) \to G(C)$ itself is an isomorphism, so h^* reflects isomorphisms. □

Proposition 5.11. Let $\mathcal{C}$ and $\mathcal{D}$ be small dg categories. Then the restriction functor

(5.26)
$$h^*\colon \mathbf{R}\mathcal{H}\mathrm{om}_{\mathrm{cont}}\left(\operatorname{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k), \operatorname{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right) \to \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{C}, \operatorname{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)$$

is an isomorphism.

Proof. Using the adjunction from remark 4.31 and the symmetry of the monoidal structure we see that for $\mathcal{E}$ an arbitrary dg category we have the isomorphisms

(5.27)
$$\begin{aligned}
&\operatorname{Hom}_{\operatorname{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{E}, \mathbf{R}\mathcal{H}\mathrm{om}\left(\operatorname{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k), \operatorname{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right)\right)_{\mathrm{cont}} \\
&\quad\cong \operatorname{Hom}_{\operatorname{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\operatorname{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k) \otimes^{\mathbf{L}} \mathcal{E}, \operatorname{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right)_{\mathrm{cont}}
\end{aligned}$$

and

(5.28)
$$\begin{aligned}
&\operatorname{Hom}_{\operatorname{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathcal{E}, \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{C}, \operatorname{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right)\right) \\
&\quad\cong \operatorname{Hom}_{\operatorname{Ho}(\mathrm{dg}\,\mathrm{Cat}_k}\left(\mathcal{C} \otimes^{\mathbf{L}} \mathcal{E}, \operatorname{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right).
\end{aligned}$$

The index cont for the right-hand of (5.27) side is the analogue for the condition on the left, i.e. we restrict ourselves to morphisms

(5.29) $F\colon \operatorname{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k) \otimes^{\mathbf{L}} \mathcal{E} \to \operatorname{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)$

such that for all $X \in \operatorname{Obj}(\mathcal{E})$ the partial functor

(5.30) $F(-, X)\colon \operatorname{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k) \to \operatorname{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)$

is continuous. So it suffices to prove the bijection of the right-hand sides by an application of the Yoneda lemma.

Using lemmas 5.6 and 5.7 this is moreover equivalent to proving that the induced morphism in (5.33) is a bijection. To do so we have to consider the quasi-fully faithful embedding induced by the inclusion of universes $\mathbb{U} \in \mathbb{V}$, see remark 4.13. By definition the category $\mathrm{Int}((\mathcal{E} \otimes^{\mathbf{L}} \mathcal{D}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k)$ consists of the dg functors

(5.31) $\mathcal{E} \otimes^{\mathbf{L}} \mathcal{D}^{\mathrm{op}} \to \mathrm{Ch}_{\mathrm{dg}}(k\text{-Mod})$

which are small relative to $\mathbb{U}$. These are embedded quasi-fully faithfully in the category of dg functors that are small relative to $\mathbb{V}$, which we will denote

(5.32) $\mathrm{Int}\left((\mathcal{E} \otimes^{\mathbf{L}} \mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k\right)_{\mathbb{V}}$.

So the morphism (5.33) fits into the commutative diagram (5.34) and we wish to prove the bijectivity of the left vertical morphism. Using lemma 5.8 for

(5.35) $\mathcal{M} = (\mathcal{C} \otimes^{\mathbf{L}} \mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathbb{V}}$

we obtain a bijection for the right vertical morphism.

Using lemma 5.10 the horizontal morphisms are injective, where we have used the quasi-fully faithfulness from remark 4.13.

To conclude we wish to show that we do not leave our small universe $\mathbb{U}$ when considering the restriction functor. So let

(5.36) $f\colon \mathrm{Int}(\mathcal{E}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k) \to \mathrm{Int}\left((\mathcal{E} \otimes^{\mathbf{L}} \mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathbb{V}}\right)$

be a morphism such that the restriction $h^*(f)$ factors as

(5.37)
$$\begin{tikzcd}
\mathrm{Int}(\mathcal{E}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k) \arrow[rr, "h^*(f)"] \arrow[dr] & & \mathrm{Int}\left((\mathcal{E} \otimes^{\mathbf{L}} \mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_{k,\mathbb{V}}\right) \\
& \mathrm{Int}\left((\mathcal{E} \otimes^{\mathbf{L}} \mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k\right). \arrow[ur] &
\end{tikzcd}$$

Then we would like a similar factorisation for f. Using lemmas 5.8 and 5.9 the image of an object in $\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k)$ is a small homotopy colimit of objects

(5.38) $h^*(f)\left(X_i^\bullet \otimes^{\mathbf{L}} C_i\right) \cong X_i^\bullet \otimes^{\mathbf{L}} h^*(f)(C_i) \cong X_i^\bullet \otimes^{\mathbf{L}} f\,\left(h(C_i)\right)$

for $i \in I$, where $X_i^\bullet \in \mathrm{Obj}(\mathrm{Ch}(k\text{-Mod}))$ and $C_i \in \mathrm{Obj}(\mathcal{C})$. So because $\mathcal{C}$ is assumed to be a small dg category, its images are small objects and the restriction $h^*(f)$ commutes with the colimits that describe everything, we see that f itself actually has small images, hence we get the desired factorisation. □

We can now put all this together to obtain a Morita theory for dg categories.

Theorem 5.12 (Derived Morita theory). Let $\mathcal{C}$ and $\mathcal{D}$ be small dg categories. There exists a natural isomorphism

(5.39) $\mathrm{Morita}_k(\mathcal{C}, \mathcal{D}) \cong \mathrm{Int}\left((\mathcal{C} \otimes^{\mathbf{L}} \mathcal{D})\text{-}\mathrm{dg\,Mod}_k\right)$

in $\mathrm{Ho}(\mathrm{dg\,Cat}_k)$.

$$h^*\colon \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k), \mathrm{Int}\left((\mathcal{E}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k\right)\right)_{\mathrm{cont}} \to \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left(\mathcal{C}, \mathrm{Int}\left((\mathcal{E}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-}\mathrm{dg\,Mod}_k\right)\right) \tag{5.33}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k),\mathrm{cont}}\left(\mathrm{Int}\left(\mathcal{C}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k\right), \mathrm{Int}\left((\mathcal{E}^{\mathrm{op}}\otimes^{\mathbf{L}}\mathcal{D})^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k\right)\right) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k),\mathrm{cont}}\left(\mathrm{Int}\left(\mathcal{C}^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k\right), \mathrm{Int}\left((\mathcal{E}^{\mathrm{op}}\otimes^{\mathbf{L}}\mathcal{D})^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k\right)_{\mathbb{V}}\right) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left(\mathcal{C}^{\mathrm{op}}, \mathrm{Int}\left((\mathcal{E}^{\mathrm{op}}\otimes^{\mathbf{L}}\mathcal{D})^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k\right)\right) \arrow[r] & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}_k)}\left(\mathcal{C}, \mathrm{Int}\left((\mathcal{E}^{\mathrm{op}}\otimes^{\mathbf{L}}\mathcal{D})^{\mathrm{op}}\text{-}\mathrm{dg\,Mod}_k\right)_{\mathbb{V}}\right)
\end{tikzcd} \tag{5.34}$$

Proof. We have the string of isomorphisms

$$
\begin{aligned}
&\mathrm{Morita}_k(\mathcal{C},\mathcal{D}) \\
&\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}_{\mathrm{cont}}\left(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k),\mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right) && \text{definition} \\
&\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{C},\mathrm{Int}(\mathcal{D}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right) && \text{proposition 5.11} \\
&\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C},\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{D}^{\mathrm{op}},\mathrm{Int}(\mathrm{Ch}(k\text{-Mod})))) && \text{lemma 5.1} \\
&\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}},\mathrm{Int}(\mathrm{Ch}(k\text{-Mod}))\right) && \text{adjunction (4.104)} \\
&\quad\cong \mathrm{Int}\left((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{D}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k\right) && \text{lemma 5.1}
\end{aligned}
\tag{5.40}
$$

which proves (5.39). □

Remark 5.13. There is also a version concerning perfect dg modules. The proof is similar, but we will not use the result in the proofs of what remains.

5.2 Hochschild cohomology

Hochschild cohomology for algebras is a classical tool in homological algebra [CSAM29, chapter 9]. As dg categories are in a sense generalisations of (dg) algebras one can ask whether it is possible to define Hochschild cohomology in this situation. Given the rich structure of dg categories it is straightforward to give a definition.

Definition 5.14. Let $\mathcal{C}$ be a dg category. Consider it as an object in $(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{C}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k$ by the rule $\mathcal{C}(X,Y)^\bullet := \mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet$, where $X,Y\in\mathrm{Obj}(\mathcal{C})$. We define the *Hochschild complex* of $\mathcal{C}$ by

$$\mathbf{HH}(\mathcal{C})^\bullet := \mathbf{R}\mathrm{Hom}_{(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{C}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k}(\mathcal{C},\mathcal{C})^\bullet \tag{5.41}$$

where $\mathbf{R}\mathrm{Hom}_{(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{C}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k}(-,-)^\bullet$ denotes the $\mathrm{Ho}(\mathrm{Ch}(k\text{-Mod}))$-enriched Hom in the category $\mathrm{Ho}((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{C}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k)$. The *ith Hochschild cohomology group* is then

$$\mathbf{HH}^i(\mathcal{C})^\bullet := \mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{(\mathcal{C}\otimes^{\mathbf{L}}\mathcal{C}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k}(\mathcal{C},\mathcal{C})^\bullet\right). \tag{5.42}$$

The first result is then that Hochschild cohomology can be computed using the internal Hom of $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$. Instead of considering $\mathcal{C}$ as a $\mathcal{C}$-$\mathcal{C}$-bimodule it is more natural to consider it as a dg category. The internal Hom then yields the dg category of dg endofunctors on $\mathcal{C}$, of which the identity functor is a special case. By looking at its endomorphisms we obtain the following result.

Theorem 5.15 (Hochschild cohomology as endomorphisms)**.** Let $\mathcal{C}$ be a dg category. We have an isomorphism

$$\mathbf{HH}(\mathcal{C})^\bullet \cong \mathrm{Hom}_{\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}},\mathrm{id}_{\mathcal{C}})^\bullet \tag{5.43}$$

in $\mathrm{Ho}(\mathrm{Ch}(k\text{-Mod}))$.

Proof. By theorem 4.30 we have the characterisation of the internal Hom as

$$\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C},\mathcal{C}) \cong \mathrm{Int}((\mathcal{C}\otimes^{\mathbf{L}}\mathcal{C}^{\mathrm{op}})\text{-dg}\,\mathrm{Mod}_k^{\mathrm{rqr}}). \tag{5.44}$$

The identity functor $\mathrm{id}_{\mathcal{C}}$ on the left goes to the $\mathcal{C}$-$\mathcal{C}$-bimodule $\mathcal{C}$ on the right. By the enrichment provided by the internal dg category this proves the result. □

Theorem 5.16 (Morita invariance of Hochschild cohomology)**.** Let $\mathcal{C}$ be a dg category. Then we have that $\mathbf{HH}(\mathcal{C})^\bullet \cong \mathbf{HH}(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k))^\bullet$.

Proof. The identity functor is a continuous morphism in the sense of §5.1, hence we find

$$
\begin{aligned}
&\mathbf{HH}\left(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right)^\bullet \\
(5.45)\qquad &\cong \mathrm{Hom}_{\mathbf{R}\mathcal{H}\mathrm{om}_{\mathrm{cont}}\left(\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k),\mathrm{Int}(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k)\right)}(\mathrm{id},\mathrm{id})^\bullet && \text{theorem 5.15} \\
&\cong \mathrm{Hom}_{\mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{C},\mathrm{Int}\left(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)\right)}(\mathrm{h}^-,\mathrm{h}^-)^\bullet && \text{proposition 5.11}
\end{aligned}
$$

because h^* in proposition 5.11 corresponds to composition with the Yoneda embeding. As the Yoneda embedding is quasi-fully faithful we get that

$$(5.46)\quad \mathrm{h}^*\colon\ \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{C},\mathrm{Int}\left(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)\right)\to\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C},\mathcal{C})$$

is again quasi-fully faithful by corollary 4.35, so we find

$$
\begin{aligned}
(5.47)\qquad \mathrm{Hom}_{\mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{C},\mathrm{Int}\left(\mathcal{C}^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)\right)}(\mathrm{h}^-,\mathrm{h}^-)^\bullet &\cong \mathrm{Hom}_{\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}},\mathrm{id}_{\mathcal{C}})^\bullet \\
&\cong \mathbf{HH}(\mathcal{C})^\bullet && \text{theorem 5.15}
\end{aligned}
$$

as desired. □

5.3 An isomorphism of integral transforms

We will now prove theorem 5.19. To do so we need to discuss the setup from [BV03] first. Remark that for a quasicompact and (quasi)separated scheme X the (model) category $\mathrm{Ch}(\mathrm{Qcoh}_X)$ is a stable, proper and cofibrantly generated model category [Hov01, theorem 4.2] (see also appendix A.3) admitting a compact generator [BV03, corollary 3.1.8]. By the analogue of the Popescu-Gabriel theorem [SS02, theorem 3.1.1] we can find a dg algebra $E_X^\bullet$ in $\mathrm{Int}(\mathrm{Ch}(\mathrm{Qcoh}_X))$ such that if we denote

$$(5.48)\quad \mathcal{A}_X^{\mathrm{op}} \coloneqq \mathbf{R}\mathcal{H}\mathrm{om}(E_X^\bullet,E_X^\bullet)\text{-dg}\,\mathrm{Mod}_k$$

the dg modules over the dg category $\mathbf{R}\mathcal{H}\mathrm{om}(E_X^\bullet,E_X^\bullet)$ with a single object and the endomorphisms of $E_X^\bullet$ as endomorphism ring we obtain the Quillen equivalence

$$(5.49)\quad \mathrm{Ch}(\mathrm{Qcoh}_X)\cong\mathcal{A}_X^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k.$$

As this Quillen equivalence is compatible with the $\mathrm{Ch}(k\text{-Mod})$-enrichment we obtain the isomorphism

$$(5.50)\quad \mathrm{Int}\left(\mathrm{Ch}(\mathrm{Qcoh}_X)\right)\cong\mathrm{Int}\left(\mathcal{A}_X^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right).$$

In lemmas 5.17 and 5.18 we prove two properties related to the category $\mathcal{A}_X^{\mathrm{op}}$, and these will yield theorem 5.19. The first lemma is a trick to take the opposite of the dg category $\mathcal{A}_X$ without any harm.

Lemma 5.17. Let $\mathcal{A}_X$ be as above. Then we have an isomorphism

$$(5.51)\quad \mathrm{Int}\left(\mathcal{A}_X^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)\cong\mathrm{Int}\left(\mathcal{A}_X\text{-dg}\,\mathrm{Mod}_k\right)$$

in $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$.

Proof. We use [SS02, theorem 3.3.3] to reduce the statement to studying the compact generator $E_X^\bullet$ of the category $\mathrm{Int}(\mathrm{Ch}(\mathrm{Qcoh}_X))$, and its homotopy category which is equivalent to $\mathbf{D}(\mathrm{Qcoh}_X)$. By taking $E_X^{\mathrm{op},\bullet}$ to be the perfect dual of the complex $E_X^\bullet$, we can consider the smallest épaisse triangulated subcategory $\langle E_X^{\mathrm{op},\bullet}\rangle$ of $(\mathbf{D}_{\mathrm{perf}}(\mathrm{Qcoh}_X)$ containing this perfect dual.

Consider the dualising morphism

$$(5.52)\quad -^\wedge\colon(\mathbf{D}_{\mathrm{perf}}(\mathrm{Qcoh}_X))^{\mathrm{op}}\to\mathbf{D}_{\mathrm{perf}}(\mathrm{Qcoh}_X).$$

Then we observe that f sends $\langle E_X^{\mathrm{op},\bullet}\rangle$ to $\langle E_X^\bullet\rangle$, which is exactly $\mathbf{D}_{\mathrm{perf}}(\mathrm{Qcoh}_X)$. So the dual complex $E_X^{\mathrm{op},\bullet}$ is a classical generator for the subcategory of perfect complexes, hence by [BV03, theorem 2.1.2] it is a compact generator for the derived category $\mathbf{D}(\mathrm{Qcoh}_X)^{\mathrm{op}}$. Then by applying [SS02, theorem 3.3.3] this is lifted to the dg enrichment. □

The next lemma is a dg enrichment of [BV03, theorem 2.1.2]. Remark that from this point on we start using the conditions from theorem 5.19 as we will need the flatness.

Lemma 5.18. Let X and Y be quasicompact and separated schemes over a ring k such that at least one of them is flat over $\operatorname{Spec} k$. Then we have the isomorphism

$$\text{(5.53)}\quad \mathrm{Int}\left((\mathcal{A}_X \otimes^{\mathbf{L}} \mathcal{A}_Y)^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right) \cong \mathrm{Int}\left(\mathrm{Ch}(\mathrm{Qcoh}_{X\times_k Y})\right)$$

in $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$.

Proof. By [BV03, lemma 3.4.1] the product complex $E_X^\bullet \boxtimes E_Y^\bullet$ (where $\boxtimes$ is a notation for the derived tensor product of the pullbacks through the projections) on $X \times_k Y$ is a compact generator for $\mathbf{D}(\mathrm{Qcoh}_{X\times_k Y})$. And because one of them is assumed to be flat over k we get a quasi-isomorphism of dg algebras, i.e. we obtain

$$\text{(5.54)}\quad \mathbf{R}\mathcal{H}\mathrm{om}(E_X^\bullet \boxtimes E_Y^\bullet, E_X^\bullet \boxtimes E_Y^\bullet) \cong \mathbf{R}\mathcal{H}\mathrm{om}(E_X^\bullet, E_X^\bullet) \otimes_k^{\mathbf{L}} \mathbf{R}\mathcal{H}\mathrm{om}(E_Y^\bullet, E_Y^\bullet)$$

so the generating object for the category on the right is quasi-isomorphic to the dg algebra that by definition generates the category on the left. □

Theorem 5.19 (Continuous integral transforms are enriched representable)**.** Let X and Y be quasicompact and separated schemes over a ring k such that at least one of them is flat over $\operatorname{Spec} k$. Then we have the isomorphism

$$\text{(5.55)}\quad \mathbf{R}\mathcal{H}\mathrm{om}_{\mathrm{cont}}\left(\mathrm{Int}(\mathrm{Ch}(\mathrm{Qcoh}_X)), \mathrm{Int}(\mathrm{Ch}(\mathrm{Qcoh}_Y))\right) \cong \mathrm{Int}\left(\mathrm{Ch}(\mathrm{Qcoh}_{X\times_k Y})\right)$$

in $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)$.

Proof. We have the string of isomorphisms

$$\text{(5.56)}\quad \begin{aligned} &\mathbf{R}\mathcal{H}\mathrm{om}_{\mathrm{cont}}\left(\mathrm{Int}\left(\mathrm{Ch}(\mathrm{Qcoh}_X)\right), \mathrm{Int}\left(\mathrm{Ch}(\mathrm{Qcoh}_Y)\right)\right) && \\ &\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}_{\mathrm{cont}}\left(\mathrm{Int}\left(\mathcal{A}_X^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right), \mathrm{Int}\left(\mathcal{A}_Y^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)\right) && (5.50) \\ &\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{A}_X, \mathrm{Int}\left(\mathcal{A}_Y^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right)\right) && \text{proposition 5.11} \\ &\quad\cong \mathbf{R}\mathcal{H}\mathrm{om}\left(\mathcal{A}_X, \mathrm{Int}\left(\mathcal{A}_Y\text{-dg}\,\mathrm{Mod}_k\right)\right) && \text{lemma 5.17} \\ &\quad\cong \mathrm{Int}\left((\mathcal{A}_X \otimes^{\mathbf{L}} \mathcal{A}_Y)^{\mathrm{op}}\text{-dg}\,\mathrm{Mod}_k\right) && \text{adjunction} \\ &\quad\cong \mathrm{Int}\left(\mathrm{Ch}(\mathrm{Qcoh}_{X\times_k Y})\right) && \text{lemma 5.18} \end{aligned}$$

which proves the theorem. □

By considering the homotopy categories it is then possible to deduce a generalisation of the Bondal-Orlov result in the language of classical algebraic geometry.

Corollary 5.20 (Continuous integral transforms are representable)**.** Let X and Y be quasicompact and separated schemes over a ring k such that at least one of them is flat over $\operatorname{Spec} k$. Then we have the bijection

$$\text{(5.57)}\quad \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}_k)}\left(\mathrm{Int}\left(\mathrm{Ch}(\mathrm{Qcoh}_X)\right), \mathrm{Int}\left(\mathrm{Ch}(\mathrm{Qcoh}_Y)\right)\right) \cong \mathrm{Isom}\left(\mathbf{D}(\mathrm{Qcoh}_{X\times_k Y})\right).$$

For completeness' sake we give the analogue of theorem 5.19 in the case smooth case, where the notion of a perfect complex is well-behaved. Recall that a perfect complex is a cochain complex of sheaves that is quasi-isomorphic to a finite complex of vector bundles.

Theorem 5.21 (Perfect integral transforms are enriched representable)**.** Let X and Y be smooth and proper schemes over a ring k such that at least one of them is flat over $\operatorname{Spec} k$. Then we have the isomorphism

$$\mathbf{R}\mathcal{H}om\left(\mathrm{Int}_{\mathrm{perf}}\left(\mathrm{Ch}(\mathrm{Qcoh}_X)\right), \mathrm{Int}_{\mathrm{perf}}\left(\mathrm{Ch}(\mathrm{Qcoh}_Y)\right)\right) \cong \mathrm{Int}_{\mathrm{perf}}\left(\mathrm{Ch}(\mathrm{Qcoh}_{X\times_k Y})\right) \tag{5.58}$$

CHAPTER 6

The derived moduli stack of quiver representations

> "Finally, know that an unshot skeet's movement against the vast lapis lazuli dome of the open ocean's sky is sun-like—i.e. orange and parabolic and right-to-left—and that its disappearance into the sea is edge-first and splashless and sad."
>
> *A Supposedly Fun Thing I'll Never Do Again*
> DAVID FOSTER WALLACE

We now take a leap through the theory. Up to now, most of the things have been developed in detail (modulo some text book material). But the results in the previous chapters, together with the formalism of derived algebraic geometry, yields the study of derived moduli stacks [TV07]. This builds on the works [HAG-I; HAG-II; HAG2DAG] and for all results in derived algebraic geometry one is referred there.

The goal of this chapter is to find an answer to a remark made in [KS, §2.6]. In this paper a *stratification functor* is constructed, which yields a correspondence between isomorphism classes of (complexes of) quiver representations and modules on the regular and singular Nakajima categories, and their moduli varieties. After stating [KS, theorem 2.7] which gives the correspondence, one reads (notation is changed)

> "It suggests that the varieties $\mathfrak{M}_0(w)$ should be related to the moduli stack of objects in $\mathbf{D}^{\mathrm{b}}(kQ\text{-Mod})$ introduced and studied by Toën-Vaquié."

In the first section we develop the necessary terminology and notation, trying to keep this to a minimum, referring to [TV07; KS] whenever needed. The second section is devoted to the construction of the morphism which in [KS] corresponds to the stratification functor. This functor now becomes a morphism between derived stacks, hence it is more geometric in nature than the original stratification functor. The last section is devoted to studying the properties of this morphism, and interpretations of the results.

Unlike the previous chapters which were developed in full generality we will assume to work over the field of complex numbers $\mathbb{C}$. This condition is mostly related to the quiver side of the story, but it also allows us to use commutative differential graded $\mathbb{C}$-algebras (or $\mathbb{C}$-cdga's) instead of simplicial commutative $\mathbb{C}$-algebras. There is no difference as far as derived algebraic geometry is concerned, but it will make notation easier.

For the results in this chapter I am greatly indebted to Bernhard Keller, who both posed the problem and helped in finding the solution.

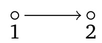

Figure 6.1: The quiver A_2

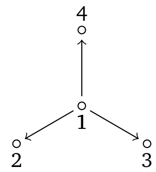

Figure 6.2: The quiver D_4

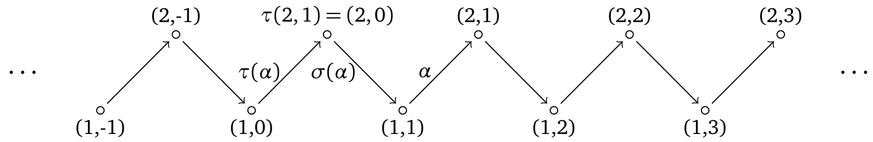

Figure 6.3: The repetition quiver $\mathbb{Z}A_2$

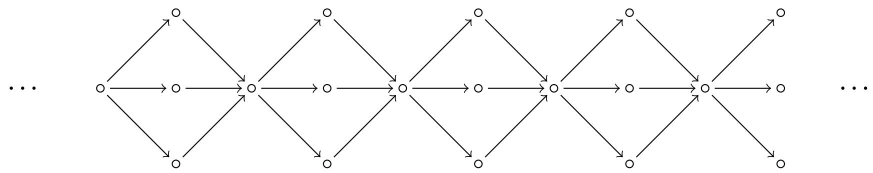

Figure 6.4: The repetition quiver $\mathbb{Z}D_4$

6.1 Preliminaries

Throughout this chapter we take Q a finite quiver without (oriented) cycles. We denote Q_0 its set of vertices, and Q_1 its set of arrows, so the finiteness condition entails that $\#Q_0, \#Q_1 < +\infty$. Because there are no oriented cycles we know that the path algebra $\mathbb{C}Q$ is finite-dimensional, which makes it easy to study in the context of derived moduli stacks.

Two examples we will keep in mind are the quivers A_2 and D_4, as depicted in figures 6.1 and 6.2.

To every quiver we wish to associate a category: the singular Nakajima category $\mathcal{S}$. This is done in definition 6.5. Before we can define this category we need to introduce some preliminary notions, which will be illustrated in case of our two recurring examples.

Definition 6.1. Let Q be a finite acyclic quiver. Its *repetition quiver* $\mathbb{Z}Q$ has as vertices the pairs (i,n) where $i \in Q_0$ and $n \in \mathbb{Z}$. Its arrows are the pairs $(\alpha,n)\colon (i,n) \to (j,n)$ and $\sigma(\alpha,n) = (j,n-1) \to (i,n)$, where $\alpha\colon i \to j \in Q_1$ and $n \in \mathbb{Z}$. The translation τ by one position to the left then satisfies the relation $\sigma^2 = \tau$ for arrows.

For the recurring example A_2 this yields figure 6.3 while D_4 has the repetition quiver depicted in figure 6.4. Remark that in case of the repetition quiver $\mathbb{Z}D_4$ we have changed the layout (but not the structure) of the quiver to make the picture easier to comprehend.

Another step in the construction of the singular Nakajima category is the framing of the quiver.

Definition 6.2. Let Q be a finite acyclic quiver. Its associated *framed quiver* $\widetilde{Q}$ has as vertices the vertices Q_0 and a copy of Q_0, whose elements are denoted i'. Its arrows are the arrows Q_1 and for each $i \in Q_0$ we have an arrow $i \to i'$. The vertices i' will be called *frozen* vertices.

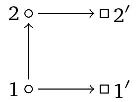

Figure 6.5: The framed quiver $\widetilde{\mathrm{A}_2}$

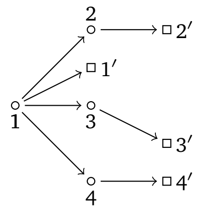

Figure 6.6: The framed quiver $\widetilde{\mathrm{D}_4}$

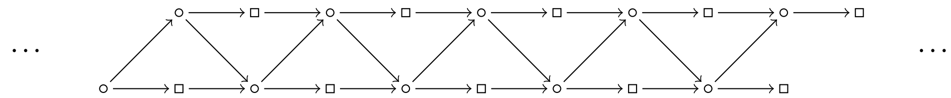

Figure 6.7: The regular Nakajima category $\mathbb{C}(\mathbb{Z}\widetilde{\mathrm{A}_2})$

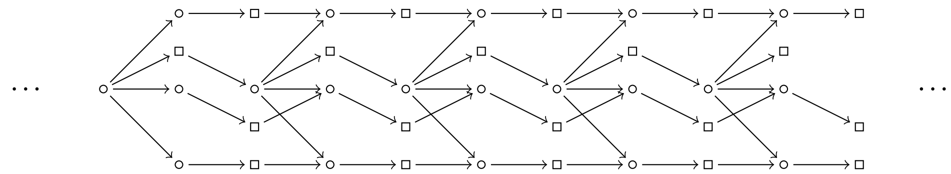

Figure 6.8: The regular Nakajima category $\mathbb{C}(\mathbb{Z}\widetilde{\mathrm{D}_4})$

For the recurring example A_2 this yields figure 6.5 while D_4 has the framed quiver depicted in figure 6.6. The frozen vertices will be indicated using a square vertex, as opposed to a circle vertex for the non-frozen vertices.

We can associate a $\mathbb{C}$-linear category to a repetition quiver, which originates in Auslander-Reiten theory.

Definition 6.3. Let Q be a quiver. The *mesh category* $\mathbb{C}(\mathbb{Z}Q)$ is a category whose objects are the vertices of $\mathbb{Z}Q$. For vertices a, b is the morphism space $\mathrm{Hom}_{\mathbb{C}(\mathbb{Z}Q)}(a, b)$ the $\mathbb{C}$-vectorspace generated by all possible paths from a to b, where we mod out the subspace of *mesh relators*. These are the linear combinations $u\mathrm{r}_x v$ of paths from a to b where u and v are paths and r_x is the linear combination $\sum_{\beta\colon y\to x}\sigma(\beta)\beta$.

Definition 6.4. Let Q be a finite acyclic quiver. The associated *regular Nakajima category* $\mathcal{R}$ is the mesh category $\mathbb{C}(\mathbb{Z}\widetilde{Q})$ on the framed quiver, where we only impose the mesh relators associated to non-frozen vertices.

For the recurring example A_2 this yields figure 6.7 while D_4 has the regular Nakajima category depicted in figure 6.8. The frozen vertices are again indicated using squares. Remark that it is not possible to draw the mesh relators.

Definition 6.5. Let Q be a finite acyclic quiver. The associated *singular Nakajima category* $\mathcal{S}$ is the full subcategory of the regular Nakajima category of Q on the frozen vertices.

For the recurring example A_2 this yields figure 6.9, in which the relations $ab - ba$ and $a^3 - cb$ hold. The singular Nakajima category for D_4 on the other hand has a very complex structure, which doesn't yield a good picture, which is why we won't try to reproduce it here.

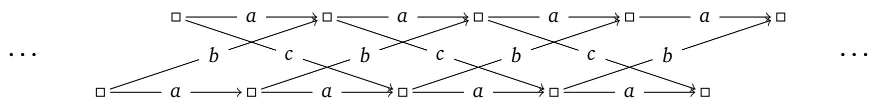

Figure 6.9: The singular Nakajima category for A_2

The last step in the construction of [KS] is to consider modules over the singular Nakajima category. These are functors $\mathcal{S} \to \mathbb{C}$-Vect such that there are only finitely many non-zero images, and the dimension in each point is finite. For more information on the construction one is referred to [KS].

6.2 The geometric stratification morphism

We are now capable of defining the generalisation of the stratification functor from [KS]. This result is given in theorem 6.19. The geometric object corresponding to $\mathbf{D}^{\mathrm{b}}(\mathbb{C}Q\text{-Mod})$ is the derived stack $\mathbf{R}\mathrm{Perf}_Q$ from [TV07]. One can interpret the varieties $\mathfrak{M}_0(w)$ as derived stacks using the fully faithful embedding (see also remark 6.20), but in order to construct the stratification morphism (6.15) we have to construct an intermediate derived stack.

Definition 6.6. Let Q be a quiver. Let $\mathcal{S}$ be its singular Nakajima category. Let s be a finite subset of $\mathrm{Obj}(\mathcal{S})$. Then we define the *finitely supported singular Nakajima dg category* $\mathrm{supp}(\mathcal{S}, s)$ by taking the full $\mathbb{C}$-linear subcategory on the objects s which we equip with the trivial differential to obtain a dg category.

The categories $\mathrm{supp}(\mathcal{S}, s)$ are a special kind of category: they are both finite (as in, finitely many objects and Hom-finite, i.e. the morphism spaces are finite-dimensional vectorspaces) and directed. These finiteness properties yield the result in corollary 6.10. Remark that the dg category $\mathrm{supp}(\mathcal{S}, s)$ is immediately a compact dg category, as defined in [Tab05b].

Lemma 6.7. For every finite subset s of $\mathcal{S}$ as in definition 6.6 we have that $\mathrm{supp}(\mathcal{S}, s)$ is a smooth dg category.

Proof. Because the category $\mathrm{supp}(\mathcal{S}, s)$ is acyclic and directed, and it only has finitely many objects, we obtain that its associated category of dg modules is of finite global dimension. As we are working over the field $\mathbb{C}$ this implies that the identity bimodule is a perfect object. □

Lemma 6.8. For every finite subset s of $\mathcal{S}$ as in definition 6.6 we have that $\mathrm{supp}(\mathcal{S}, s)$ is a proper dg category.

Proof. For every $x, y \in \mathrm{Obj}(\mathrm{supp}(\mathcal{S}, s))$ we have that $\mathrm{Hom}_{\mathrm{supp}(\mathcal{S},s)}(x, y)^\bullet$ is a finite-dimensional vectorspace, hence the dg category $\mathrm{supp}(\mathcal{S}, s)$ is locally perfect in the sense of [TV07, definition 2.4].

Because the set of objects of $\mathrm{supp}(\mathcal{S}, s)$ is finite, we can construct a compact object by taking the finite direct sum of the simple modules. This object serves as a compact generator because it acts as an indicator for the simple submodules. □

Remark 6.9. The previous lemmas could (and maybe should) be stated in a more abstract form. The first lemma would probably reduce to a statement on "very small dg categories", while the conditions in the second would be "finite and Hom-finite".

Corollary 6.10. For every finite subset s of $\mathcal{S}$ as in definition 6.6 we have that $\mathrm{supp}(\mathcal{S}, s)$ is a dg category of finite type.

Proof. Using lemmas 6.7 and 6.8 we can apply [TV07, corollary 2.13] to obtain the result. □

Of course, we can similarly consider $\mathcal{S}$ as a dg category. Unfortunately this dg category is *not* suited for the formalism of derived moduli stacks of objects in dg categories as introduced in [TV07], as it is not of finite type. So we cannot consider the naive derived moduli stack of $\mathcal{S}$, as it would not (immediately) be a derived algebraic stack, locally of finite type.

On the other hand we see that using corollary 6.10 the dg categories $\mathrm{supp}(\mathcal{S},s)$ are of finite type, hence we can consider the derived moduli stacks $\mathbf{R}\mathrm{Perf}_{\mathrm{supp}(\mathcal{S},s)}$ of the dg categories $\mathrm{supp}(\mathcal{S},s)$, where $s\subset\mathrm{Obj}(\mathcal{S})$ finite.

Definition 6.11. Let Q be a finite and acyclic quiver. The *derived moduli stack of objects in its singular Nakajima category* $\mathcal{S}$ is

$$(6.1)\quad \mathbf{R}\mathrm{Perf}_{\mathcal{S}} := \bigcup_{\substack{s\subset\mathrm{Obj}(\mathcal{S})\\ \#s<+\infty}} \mathbf{R}\mathrm{Perf}_{\mathrm{supp}(\mathcal{S},s)}.$$

Remark 6.12. This decomposition of a dg category into dg categories of finite type is a way of generalising the result in [TV07], as indicated in [TV07, remark 3.30(2)]. The extent to which this is fruitful has not been investigated.

Now we will describe the construction of the morphism given in theorem 6.19. This goes through several steps, and relies on the results obtained in [KS]. Recall that we have the category $\mathrm{gpr}(\mathcal{S})$ of Gorenstein-projective modules of finite presentation [KS, §5.12].

The construction of the regular Nakajima category allows for a choice of embedding of Q in $\mathbb{Z}\widetilde{Q}$, hence for a vertex $x\in Q_0$ we can consider the object in $\mathcal{R}$-Mod represented by x, i.e. $\mathrm{Hom}_{\mathcal{R}}(-,x)$. If we restrict this representable object to the singular Nakajima category $\mathcal{S}$ we define

$$(6.2)\quad \mathrm{G}_x := \mathrm{Hom}_{\mathcal{R}}(-,x)|_{\mathcal{S}}$$

which is Gorenstein projective by [KS, lemma 5.13].
Using [KS, theorem 5.18] we obtain an equivalence of categories

$$(6.3)\quad \mathbf{D}^{\mathrm{b}}(\mathbb{C}Q\text{-Mod}) \to \underline{\mathrm{gpr}}(\mathcal{S}) : \mathrm{H}(x)\mapsto \mathrm{G}_x$$

and hence we can define a functor

$$(6.4)\quad \mathbb{C}Q \to \mathrm{gpr}(\mathcal{S}) : x\mapsto \mathrm{G}_x$$

where $\mathbb{C}Q$ is the *path category* of the quiver Q.

Lemma 6.13. Let Q be a finite and acyclic quiver. There exists a functor

$$(6.5)\quad \mathbb{C}Q \to \mathrm{Ch}_{\mathrm{ac,proj}}(\mathrm{gpr}(\mathcal{S}))$$

which is canonical up to the choice of a projective resolution.

Proof. By [KS, theorem 3.9] we find (minimal) projective Tate resolutions of the objects G_x. These resolutions will be denoted $\mathrm{Tate}_x^\bullet$. Hence the functor in (6.4) can be extended to the category of acyclic complexes with projective components, because there are no relations in the path category. □

Remark 6.14. By choosing the minimal projective resolution as given in [KS, theorem 3.9] we make this functor well-defined.

Now we wish turn this functor into the functor (6.8). We do this by lifting

$$(6.6)\quad (\mathbb{C}Q)^{\mathrm{op}} \to \mathrm{Ch}^{\mathrm{b}}(\mathbb{C}\text{-Mod}) : M\mapsto \mathrm{Hom}_{\mathcal{S}}(\mathrm{Tate}_x^\bullet, M)$$

or equivalently an object in the category $\mathbf{D}^{\mathrm{b}}(\mathbb{C}Q\text{-Mod})$ to

$$(6.7)\quad (\mathbb{C}Q)^{\mathrm{op}} \to \mathrm{Ch}^{\mathrm{b}}_{\mathrm{dg}}(\mathbb{C}\text{-Mod}) : M^\bullet \mapsto \mathrm{Tot}^{\mathrm{II}}\,\mathrm{Hom}_{\mathcal{S}}(\mathrm{Tate}_x^\bullet, M^\bullet)^\bullet.$$

Lemma 6.15. The functor (6.6) is well-defined, and exact with respect to M.

Proof. The minimal projective resolution chosen in lemma 6.13 consists of increasing shifts, by [KS, theorem 3.9]. As the number of vertices on which M is supported is finite by assumption we see that the complex $\mathrm{Hom}_{\mathcal{S}}(\mathrm{Tate}_x^\bullet, M)$ is a bounded complex of finite-dimensional vectorspaces because only finitely many shifts have a non-disjoint support, hence the functor lands in the stated category.

Because $\mathrm{Tate}_x^\bullet$ is an acyclic complex of projective modules the functor is exact with respect to M. ☐

Lemma 6.16. The functor (6.7) is well-defined, and exact with respect to $M^\bullet$.

Proof. The complex $M^\bullet$ of $\mathcal{S}$-modules is bounded, hence the argument of lemma 6.15 applies to the finitely many non-zero terms.

Because $\mathrm{Tate}_x^\bullet$ is an acyclic complex of projective modules the functor is exact with respect to $M^\bullet$. ☐

Lemma 6.17. There exists a functor

$$\mathbf{D}^{\mathrm{b}}(\mathcal{S}\text{-Mod}) \to \mathbf{D}^{\mathrm{b}}(kQ\text{-Mod}) \tag{6.8}$$

which (up to a choice of a shift) corresponds to the stratification functor by considering the composition

$$\begin{tikzcd} \mathcal{S}\text{-Mod} \arrow[r, hook, "i"] \arrow[dr, "\Phi"'] & \mathbf{D}^{\mathrm{b}}(\mathcal{S}\text{-Mod}) \arrow[d] \\ & \mathbf{D}^{\mathrm{b}}(kQ\text{-Mod}). \end{tikzcd} \tag{6.9}$$

Proof. Let $M_1^\bullet \to M_2^\bullet$ be a quasi-isomorphism of $\mathcal{S}$-modules (which is defined pointwise). Then we have a quasi-isomorphism

$$\mathrm{Tot}^{\amalg}\,\mathrm{Hom}_{\mathcal{S}}(\mathrm{Tate}_x^\bullet, M_1^\bullet) \to \mathrm{Tot}^{\amalg}\,\mathrm{Hom}_{\mathcal{S}}(\mathrm{Tate}_x^\bullet, M_2^\bullet). \tag{6.10}$$

Hence the dg functor

$$\mathrm{Ch}^{\mathrm{b}}_{\mathrm{dg}}(\mathcal{S}\text{-Mod}) \to \mathrm{Ch}^{\mathrm{b}}_{\mathrm{dg}}(kQ\text{-Mod}) \tag{6.11}$$

whose associated underlying functor

$$\mathrm{Ch}^{\mathrm{b}}(\mathcal{S}\text{-Mod}) \to \mathrm{Ch}^{\mathrm{b}}(kQ\text{-Mod}) \tag{6.12}$$

preserves quasi-isomorphisms induces a functor

$$\mathbf{D}^{\mathrm{b}}(\mathcal{S}\text{-Mod}) \to \mathbf{D}^{\mathrm{b}}(kQ\text{-Mod}) \tag{6.13}$$

on the level of the homotopy categories.

By the inclusion of $\mathcal{S}$-Mod in $\mathbf{D}^{\mathrm{b}}(\mathcal{S}\text{-Mod})$ we get the diagram (6.9), which corresponds to the stratification functor of [KS] by comparing the constructions. ☐

The final step is to lift this to the context of derived algebraic geometry, i.e. the construction should be be defined for a commutative differential graded algebra $C^\bullet$ concentrated in non-positive degree (abbreviated to $\mathrm{cdga}^{\leq 0}$). To do so we simply replace $\mathrm{Ch}^{\mathrm{b}}_{\mathrm{dg}}(\mathbb{C}\text{-Mod})$ by $C^\bullet\text{-dg}\,\mathrm{Mod}_{\mathbb{C}}$ in (6.7).

To a dg $\mathcal{S} \otimes C^\bullet$-module $M^\bullet$ which is perfect over $C^\bullet$ we associate the dg $\mathbb{C}Q \otimes_{\mathbb{C}} C^\bullet$-module $\mathrm{Hom}(\mathrm{Tate}_x^\bullet, M^\bullet)^\bullet$.

Lemma 6.18. This functor is well-defined and the image is a perfect dg $C^\bullet$-module.

Proof. Using [TV07, proposition 2.20(4)] and the observation that $M^\bullet$ is finitely supported and finite-dimensional we can conclude. □

Hence we have found a morphism

$$\mathrm{Map}(\mathcal{S}, \mathrm{Int}_{\mathrm{perf}}(C^\bullet\text{-dg}\,\mathrm{Mod}_{\mathbb{C}})) \to \mathrm{Map}(\mathbb{C}Q, \mathrm{Int}_{\mathrm{perf}}(C^\bullet\text{-dg}\,\mathrm{Mod}_{\mathbb{C}})) \tag{6.14}$$

and by the definition of the derived moduli stacks this corresponds to the morphism (6.15).

Theorem 6.19. Let Q be a finite acyclic quiver. Let $\mathcal{S}$ be its singular Nakajima category. There exists a morphism of derived stacks

$$\Phi\colon \mathbf{R}\mathrm{Perf}_{\mathcal{S}} \to \mathbf{R}\mathrm{Perf}_Q. \tag{6.15}$$

Proof. By the preceding discussion. □

Remark 6.20. By the inclusion in (6.9) we also obtain a morphism

$$\mathrm{i}\left(\sqcup_w \mathfrak{M}_0(w)\right) \to \mathbf{R}\mathrm{Perf}_Q \tag{6.16}$$

where i denotes the fully faithful embedding of classical algebraic geometry into the derived setting.

6.3 Properties and conclusions

The following result is a strengthening of the properties of $\mathbf{R}\mathrm{Perf}_Q$ and the open substacks $\mathbf{R}\mathrm{Perf}_Q^{[a,b]}$ and relates to [TV07, corollary 3.35].

Proposition 6.21. The derived algebraic stack $\mathbf{R}\mathrm{Perf}_Q^{[a,b]}$ is n-geometric for $n = b - a + 1$.

Proof. In the proof of [TV07, proposition 3.13] the representing dg algebra $B^\bullet$ is the (non-dg) algebra $\mathbb{C}Q$. Hence the finite diagram of algebras $B_i^\bullet$ such that we have homotopy pushouts

$$\begin{array}{ccc} \mathbb{C}[p_i] & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ B_i^\bullet & \longrightarrow & B_{i+1}^\bullet \end{array} \tag{6.17}$$

actually has $p_i = 0$ for all i. This means that the morphism

$$\mathbf{R}\mathrm{Perf}_Q^{[a,b]} \to \mathbf{R}\mathrm{Perf}^{[a,b]} \tag{6.18}$$

is n-representable for $n = b-a$ and hence the derived algebraic stack $\mathbf{R}\mathrm{Perf}_Q^{[a,b]}$ is $(b-a+1)$-geometric. □

Using this proposition we can strengthen [TV07, corollary 3.35].

Corollary 6.22. Let $v\colon \mathbb{Z} \to \mathbb{N}$ be a function with finite support as in [TV07, proposition 3.20]. The derived open substack $\mathbf{R}\mathrm{Perf}_Q^{v}$ is n-geometric for n equal to

$$\min\{i \mid \forall j \geq i\colon v(j) = 0\} - \max\{i \mid \forall j \leq i\colon v(j) = 0\} + 1. \tag{6.19}$$

Proof. In the introduction to [TV07, §3.3] it is shown that for any choice of $[a,b]$ such that $\mathrm{supp}(v) \subseteq [a,b]$ we have an inclusion of derived algebraic stacks $\mathbf{R}\mathrm{Perf}_Q^{v} \subset \mathbf{R}\mathrm{Perf}_Q^{[a,b]}$. By choosing a and b minimal we obtain the result using proposition 6.21. □

Appendix A

Model category results

The purpose of this chapter is to discuss some more results in model categories which are interesting in the context of dg categories. The main object of study in this appendix are (co)chain complexes: their model category structures are discussed. This is interesting for several reasons:

(i) it is an important and nice example of a model category;

(ii) it depends on the small object argument, which is elaborated upon in appendix A.1;

(iii) it shows that there can be several (non-trivial) model category structures on a given category appendix A.2;

(iv) the generalisation to the case of an arbitrary Grothendieck category [Hov01] is discussed in appendix A.3.

A.1 The small object argument

For the proof and applications of the small object argument we will need some classes of morphisms, defined relative to a given class I of morphisms. In the application of the small object argument this class I will be the set of generating (acyclic) cofibrations, and in this case there is a concrete interpreation of these classes, as given in example A.2. This section is based on [MSM63, §2.1].

Definition A.1. Let $\mathcal{C}$ be a category. Let I be a class of morphisms in $\mathcal{C}$. Let f be a morphism in $\mathcal{C}$. We say that

(i) f is *I-injective* if it has the right lifting property for every morphism in I;

(ii) f is *I-projective* if it has the left lifting property for every morphism in I;

(iii) f is *I-fibration* if it has the right lifting property for every I-projective morphism;

(iv) f is *I-cofibration* if it has the left lifting property for every I-injective morphism.

The class of I-injective (resp. I-projective) morphisms is denoted I-inj (resp. I-proj). The class of I-fibrations (resp. I-cofibrations) is denoted I-fib (resp. I-cof). We moreover observe that

$$\begin{aligned} I\text{-fib} &= (I\text{-proj})\text{-inj}, \\ I\text{-cof} &= (I\text{-inj})\text{-proj}. \end{aligned} \tag{A.1}$$

Example A.2. Let $\mathcal{C}$ be a model category. Let I be the class of cofibrations. Then using the characterisation [MSM63, lemma 1.1.10] we see that I-inj is the class of acyclic fibrations, and I-cof $= I$. We also have the dual statement. Now the main idea is to replace this choice of the class I by a class of "generating" cofibrations. This is useful for the recognition theorem [MSM63, theorem 2.1.19] and therefore for the proof of the model category structure on $\mathrm{dg}\,\mathrm{Cat}_k$ as discussed in appendix B.

There is one more class of morphisms relative to a given class we need to define. For this we need the notion of a transfinite composition, which is the morphism from the first element of a sequence to the colimit, where sequence must be understood in the context of ordinal and cardinal numbers.

Definition A.3. Let $\mathcal{C}$ be a cocomplete category. Let I be a set of morphisms in $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of elements in I. The collection of relative I-cell complexes is denoted I-cell.

Let $C \in \mathrm{Obj}(\mathcal{C})$. Then C is an I-cell complex if $0 \to A$ is a relative I-cell complex.

Remark A.4. In other words, for $f : C \to D$ a morphism in $\mathcal{C}$ to be a relative I-cell complex, we need the existence of an ordinal λ and a λ-sequence $X\colon \lambda \to \mathcal{C}$, such that f is the morphism $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ and such that for each β for which $\beta+1 < \lambda$ we have a pushout square

$$\begin{tikzcd} A_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ B_\beta \arrow[r] & X_{\beta+1} \end{tikzcd} \tag{A.2}$$

where $g_\beta : A_\beta \to B_\beta \in I$.

The proof relies on a transfinite induction argument. The required set theory is not reproduced, a good reference for this is [Jec03], while [MSM99, chapter 10] discusses all the fine points of using this in a model category theory context. For the author [AMS148] has proven to be the most lucid reference for this type of ideas.

Lemma A.5 (Small object argument)**.** Let $\mathcal{C}$ be a cocomplete category. Let I be a set of morphisms in $\mathcal{C}$. Assume that the domains of the morphisms in I are small with respect to I-cell.

Then there exists a functorial factorisation (γ, δ) on $\mathcal{C}$ such that $\gamma(f) \in I$-cell and $\delta(f) \in I$-inj for all $f \in I$.

Proof. Let κ be a cardinal such that the domains of the morphisms in I are κ-small with respect to I-cell. Let λ be a κ-filtered ordinal.

We define the factorisation of the morphism $f : X \to Y$ in $\mathcal{C}$ by a functorial λ-sequence

$$Z^f : \lambda \to \mathcal{C}, \tag{A.3}$$

starting with $Z_0^f := X$ and $\rho_0^f := f$. For the induction steps, first assume that Z_α^f and ρ_α^f are defined for all $\alpha < \beta$, where β is a limit ordinal. Then we take

$$Z_\beta^f := \mathrm{colim}_{\alpha<\beta} Z_\alpha^f \tag{A.4}$$

and

$$\rho_\beta^f := \mathrm{colim}_{\alpha<\beta} \left(\rho_\alpha^f : Z_\alpha^f \to Y\right). \tag{A.5}$$

Now we have to define $Z^f_{\beta+1}$ and $\rho^f_{\beta+1}$. To do so we consider the set S of commutative squares

$$\text{(A.6)}\quad S := \left\{ s\colon \begin{tikzcd} A_s \arrow[r] \arrow[d, "g_s"'] & Z^f_\beta \arrow[d, "\rho^f_\beta"] \\ B_s \arrow[r] & Y \end{tikzcd} \mid g_s \in I \right\}.$$

We then consider the pushout diagram

$$\text{(A.7)}\quad \begin{tikzcd} \coprod_{s\in S} A_s \arrow[r] \arrow[d, "\coprod_{s\in S} g_s"'] & Z^f_\beta \arrow[d, "i_\beta"] \arrow[ddr, "\rho^f_\beta"] & \\ \coprod_{s\in S} B_s \arrow[r] \arrow[drr] & Z^f_{\beta+1} \arrow[dr, dashed, "\rho^f_{\beta+1}"] & \\ & & Y \end{tikzcd}$$

which yields $Z^f_{\beta+1}$ and $\rho^f_{\beta+1}$.

This construction allows us to define $\gamma(f)$ as the transfinite composition of the λ-regular sequence Z^f, i.e.

$$\text{(A.8)}\quad \gamma(f)\colon Z^f_0 = X \to \operatorname{colim}_\beta Z^f_\beta.$$

By [MSM63, lemmas 2.1.12 and 2.1.13] we have that $\gamma(f)$ is a relative I-cell complex, which in terms of model categories means that it is a cofibration (if I is taken to be a set of generating cofibrations).

Then we can take $\delta(f)$ to be the morphism $\operatorname{colim}_\beta \rho^f_\beta$, i.e.

$$\text{(A.9)}\quad \delta(f)\colon \operatorname{colim}_\beta Z^f_\beta \to Y.$$

To conclude the proof we have to show that this choice of $\delta(f)$ has the right lifting property with respect to I, which in terms of model categories means that it is a fibration (if I is taken to be the set of generating cofibrations). Consider a commutative square

$$\text{(A.10)}\quad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & \operatorname{colim}_\beta Z^f_\beta \arrow[d, "\delta(f)"] \\ B \arrow[r, "k"'] & Y \end{tikzcd}$$

where $g \in I$. By assumption A is κ-small with respect to I-cell, so we can choose $\beta < \kappa$ such that h factors as

$$\text{(A.11)}\quad \begin{tikzcd} A \arrow[r, "h_\beta"] \arrow[dr, "h"'] & Z^f_\beta \arrow[d] \\ & \operatorname{colim}_\beta Z^f_\beta. \end{tikzcd}$$

But by construction (A.7) we have a commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "h_\beta"] \arrow[d, "g"'] & Z^f_\beta \arrow[d, "i_\beta"] \\
B \arrow[r, "k_\beta"] \arrow[dr, "k"'] & Z^f_{\beta+1} \arrow[d] \\
& Y.
\end{tikzcd}
\tag{A.12}
$$

Now it suffices to consider the composition

$$
\begin{tikzcd}
B \arrow[r, "k_\beta"] \arrow[dr] & Z^f_{\beta+1} \arrow[d] \\
& \operatorname{colim}_\beta Z^f_\beta
\end{tikzcd}
\tag{A.13}
$$

to obtain the lifting in (A.10). □

A.2 Model category structures on Ch(k-Mod)

The model category structure on $\mathrm{Ch}_{\geq 0}(k\text{-Mod})$ was one of the motivating examples in the development of homotopical algebra [LNM43, examples I.2.B]. Remark that Quillen restricts himself to non-negatively graded chain complexes, as the description of the cofibrant objects is nicer [MSM63, lemma 2.3.6] and there is the Dold-Kan correspondence to relate things to simplicial objects. But it is nevertheless possible to construct (non-trivial) model category structures on Ch(k-Mod), the category of *unbounded* cochain complexes, as is done in proposition A.7 and proposition A.9. Remark that again the cochain convention is used, which makes this section differ from most texts on this subject: the "standard" structure, which we will discuss first, is now an injective structure.

This first structure is the one Quillen originally constructed, but now applied to the more general context of unbounded complexes. As both structures on Ch(k-Mod) are cofibrantly generated we will only discuss the generating (acyclic) cofibrations and refer to [MSM63, §2.3] for the proofs. The main properties of these structures are described in table A.1.

Definition A.6. Let M be a k-module. Let n be an integer. The *nth disk object* is the cochain complex $\mathrm{D}_n(M)^\bullet$, defined by

$$
\mathrm{D}_n(M)^\bullet := \begin{cases} M & m = n-1, n \\ 0 & \text{otherwise} \end{cases}
\tag{A.14}
$$

with the only non-trivial differential $\mathrm{d}^{n-1} = \mathrm{id}$.

The *nth sphere object* is the cochain complex $\mathrm{S}_n(M)^\bullet$, defined by

$$
\mathrm{S}_n(M)^\bullet := \begin{cases} M & m = n \\ 0 & \text{otherwise.} \end{cases}
\tag{A.15}
$$

We have the injection $\mathrm{i}_n \colon \mathrm{S}_n(M)^\bullet \to \mathrm{D}_n(M)^\bullet$. We then define

$$
\begin{aligned}
\mathrm{I}_{\mathrm{inj}} &:= \left\{ \mathrm{S}_n(M)^\bullet \to \mathrm{D}_n(M)^\bullet \mid M \in \mathrm{Obj}(k\text{-Mod}) \right\}, \\
\mathrm{J}_{\mathrm{inj}} &:= \left\{ 0 \to \mathrm{D}_n(M)^\bullet \mid M \in \mathrm{Obj}(k\text{-Mod}) \right\}.
\end{aligned}
\tag{A.16}
$$

Proposition A.7 (Injective model category structure on Ch(k-Mod))**.** If we take $\mathrm{I}_{\mathrm{inj}}$ (resp. $\mathrm{J}_{\mathrm{inj}}$) as generating cofibrations (resp. generating acyclic cofibrations) and quasi-isomorphisms as weak equivalences we have a cofibrantly generated model category structure on Ch(k-Mod).

Besides this "standard" model category structure we also have a projective model category structure on Ch(k-Mod). Whereas the standard one has injective resolutions as the cofibrant replacement, we now have projective resolutions playing the lead role. The reason why this model structure is less standard is that the definition of the generating (acyclic) cofibrations is more involved.

Definition A.8. Let $M^\bullet$ be a cochain complex of k-modules. We define the *cardinality* $|M^\bullet|$ of $M^\bullet$ by setting it equal to the cardinality of $\bigcup_{n\in\mathbb{Z}} M^n$. We also set

$$\gamma := \sup\{|k|, +\infty\}\,. \tag{A.17}$$

Then we define $\mathrm{I}_{\mathrm{proj}}$ to be the set of morphisms containing a representative of each isomorphism class of monomorphisms $M^\bullet \to N^\bullet$ such that $|N^\bullet| \leq \gamma$, and $\mathrm{I}_{\mathrm{proj}}$ is the intersection of $\mathrm{I}_{\mathrm{inj}}$ and the class of quasi-isomorphisms.

Proposition A.9 (Projective model category structure on Ch(k-Mod))**.** If we take $\mathrm{I}_{\mathrm{proj}}$ (resp. $\mathrm{J}_{\mathrm{proj}}$) as generating cofibrations (resp. generating acyclic cofibrations) and quasi-isomorphisms as weak equivalences we have a cofibrantly generated model category structure on Ch(k-Mod).

In table A.1 a summary of the properties of these two structures is given. Remark that the identity functor on Ch(k-Mod) yields a Quillen equivalence between the two model category structures, hence they are compatible, and they can both be considered as a generalisation of homological algebra.

Remark A.10. In [LNM43, §II.5] one finds the following quote.

> "If homotopical algebra is thought of as 'non-linear' or 'non-additive' homological algebra, then it is natural to ask what is the 'linearization' of 'abelianization' of this non-linear situation."

In other words: how does the formalism of homotopical algebra encode the classical case of homological algebra, or how can we interpret the results from homological algebra from a homotopical point of view? A result depicting this idea is [LNM43, theorem II.5.5].

In more down-to-earth terms: in case of Ch(k-Mod) this means that the (co)fibrant replacements correspond to resolutions. By the interpretation of the Ext-groups as Hom's in the derived category [CSAM29, corollary 10.7.5] we see that

$$\mathrm{Hom}_{\mathrm{Ho(Ch}(k\text{-Mod}))}(M[n], N) \cong \mathrm{Ext}_k^n(M,N) \cong \mathrm{Hom}_{\mathrm{Ho(Ch}(k\text{-Mod}))}(M, N[n]) \tag{A.18}$$

where M and N are k-modules considered as cochain complexes in degree zero.

A.3 The case of an arbitrary Grothendieck category

In §5.3 a model category structure on $\mathrm{Ch}(\mathrm{Qcoh}_X)$ is used. This structure is a special case of the model category structure on the cochain complexes of objects in a Grothendieck category [Hov01]. To apply this result we need to impose some conditions on X, namely it should be quasicompact and quasiseparated. These conditions are exactly the conditions imposed in §5.3. In [Hov01] there are moreover two other model category structures introduced, but only one of them applies (partially) to our situation. In this section we will discuss these two model category structures.

The *projective* model category structure on $\mathrm{Ch}(\mathrm{Qcoh}_X)$ is constructed in the same way of the projective model category structure on Ch(A-Mod), as done in appendix A.2. One uses the Popescu-Gabriel theorem to reduce the statement to a statement on modules, while taking care of all the cardinality arguments. For its proof we refer to [Hov01, §2].

Another possibility is the *locally free* model category structure on $\mathrm{Ch}(\mathrm{Qcoh}_X)$ [Hov01, theorem 4.4], which is similar (but not equal to) the injective model category structure as discussed in appendix A.2. A comparison between the two structures is given in table A.2.

		injective model structure	projective model structure
weak equivalence	$=$	quasi-isomorphism	quasi-isomorphism
fibration	$=$	split epi with injective kernel	epimorphism
cofibration	$=$	monomorphism	split mono with projective cokernel
fibrant object	$\Rightarrow$	positionwise injective	
bounded below fibrant	$\Leftrightarrow$		
cofibrant object	$\Rightarrow$		positionwise projective
bounded above fibrant	$\Leftrightarrow$		

Table A.1: Comparison of the injective and projective model category structures on $\mathrm{Ch}(k\text{-Mod})$

		locally free model structure	projective model structure
weak equivalence	$=$	quasi-isomorphism	quasi-isomorphism
fibration	$=$	split epi with injective kernel	epimorphism
cofibration	$=$	*unknown*	split mono with injective kernel

Table A.2: Comparison of the locally free and projective model category structure on $\mathrm{Ch}(\mathrm{Qcoh}_X)$

Appendix B

A model category structure on $\mathrm{dg\,Cat}_k$

In this appendix we discuss the proof of the model category structure on $\mathrm{dg\,Cat}_k$, which is used in §3.1. To not disrupt the discussion in that section we give the full proof here. It also ties in with the results from appendix A. It is probably the first proof of this result in English, the original article is in French [Tab05a]. In remark B.22 we introduce two other model category structures and compare them.

We will use the recognition theorem [MSM63, theorem 2.1.19]. Hence we need to define a class of generating cofibrations, and a class of generating trivial cofibrations. The class of weak equivalences will be the quasi-equivalences.

Definition B.1. We define $\mathcal{K}$ to be the dg category with as its objects $\mathrm{Obj}(\mathcal{K}) = \{1, 2\}$, whose morphism complexes are generated by

$$\mathcal{K}\colon \begin{tikzcd} 1 \arrow[loop left, "r_1"] \arrow[loop below, "\mathrm{id}_1"] \arrow[r, shift left, "f"] \arrow[r, bend left=60, "r_{1,2}"] & 2 \arrow[l, shift left, "g"] \arrow[loop right, "r_2"] \arrow[loop below, "\mathrm{id}_2"] \end{tikzcd} \tag{B.1}$$

where

$$\begin{gathered} f \in \mathrm{Hom}_{\mathcal{K}}(1,2)^0, \qquad g \in \mathrm{Hom}_{\mathcal{K}}(2,1)^0, \\ r_1 \in \mathrm{Hom}_{\mathcal{K}}(1,1)^{-1}, \qquad r_2 \in \mathrm{Hom}_{\mathcal{K}}(2,2)^{-1}, \qquad r_{1,2} \in \mathrm{Hom}_{\mathcal{K}}(1,2)^{-2}, \end{gathered} \tag{B.2}$$

and whose differentials are given by

$$\begin{aligned} \mathrm{d}(f) = \mathrm{d}(g) &= 0, \\ \mathrm{d}(r_1) &= g \circ f - \mathrm{id}_1, \\ \mathrm{d}(r_2) &= f \circ g - \mathrm{id}_2, \\ \mathrm{d}(r_{1,2}) &= f \circ r_1 - r_2 \circ f. \end{aligned} \tag{B.3}$$

The definition of this category might seem rather ad-hoc, but its use will become clear in lemma B.2. Its origin can be traced to [Dri04, §3.7], who cites Kontsevich as the discoverer, which might be an explanation of the name $\mathcal{K}$. In the statement of the next lemma, recall that a *contraction* is the morphism in degree -1 that is used to define a null-homotopy of the identity morphism (which lives in degree 0).

Lemma B.2. Let $\mathcal{C}$ be a dg category. Then there exists a bijection between $\mathrm{Hom}_{\mathrm{dg\,Cat}_k}(\mathcal{K}, \mathcal{C})$ and the set

$$\left\{(s,h) \mid s \in \mathrm{Z}^0(\mathrm{Hom}_{\mathcal{C}}(X,Y)^\bullet), h \text{ contraction of } \mathrm{cone}(s^*) \text{ in } \mathcal{C}\text{-}\mathrm{dg\,Mod}_k\right\}. \tag{B.4}$$

Proof. Let $F\colon \mathcal{K} \to \mathcal{C}$ be a dg functor. The image of $f : 1 \to 2$ in $\mathcal{K}$ must be a morphism $s\colon X \to Y$ in $\mathcal{C}$ such that $\mathrm{d}(s) = 0$.
Now consider this morphism in $\mathcal{C}$-dg Mod_k under the Yoneda embedding. We will denote it $s^*\colon \mathrm{h}^X \to \mathrm{h}^Y$. We can then consider its mapping cone in $\mathcal{C}$-dg Mod_k, which we'll denote $\mathrm{cone}(s^*)$. It is the $\mathcal{C}$-dg module $\mathrm{h}^Y \oplus \mathrm{h}^X[1]$ whose differential is given by

$$\text{(B.5)}\quad \begin{pmatrix} \mathrm{d}_Y & s \\ 0 & -\mathrm{d}_X \end{pmatrix}^*$$

where $\mathrm{d}_X[1] = -\mathrm{d}_X$ as in definition 1.8.
Let h be a morphism in $\mathrm{Hom}_{\mathcal{C}\text{-dg}\,\mathrm{Mod}_k}(\mathrm{cone}(s^*), \mathrm{cone}(s^*))^{-1}$. Because we are considering representable objects, this morphism can be written as

$$\text{(B.6)}\quad \begin{pmatrix} r_2' & r_{1,2}' \\ g' & r_1' \end{pmatrix}^*$$

with

$$\text{(B.7)}\quad \begin{gathered} g' \in \mathrm{Hom}_{\mathcal{C}}(Y,X)^0, \\ r_1' \in \mathrm{Hom}_{\mathcal{C}}(X,Y)^{-1}, \qquad r_2' \in \mathrm{Hom}_{\mathcal{C}}(Y,X)^{-1}, \qquad r_{1,2}' \in \mathrm{Hom}_{\mathcal{C}}(X,Y)^{-2}. \end{gathered}$$

The condition that h is a contraction of $\mathrm{cone}(s^*)$ can now be written as the equation

$$\text{(B.8)}\quad \begin{pmatrix} r_2' & r_{1,2}' \\ g' & r_1' \end{pmatrix}^* \begin{pmatrix} \mathrm{d}_Y & s \\ 0 & -\mathrm{d}_X \end{pmatrix}^* - (-1)^{-1} \begin{pmatrix} \mathrm{d}_Y & s \\ 0 & -\mathrm{d}_X \end{pmatrix}^* \begin{pmatrix} r_2' & r_{1,2}' \\ g' & r_1' \end{pmatrix}^* = \begin{pmatrix} \mathrm{id}_Y & 0 \\ 0 & \mathrm{id}_X \end{pmatrix}^*$$

which encodes the fact that h yields a cochain homotopy from the identity to the zero morphism. We can write it equivalently as a linear system

$$\text{(B.9)}\quad \begin{cases} \mathrm{id}_Y = r_2' \circ \mathrm{d}_Y + \mathrm{d}_Y \circ r_2' + s \circ g' \\ \quad 0 = r_2' \circ s - r_{1,2}' \circ \mathrm{d}_X + \mathrm{d}_Y \circ r_{1,2}' - s \circ r_1' \\ \quad 0 = g' \circ \mathrm{d}_Y - \mathrm{d}_X \circ g' \\ \mathrm{id}_X = g' \circ s - r_1' \circ \mathrm{d}_X - \mathrm{d}_X \circ r_1'. \end{cases}$$

of morphisms in $\mathcal{C}$. Remark the sign for $s \circ r_1'$: it is negative by the Koszul sign rule (1.6). Given the differential for $\mathrm{Ch}_{\mathrm{dg}}(k\text{-Mod})$ as in example 1.11 we can rewrite this to (mind the degree of the morphisms)

$$\text{(B.10)}\quad \begin{cases} \mathrm{id}_Y = -\mathrm{d}(r_2') + s \circ g' \\ \quad 0 = r_2' \circ s + \mathrm{d}(r_{1,2}') - s \circ r_1' \\ \quad 0 = -\mathrm{d}(g') \\ \mathrm{id}_X = g' \circ s - \mathrm{d}(r_1'). \end{cases}$$

But these are exactly the conditions as given in (B.3) (we have chosen s such that $\mathrm{d}(s) = 0$), hence the data of a pair (s, h) corresponds bijectively to a dg functor in $\mathrm{Hom}_{\mathrm{dg}\,\mathrm{Cat}_k}(\mathcal{K}, \mathcal{C})$. $\square$

We now introduce two easy dg categories: one is just the unit for the monoidal structure while the other is the category with two objects and the trivial morphism complexes.

Definition B.3. We denote k the dg category with as its objects $\mathrm{Obj}(k) = \{3\}$, whose morphism complex is generated by

$$\text{(B.11)}\quad k\colon\ \begin{tikzcd} 3 \arrow[loop below, "\mathrm{id}_3"] \end{tikzcd}$$

where $\mathrm{Hom}_k(3,3)^0 = k$, together with the trivial differential. This is nothing but the dg category corresponding to the (trivial) dg algebra k, but to keep up with the notation from [Tab05a] we use 3 instead of $*$ for the unique object.

Definition B.4. We define $\mathcal{B}$ to be the dg category with as its objects $\mathrm{Obj}(\mathcal{B}) = \{4,5\}$, whose morphism complexes are generated by

$$\text{(B.12)}\quad \mathcal{B}\colon \begin{tikzcd} 4 \arrow[loop below, "\mathrm{id}_4"] & 5 \arrow[loop below, "\mathrm{id}_5"] \end{tikzcd}$$

where $\mathrm{Hom}_{\mathcal{B}}(4,4)^0 = \mathrm{Hom}_{\mathcal{B}}(5,5)^0 = k$, together with the trivial differentials.

We now mimick the definitions from I_{inj} and J_{inj} from definition A.6, and we will also reuse the notation of the sphere and disk objects introduced there. We first introduce analogues of the sphere and disk objects, which will act as sphere and disk dg categories.

Definition B.5. Let n be an integer. We define $\mathcal{D}(n)$ to be the dg category with as its objects $\mathrm{Obj}(\mathcal{D}(n)) = \{6,7\}$, whose morphism complexes are generated by

$$\text{(B.13)}\quad \mathcal{D}(n)\colon \begin{tikzcd} 6 \arrow[loop below, "\mathrm{id}_6"] \arrow[r, shift left] \arrow[r, shift right] & 7 \arrow[loop below, "\mathrm{id}_7"] \end{tikzcd}$$

where $\mathrm{Hom}_{\mathcal{D}(n)}(6,6)^0 = \mathrm{Hom}_{\mathcal{D}(n)}(7,7)^0 = k$ and $\mathrm{Hom}_{\mathcal{D}(n)}(6,7)^\bullet = \mathrm{D}_n(k)^\bullet$, together with the trivial differential for the endomorphism complexes.

One could add the differential to the diagram in (B.13), it would result in

$$\text{(B.14)}\quad \mathcal{D}(n)\colon \begin{tikzcd} 6 \arrow[loop below, "\mathrm{id}_6"] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 7. \arrow[loop below, "\mathrm{id}_7"] \end{tikzcd}$$

The convention is that the trivial differentials are not indicated. A similar diagram for (B.1) would unfortunately be infeasible.

Definition B.6. Let n be an integer. We define $\mathcal{S}(n)$ to be the dg category with as its objects $\mathrm{Obj}(\mathcal{S}(n)) = \{8,9\}$, whose morphism complexes are generated by

$$\text{(B.15)}\quad \mathcal{S}(n)\colon \begin{tikzcd} 8 \arrow[loop below, "\mathrm{id}_8"] \arrow[r] & 9 \arrow[loop below, "\mathrm{id}_9"] \end{tikzcd}$$

where $\mathrm{Hom}_{\mathcal{S}(n)}(8,8) = \mathrm{Hom}_{\mathcal{S}(n)}(9,9) = k$ and $\mathrm{Hom}_{\mathcal{S}(n)}(8,9) = \mathrm{S}_n(k)^\bullet$, together with the trivial differential for the endomorphism complexes.

We can now define the dg functors that will constitute the generating (acyclic) cofibrations.

Definition B.7. We define $I\colon k \to \mathcal{K}$ to be the dg functor with $3 \mapsto 1$, i.e. the inclusion of k in $\mathcal{K}$ on the object 1.

We define $R(n)\colon \mathcal{B} \to \mathcal{D}(n)$ to be the dg functor with $4 \mapsto 6$ and $5 \mapsto 7$, i.e. the inclusion of $\mathcal{B}$ in $\mathcal{D}(n)$ in the "constant" part of $\mathcal{D}(n)$.

We define $Q\colon 0 \to k$ to be the unique dg functor from the initial object to k.

We define $S(n)\colon \mathcal{S}(n) \to \mathcal{D}(n)$ to be the dg functor with $8 \mapsto 6$, $9 \mapsto 7$ and $S(n)_{8,9} = \mathrm{i}_n$, as in definition A.6.

Finally, we set

$$\text{(B.16)}\quad \begin{aligned} \mathrm{I}_{\mathrm{dg}} &:= \{Q\} \cup \{S(n) \mid n \in \mathbb{Z}\}, \\ \mathrm{J}_{\mathrm{dg}} &:= \{I\} \cup \{R(n) \mid n \in \mathbb{Z}\}. \end{aligned}$$

We can now check all the conditions from [MSM63, theorem 2.1.19]. These conditions correspond to lemmas B.8 to B.10, B.14, B.18 and B.19, whose ordering is the same as in [MSM63]. The class of all quasi-equivalences in $\mathrm{dg\,Cat}_k$ will be denoted $\mathcal{W}$.

The first three properties are straight-forward.

Lemma B.8. The class of quasi-equivalences $\mathcal{W}$ satisfies the 3-out-of-2 property and is stable under retracts.

Proof. The class of quasi-fully faithful dg functors is stable under the 3-out-of-2 property because it reduces to the 3-out-of-2 property for quasi-isomorphisms, which is characterised using actual isomorphisms and these satisfy the 3-out-of-2 property trivially. The class of quasi-essentially surjective dg functors is stable under the 3-out-of-2 property because it reduces to a statement about the essential surjectivity of homotopy categories.

The class of quasi-fully faithful dg functors is stable under retracts because it reduces to a statement on quasi-isomorphisms, which reduces to a statement on actual isomorphisms which are stable under retracts. The class of quasi-essentially surjective dg functors is stable under retracts because retracts are preserved by functors, and then the statement is again reduced to a classical categorical statement. □

Lemma B.9. The domains of the elements of I_{dg} are small with respect to I_{dg}-cell.

Proof. The domains of the morphisms in I_{dg} have the same cardinality as k and $k \oplus k$, hence they are small in $\mathrm{dg\,Cat}_k$ and I_{dg}-cell. □

Lemma B.10. The domains of the elements of J_{dg} are small with respect to J_{dg}-cell.

Proof. The domains of the morphisms in J_{dg} have the same cardinality as $k \oplus k$, hence they are small in $\mathrm{dg\,Cat}_k$ and J_{dg}-cell. □

To prove lemma B.14 (which corresponds to condition 4 in [MSM63, theorem 2.1.19]) we will prove separately that $\mathrm{J}_{\mathrm{dg}}\text{-cell} \subseteq \mathcal{W}$ and $\mathrm{J}_{\mathrm{dg}}\text{-cell} \subseteq \mathrm{I}_{\mathrm{dg}}\text{-cof}$. This is done in lemmas B.11 and B.13.

Lemma B.11. We have that

$$\mathrm{J}_{\mathrm{dg}}\text{-cell} \subseteq \mathcal{W}. \tag{B.17}$$

Proof. The proof consists of three steps: proving that $\mathcal{W}$ is closed under transfinite composition, that a pushout along $R(n)$ induces a quasi-equivalence en that a pushout along I induces a quasi-equivalence.

To see that $\mathcal{W}$ is closed under transfinite composition, recall that taking cohomology in an AB5 category commutes with colimits [Gro57]. Hence the quasifully faithfullness and quasi-essential surjectivity, which are described using cohomology operations, are stable under transfinite compositions. A similar argument is used in [Hov01, §1].

Now we prove that pushout along $R(n)$ induces a quasi-equivalence. Let $F\colon \mathcal{B} \to \mathcal{C}_1$ be a dg functor. We define the dg category $\mathcal{C}_2$ by the pushout diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "F"] \arrow[d, "R(n)"'] & \mathcal{C}_1 \arrow[d, "R(n)'"] \\
\mathcal{D}(n) \arrow[r] & \mathcal{C}_2 \coloneqq \mathcal{C}_1 \sqcup_{\mathcal{B}} \mathcal{D}(n)
\end{tikzcd} \tag{B.18}$$

We wish to show that $R(n)'$ is a quasi-equivalence. By the description of $\mathcal{B}$ and $\mathcal{D}(n)$ we can construct an explicit model for $\mathcal{C}_2$. We first construct the disjoint union, for this it suffices to add a morphism f to $\mathrm{Hom}_{\mathcal{C}_1}(F(4),F(5))^{n-1}$ and g to $\mathrm{Hom}_{\mathcal{C}_1}(F(4),F(5))^{n}$ such that $\mathrm{d}(f) = g$.

Then we take the dg quotient [Dri04, §3.1], which for objects $X, Y \in \mathrm{Obj}(\mathcal{C}_1) = \mathrm{Obj}(\mathcal{C}_2)$ yields a decomposition (as graded k-modules)

$$\text{(B.19)}\quad \mathrm{Hom}_{\mathcal{C}_2}(X,Y)^\bullet \cong \bigoplus_{m=0}^{+\infty} \mathrm{Hom}^{(m)}_{\mathcal{C}_2}(X,Y)^\bullet$$

where

(B.20)

$$\mathrm{Hom}^{(m)}_{\mathcal{C}_2}(X,Y)^\bullet \cong \mathrm{Hom}_{\mathcal{C}_1}(F(5),Y)^\bullet \otimes_k \mathrm{S}_1(k)^\bullet \otimes_k \mathrm{Hom}_{\mathcal{C}_1}(F(5),F(4))^\bullet \otimes_k \mathrm{S}_1(k)^\bullet \otimes_k \cdots$$
$$\cdots \otimes_k \mathrm{S}_1(k)^\bullet \otimes_k \mathrm{Hom}_{\mathcal{C}_1}(X,F(4))^\bullet.$$

This is a special case of the decomposition of the morphism complex of a dg quotient, because we can only have one term in the direct sum by the structure of the dg category k: this means that (up to permutation) we have this isomorphism. There are m factors of $\mathrm{S}_1(k)^\bullet$ in the tensor product, hence for $m = 0$ we get an isomorphism with the original complex. This yields an inclusion

$$\text{(B.21)}\quad \mathrm{Hom}_{\mathcal{C}_1}(X,Y)^\bullet \to \mathrm{Hom}_{\mathcal{C}_2}(X,Y)^\bullet.$$

As $\mathrm{D}_n(k)^\bullet$, which corresponds to the third factor, is a contractible chain complex we obtain the vanishing of the cohomology for $m \geq 1$ (take tensor products of the cochain contraction with the identities in the other terms), hence this inclusion is a quasi-isomorphism. As the dg functor $R(n)$ was the identity on the objects, we obtain that it is indeed a quasi-equivalence.
Finally we prove that a pushout along I induces a quasi-equivalence. Let $F\colon k \to \mathcal{C}_1$ be a dg functor. We define the dg category $\mathcal{C}_2$ by the pushout diagram

$$\text{(B.22)}\quad \begin{tikzcd} k \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{C}_1 \arrow[d, "I'"] \\ \mathcal{K} \arrow[r, "F'"'] & \mathcal{C}_2 \coloneqq \mathcal{C}_1 \sqcup_k \mathcal{K} \end{tikzcd}$$

We wish to show that I' is a quasi-equivalence. We can write down two models for the category $\mathcal{C}_2$.

(i) We add $\mathcal{K}$ to $\mathcal{C}_1$ and identify the objects $F(3)$ and $I(3)$, as a very special case of the construction of the dg quotient [Dri04, §3.1].

(ii) Let $\mathcal{C}_1^0$ be the dg category obtained from the dg category $\mathcal{C}_1$ by adding an object C to $\mathrm{Obj}(\mathcal{C}_1)$ and a morphism $s \in \mathrm{Hom}_{\mathcal{C}_1^0}(F(3),C)^0$ with $\mathrm{d}(s) = 0$.

Let $\mathcal{C}_1^1$ be the full dg subcategory of $\mathcal{C}_1^0$-dgMod_k of representable dg functors (this includes the mapping cone on s^*).

Let $\mathcal{C}_1^2$ be the dg category obtained from the dg category $\mathcal{C}_1^1$ by adding a morphism h to $\mathrm{Hom}_{\mathcal{C}_1^1}(\mathrm{cone}(s^*),\mathrm{cone}(s^*))^{-1}$ such that $\mathrm{d}(h) = \mathrm{id}_{\mathrm{cone}(s^*)}$. This means we have obtained the following situation

$$\text{(B.23)}\quad \begin{tikzcd} \mathrm{h}^{I'(2)} \arrow[r, shift left, "I'(g)^*"] & \mathrm{h}^{I'(1)} \arrow[l, shift left, "I'(f)^*"] \arrow[r, "s^*"] & \mathrm{h}^C \arrow[r] & \mathrm{cone}(s^*) \arrow[loop right, "h"] \end{tikzcd}$$

Let $\mathcal{C}_1^3$ be the full dg subcategory of $\mathcal{C}_1^2$ whose objects are the images of $\mathcal{C}_1^0$ under the Yoneda embedding. We can then prove that $\mathcal{C}_2$ is isomorphic to $\mathcal{C}_1^3$. This is done in lemma B.12, and we will assume this fact from now on.

Let X and Y be objects of $\mathcal{C}_1$. As in the previous part of this proof where $R(n)'$ is shown to be a quasi-equivalence we can apply [Dri04, §3.1] which yields a decomposition of graded k-modules

$$\text{(B.24)}\quad \mathrm{Hom}_{\mathcal{C}_2}(X,Y)^\bullet \cong \mathrm{Hom}_{\mathcal{C}_1^2}(\mathrm{h}^X,\mathrm{h}^Y)^\bullet \cong \bigoplus_{m=0}^{+\infty} \mathrm{Hom}_{\mathcal{C}_1^2}^{(m)}(\mathrm{h}^X,\mathrm{h}^Y)^\bullet$$

where

(B.25)

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}_1^2}^{(m)}\left(\mathrm{h}^X,\mathrm{h}^Y\right)\\ &\quad\cong \mathrm{Hom}_{\mathcal{C}_1^1}\left(\mathrm{cone}(s^*),\mathrm{h}^Y\right)^\bullet \otimes_k \mathrm{S}_1(k)^\bullet \otimes_k \mathrm{Hom}_{\mathcal{C}_1^1}\left(\mathrm{cone}(s^*),\mathrm{cone}(s^*)\right)^\bullet \otimes_k \mathrm{S}_1(k)^\bullet \otimes_k \cdots\\ &\qquad \cdots \otimes_k \mathrm{S}_1(k)^\bullet \otimes_k \mathrm{Hom}_{\mathcal{C}_1^1}\left(\mathrm{h}^X,\mathrm{cone}(s^*)\right)^\bullet.\end{aligned}$$

Again we have no direct sum, there exists only one term.
As in [Dri04, §3.1] we have that the finite sum

$$\text{(B.26)}\quad \bigoplus_{m=0}^{n} \mathrm{Hom}_{\mathcal{C}_1^2}^{(m)}(\mathrm{h}^X,\mathrm{h}^Y)^\bullet$$

of cochain complexes yields a subcomplex of $\mathrm{Hom}_{\mathcal{C}_1^2}(\mathrm{h}^X,\mathrm{h}^Y)^\bullet$, hence we obtain an exhaustive filtration of this morphism complex (by the isomorphism (B.24)), whose mth subquotient is identified with $\mathrm{Hom}_{\mathcal{C}_1^2}^{(m)}(\mathrm{h}^X,\mathrm{h}^Y)^\bullet$.
The morphism complex $\mathrm{Hom}_{\mathcal{C}_1^1}(\mathrm{h}^X,\mathrm{cone}(s^*))^\bullet$ can be identified with the mapping cone over the isomorphism

$$\text{(B.27)}\quad \mathrm{Hom}_{\mathcal{C}_1^0}(X,I(3))^\bullet \cong \mathrm{Hom}_{\mathcal{C}_1^0}(X,C)^\bullet$$

in $\mathcal{C}_1^0$ because we are considering representable objects, hence this cochain complex is contractible. This yields that the inclusion

$$\text{(B.28)}\quad \mathrm{Hom}_{\mathcal{C}_1}(X,Y)^\bullet \to \mathrm{Hom}_{\mathcal{C}_2}(X,Y)^\bullet \cong \mathrm{Hom}_{\mathcal{C}_1^2}(\mathrm{h}^X,\mathrm{h}^Y)^\bullet$$

is a quasi-isomorphism.
There is one more morphism s, but this becomes an isomorphism in $\mathrm{H}^0(\mathcal{C}_2)$. Hence I' is *quasi-fully faithful*.
As the dg functor I' corresponding to the inclusion is the identity on the level of objects we have that I' is *quasi-essentially surjective*. We conclude that I' is a quasi-equivalence. □

Lemma B.12. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_1^i$ be as in the proof of lemma B.11. Then the natural dg functor $\mathcal{C}_1 \to \mathcal{C}_1^3$ extends uniquely to a dg functor $\mathcal{C}_2 \to \mathcal{C}_1^3$, and this second dg functor is an isomorphism.

Proof. All notation in this lemma is taken from the proof of lemma B.11.
Using the proof of lemma B.2 we see that the natural dg functor $\mathcal{C}_1 \to \mathcal{C}_1^3$ extends uniquely to a dg functor $\mathcal{C}_2 \to \mathcal{C}_1^3$ such that:

- $I'(2)$ is mapped to h^C;
- $I'(f)$ is mapped to s^*;
- $I'(g)$, $I'(r_1)$, $I'(r_2)$ and $I'(r_{1,2})$ are mapped to the components of the contraction h, as in (B.6).

This dg functor is an isomorphism, whose inverse we will construct explicitly. To do so, we extend the inclusion $\mathcal{C}_1 \to \mathcal{C}_2$ to a dg functor $\mathcal{C}_1^0 \to \mathcal{C}_2$ which sends C to $I'(2)$ and s to $I'(f)$. Then we extend this dg functor to a dg functor $\mathcal{C}_1^1 \to \mathcal{C}_2$-dg Mod_k which sends cone(s^*) to cone$(I'(f)^*)$. Using lemma B.2 we obtain a contraction h' of cone$(I'(f)^*)$.
Finally we extend this dg functor uniquely to a dg functor $\mathcal{C}_1^2 \to \mathcal{C}_2$-dg Mod_k by mapping h to h'.
After restricting this functor to $\mathcal{C}_1^3 \to \mathcal{C}_2$-dg Mod_k we obtain an inverse for the dg functor $\mathcal{C}_2 \to \mathcal{C}_1^3$. □

Lemma B.13. We have that

(B.29) $\mathrm{J_{dg}}$-cell $\subseteq \mathrm{I_{dg}}$-cof.

Proof. This holds in general if condition 5 of [MSM63, theorem 2.1.19] is satisfied. Let I and J be general, as in the recognition theorem. Assume that I-inj $\subseteq J$-inj. This implies J-cof $\subseteq I$-cof, and we have J-cell $\subseteq J$-cof. Hence in this case it will follow if we have proven lemma B.18. □

Lemma B.14. We have the inclusion

(B.30) $\mathrm{J_{dg}}$-cell $\subseteq \mathcal{W} \cap \mathrm{I_{dg}}$-cof.

Proof. This follows immediately from lemmas B.11 and B.13. □

To prove lemmas B.18 and B.19 (or conditions 5 and 6 in [MSM63, theorem 2.1.19]) we will prove that

(B.31) $\mathrm{I_{dg}}$-inj $= \mathcal{W} \cap \mathrm{J_{dg}}$-inj,

by relating both sides to yet another class of morphisms with an explicit description.

Definition B.15. Let $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ be a dg functor. We say it is *quasi-surjective* if

(i) it is a surjection on the level of the objects;

(ii) it induces a surjective quasi-isomorphism for every morphism complex.

The class of all quasi-surjective morphisms will be denoted Surj.

Lemma B.16. We have that

(B.32) $\mathrm{I_{dg}}$-inj $=$ Surj.

Proof. We can rewrite $\mathrm{I_{dg}}$-inj as

(B.33) $\mathrm{I_{dg}}$-inj $= \{Q\}$-inj $\cap \{S(n) \mid n \in \mathbb{Z}\}$-inj.

The class $\{Q\}$-inj corresponds to the dg functors which are *surjective on the level of objects*: we have to consider the diagram

(B.34)
$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "Q"'] & \mathcal{C}_1 \arrow[d] \\ k \arrow[r] \arrow[ur, dashed] & \mathcal{C}_2 \end{tikzcd}$$

in which the dashed arrow corresponds to the choice of an object in $\mathcal{C}_1$ mapping to $\mathcal{C}_2$, there are no relations on the morphisms to be satisfied. The fact that $\mathrm{Hom}_k(3,3)^\bullet = k$ corresponds to the identity morphism, hence this injects into every morphism complex of a dg category.

The class $\{S(n) \mid n \in \mathbb{Z}\}$-inj corresponds to the dg functors which *induce surjective quasi-isomorphisms on the morphism complexes*: we have to consider the diagram

$$\begin{tikzcd} \mathcal{S}(n) \arrow[r, "F"] \arrow[d, "S(n)"'] & \mathcal{C}_1 \arrow[d, "P"] \\ \mathcal{D}(n) \arrow[r, "G"'] \arrow[ur, dashed] & \mathcal{C}_2 \end{tikzcd} \tag{B.35}$$

for every $n \in \mathbb{Z}$. Because there is only one non-trivial morphism complex in both $\mathcal{S}(n)$ and $\mathcal{D}(n)$ this corresponds to the diagram

$$\begin{tikzcd} \mathrm{S}_n(k)^\bullet \arrow[r, "F_{8,9}"] \arrow[d, "\mathrm{i}_n"'] & \mathrm{Hom}_{\mathcal{C}_1}(F(8), F(9))^\bullet \arrow[d, "P_{F(8),F(9)}"] \\ \mathrm{D}_n(k)^\bullet \arrow[r, "G_{6,7}"'] \arrow[ur, dashed] & \mathrm{Hom}_{\mathcal{C}_2}(G(6), G(7))^\bullet \end{tikzcd} \tag{B.36}$$

of cochain complexes. But the right lifting property with respect to $\mathrm{S}_n(k)^\bullet \to \mathrm{D}_n(k)^\bullet$ is given in [MSM63, lemma 2.3.5], which corresponds to the characterisation we need. □

Lemma B.17. We have that

$$\mathrm{J}_{\mathrm{dg}}\text{-inj} \cap \mathcal{W} = \mathrm{Surj}. \tag{B.37}$$

Proof. We first prove the inclusion $\supseteq$. Let $H\colon \mathcal{C}_1 \to \mathcal{C}_2$ be a dg functor in Surj. Then we have that H is a quasi-equivalence, by definition of a quasi-surjection.

The class $\{R(n) \mid n \in \mathbb{Z}\}$-inj correspond to the dg functors which are *surjections on the morphism complexes*: we have to consider the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "R(n)"'] & \mathcal{C}_1 \arrow[d] \\ \mathcal{D}(n) \arrow[r, "G"'] & \mathcal{C}_2 \end{tikzcd} \tag{B.38}$$

for every $n \in \mathbb{Z}$ (the functor $\mathcal{C}_1 \to \mathcal{C}_2$ is now arbitrary). Because there is only one non-trivial morphism complex in $\mathcal{D}(n)$ this corresponds to the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}_1}(F(4), F(5))^\bullet \arrow[d] \\ \mathrm{D}_n(k)^\bullet \arrow[r] & \mathrm{Hom}_{\mathcal{C}_2}(G(6), G(7))^\bullet \end{tikzcd} \tag{B.39}$$

of cochain complexes. The right lifting property with respect to $0 \to \mathrm{D}_n(k)^\bullet$ corresponds to the surjectivity on the level of morphism complexes.

Hence it suffices to prove that $H \in \{I\}$-inj. To do so we consider the diagram

$$\begin{tikzcd} k \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{C}_1 \arrow[d, "H"] \\ \mathcal{K} \arrow[r, "G"'] & \mathcal{C}_2. \end{tikzcd} \tag{B.40}$$

As there is only one object in k, this commutative square corresponds to the choice of an object $F(3)$ in $\mathcal{C}_1$, a morphism $G(1) \to G(2)$ in $\mathcal{C}_2$ such that $F(3)$ is mapped to $G(1)$, and a contraction h of the $\mathrm{cone}(G(f)^*)$ in $\mathcal{C}_2$-dg Mod$_k$. By assumption we have that H is surjective on the level of objects, hence we can choose $C \in \mathrm{Obj}(\mathcal{C}_1)$ mapping to $G(2)$. The dg functor H is also a surjective quasi-isomorphism on the level of the morphism complexes, hence we can lift $G(f)$ to a morphism $\overline{G(f)}\colon F(3) \to C$. This corresponds to the data

(B.41)
$$\begin{tikzcd} F(3) \arrow[d, mapsto, "H"] & C \arrow[d, mapsto, "H"'] \\ G(1) \arrow[r, "G(f)"'] & G(2) \end{tikzcd}$$

Now we embed this diagram in the dg categories $\mathcal{C}_i$-dg Mod$_k$ using the Yoneda embedding. This yields

(B.42)
$$\begin{tikzcd} \mathrm{h}^{F(3)} \arrow[r, "\overline{G(f)}^*"] \arrow[d, mapsto, "H"'] & \mathrm{h}^{C} \arrow[r] \arrow[d, mapsto, "H"'] & \mathrm{cone}\left(\overline{G(f)}^*\right) \arrow[d, mapsto, "H"'] \\ \mathrm{h}^{G(1)} \arrow[r, "G(f)^*"] & \mathrm{h}^{G(2)} \arrow[r] & \mathrm{cone}\left(G(f)^*\right) \arrow[loop right, "h"] \end{tikzcd}$$

where h is the contraction from before. The fact that H induces a surjective quasi-isomorphism lifts to the representable objects in $\mathcal{C}_i$-dg Mod$_k$, and as we are considering mapping cones over representables (which are again representable, as in lemma B.2) we can lift the contraction h to a contraction $\overline{h}$ of $\mathrm{cone}(\overline{G(f)}^*)$ by applying [MSM63, lemma 2.3.5] to the pair $(h, \mathrm{id}_{\mathrm{cone}(G(f)^*)})$.

We now prove the inclusion $\subseteq$. Let $H\colon \mathcal{C}_1 \to \mathcal{C}_2$ be a functor in J_{dg}-inj $\cap\, \mathcal{W}$. Because it is contained in $\{R(n) \mid n \in \mathbb{Z}\}$-inj it is surjective on the level of the morphism complexes.

To prove that it is also surjective on the level of the objects, let C_2 be an object in $\mathcal{C}_2$. Because H is a quasi-equivalence it is quasi-surjective, hence there exists an object C_1 in $\mathcal{C}_1$ and a morphism $q \in \mathrm{Hom}_{\mathcal{C}_1}(H(C_1), C_2)^{\bullet}$ that becomes an isomorphism after taking the homotopy category $\mathrm{H}^0(\mathcal{C}_2)$. Hence we have the diagram

(B.43)
$$\begin{tikzcd} C_1 \arrow[d, mapsto, "H"'] & \\ H(C_1) \arrow[r, "q"'] & C_2. \end{tikzcd}$$

We can obtain q as the image of the morphism f in $\mathcal{K}$ by a dg functor $G\colon \mathcal{K} \to \mathcal{C}_2$. The assumption $H \in \{I\}$-inj implies that we can lift G to a functor $\mathcal{K} \to \mathcal{C}_1$. Hence we can find a lift of q to $\mathcal{C}_1$, which leads to a lift of the object C_2 to $\mathcal{C}_1$, as desired. □

Lemma B.18. We have the inclusion

(B.44) I_{dg}-inj $\subseteq \mathcal{W} \cap \mathrm{J}_{\mathrm{dg}}$-inj.

Proof. This follows immediately from lemmas B.16 and B.17. □

Lemma B.19. We have that either

(B.45) $\mathcal{W} \cap \mathrm{I}_{\mathrm{dg}}$-cof $\subseteq \mathrm{J}_{\mathrm{dg}}$-cof

or

(B.46) $\mathcal{W} \cap \mathrm{J}_{\mathrm{dg}}$-inj $\subseteq \mathrm{I}_{\mathrm{dg}}$-inj.

Proof. The second inclusion follows immediately from lemmas B.16 and B.17. □

We have done all the work to obtain the main result of this appendix.

Theorem B.20 (Model category structure on $\mathrm{dg\,Cat}_k$)**.** If we take I_{dg} (resp. J_{dg}) as generating cofibrations (resp. generating acyclic cofibrations) and quasi-equivalences as weak equivalences we have a cofibrantly generated model category structure on $\mathrm{dg\,Cat}_k$.

Proof. The conditions for [MSM63, theorem 2.1.19] are satisfied using lemmas B.8 to B.10, B.14, B.18 and B.19. □

Corollary B.21. The model category structure generated by our choices of I_{dg} and J_{dg} in theorem B.20 is the same as the model category structure described in §3.1.

Proof. The fibrations in a cofibrantly generated model category correspond to J_{dg}-inj [MSM63, definition 2.1.17]. By the description of $\{R(n) \mid n \in \mathbb{Z}\}$-inj in lemma B.17 we obtain its equivalence with condition 3.3(i).

Now let v be an isomorphism in $\mathrm{H}^0(\mathcal{D})$ as in 3.3(ii). Consider the mapping cone of v^* in $\mathcal{D}$-$\mathrm{dg\,Mod}_k$, which we will denote $\mathrm{cone}(v^*)$. As v lives in the H^0 of a morphism complex we obtain a contraction of $\mathrm{cone}(v^*)$, hence by lemma B.2 we obtain a dg functor $G\colon \mathcal{K} \to \mathcal{D}$ such that $G(f) = v$. So if a dg functor $\mathcal{C} \to \mathcal{D}$ is contained in $\{I\}$-inj we obtain condition 3.3(ii).

For the other implication we apply the ideas of the proof of the first inclusion in lemma B.17. Condition 3.3(ii) yields a lifting $\overline{G(f)}$ as in (B.41) and it implies that the endomorphism dg algebra of $\mathrm{cone}(\overline{G(f)}^*)$ is acyclic because we have an isomorphism in the homotopy category. This yields that H^* induces a surjective quasi-isomorphism

$$\mathrm{End}_{\mathcal{C}_1\text{-}\mathrm{dg\,Mod}_k}\left(\mathrm{cone}\left(\overline{G(f)}^*\right)\right) \to \mathrm{End}_{\mathcal{C}_2\text{-}\mathrm{dg\,Mod}_k}\left(\mathrm{cone}\left(G(f)^*\right)\right) \tag{B.47}$$

hence we can lift with respect to I. □

Remark B.22. There exist at least two other (non-trivial) model category structures on the category $\mathrm{dg\,Cat}_k$. One in which the *quasi-equiconical* morphisms are inverted, and another in which the *Morita morphisms* are inverted [Tab07a; Tab05b]. Without going into details, quasi-equiconical morphisms are dg functors which are quasi-fully faithful and which are essentially surjective on the level of the homotopy categories of their pretriangulated hulls. This generalises the idea of quasi-essentially surjective, where we asked essential surjectivity on the level of the homotopy categories themselves.

The Morita model category structure is obtained by applying an idempotent completion [BS01] to the quasi-equiconical morphisms. This explains the lack of references to the quasi-equiconical morphisms in the literature (none have been found), these serve only as an intermediate step in the construction of the far more important Morita model category structure.

We have the following inclusion of classes of weak equivalences

$$\{\text{quasi-equivalences}\} \subseteq \{\text{quasi-equiconical morphisms}\} \subseteq \{\text{Morita morphisms}\}. \tag{B.48}$$

The existence of an internal Hom in the Morita model category structure is less painful to obtain, as it does not depend on an ad-hoc construction (but it remains non-trivial).

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