Exceptional collections and the geometry of partial flag varieties

Sommersemester 2020

The goal of this seminar is to understand what exceptional collections are in algebra and algebraic geometry, with the focus on

1. the general theory of exceptional collections for finite-dimensional algebras and highest weight categories;
2. exceptional collections in derived categories of coherent sheaves on **partial flag varieties** G/P.

The first half of the seminar is dedicated to the general theory and getting familiar with the conepts and classical examples, whilst the second half will look in detail at current developments and open problems regarding the construction of exceptional collections for generalised Grassmannians.

Exceptional collections were introduced in [3, 16], and we can interpret them according to the following rough dictionary:

category theory	$\to$	linear algebra
$\mathcal{T}$		finite-dimensional vector space $\mathrm{K}_0(\mathcal{T})$)
$\mathrm{Hom}_{\mathcal{T}}(-,-)$		bilinear Euler form $\sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{T}}(-,-[i])$
exceptional collection		semiorthogonal basis

The first example of a (full) exceptional collection is given in [2], where the derived category of $\mathbb{P}^n$ was described via an explicit finite-dimensional algebra. Nowadays, there exist many examples of varieties with full exceptional collections. Ever since the construction of exceptional collections on $\mathbb{P}^n$, and $\mathrm{Gr}(k,n)$ resp. a smooth quadric Q^n [16], the following conjecture has been an important motivation in the study of exceptional collections and the geometry of partial flag varieties (also known as homogeneous spaces):

Conjecture. Let G be a semisimple algebraic group, and P a parabolic subgroup. Then there exists a full exceptional collection in the derived category $\mathbf{D}^{\mathrm{b}}(G/P)$.

For this conjecture it in fact suffices to consider the case where P is a *maximal* parabolic subgroup, so that we can call G/P a *generalised Grassmannian.*

To study the above objects, we can conveniently use the representation theory of the algebraic groups G and P.

1 Exceptional collections

The first goal is to get familiar with the notion of exceptional collections, in the easiest possible setting: that of finite-dimensional algebras. We also discuss perverse sheaves on Grassmannians as an alternative source of exceptional collections with a representation-theoretic flavour, before we move on to coherent sheaves on varieties.

April 9: Derived categories and exceptional collections for finite-dimensional algebras I (Timm Peerenboom and Ismaele Vanni)

1. Introduce derived categories of abelian categories using [15, §1, §2]. For a detailed introduction, see [20], [21]. The examples of interest in this lecture are module categories, but later in the seminar we will consider categories of coherent sheaves, hence the need to work with general abelian categories.

2. Define exceptional collections (i.e. exceptional objects, full collections, strong collections), using [12, §2], and [10, 22].
3. Discuss finite-dimensional algebras of finite global dimension: quivers with relations, exceptional objects from injectives, projectives and simples using [10, 22].

Additional references: [9, §1], [12]

April 16: Derived categories and exceptional collections for finite-dimensional algebras II (Ismaele Vanni and Zbigniew Wojciechowski)

1. Mutation of exceptional collections [12, §1.4, §2].
2. Euler forms: introduce them, discuss upper-triangularity in the presence of an exceptional collection, give examples (e.g. kQ for Q a Dynkin quiver, the Beilinson quiver for $\mathbb{P}^2$, ...) [9, §2.3].
3. Discuss what having a full exceptional collection implies for the Grothendieck group, Hochschild homology, ... (and mention that these always hold for finite-dimensional algebras of finite global dimension, without a full exceptional collection), [9, theorem 1.12] (acyclicity doesn't matter here!)

References: [9, §1], [10] and references therein
Interesting examples to discuss:

1. No exceptional objects at all: Fibonacci algebras [13]
2. No Jordan–Hölder: [18] and `https://pbelmans.ncag.info/blog/2017/04/08/bondals-example/`, `https://pbelmans.ncag.info/blog/2017/04/09/bondals-example-2/`

April 23: Quasi-hereditary algebras and (ϵ)-highest weight categories (Jonas Mehme)

1. Introduce the 6 functor formalism and recollements for abelian and triangulated categories.
2. Define (ϵ)-highest weight categories, and discuss examples coming from quivers with relations.
3. Discuss the characterisation of highest weight categories from [5].

References: [17], [23], [5]

April 30: Ringel duality and tilting modules (Amine Koubaa)

1. Define tilting modules, and discuss the classification of indecomposable tilting modules, with a sketch of the proof.
2. Show that Ringel duality for highest weight categories gives a highest weight category, and describe the functors $F = \mathrm{Hom}_A(T, -)$ and $G = (-)^* \circ \mathrm{Hom}_A(-, T)$ for a finite-dimensional algebra A on important classes of objects.
3. Explain that Ringel duality is a duality.
4. Discuss Happel's theorem, which shows that $\mathbf{R}F$ and $\mathbf{L}G$ are equivalences of categories.

References: [17], [14], [5]

2 Exceptional collections in algebraic geometry

May 7: Derived categories of coherent sheaves (Patrick Seifner)

1. Discuss derived categories of coherent sheaves, derived functors (pushfoward, pullback, tensor product, ...) and the framework of Fourier–Mukai transforms (which are just bimodules in disguise). [15, §3] and [8] are good starting points.

   State Serre duality. Compare with Serre duality for finite dimensional algebras in terms of tensoring with bimodules; see [14, §4.6] in terms of the Nakayama functor, and [4].
2. Explain how varieties with a tilting object are special by comparing them e.g. to the derived category of a curve of higher genus. One possible starting point is [6].

May 14: Exceptional objects on varieties and Beilinson's collection for projective space (Mingyu Ni)

1. Construct Beilinson's collections for projective space using [8, §3] and give the description of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$ in terms of a finite-dimensional algebra.
2. Mention other examples of varieties with a full exceptional collection: del Pezzo surfaces, toric varieties (link to combinatorics), ...
3. Discuss what it means for a vector bundle to be exceptional, and what needs to change if a non-locally free sheaf is exceptional. Discuss the example of the blowup of a point on a surface.

May 20 (!): The geometry of partial flag varieties (Liao Wang)

Using the notation from [19, §2] and the exposition from [1, §2, §3, §4 and §5]:

1. Discuss partial flag varieties.
2. Discuss equivariant vector bundles and explain the difference between $\operatorname{rep} P$ and $\operatorname{rep} L$.
3. State the Borel–Weil–Bott theorem.
4. Discuss examples:
   (a) show ranks of equivariant vector bundles and their cohomology for all rank 1 and 2 root systems
   (b) explain Serre duality on $\mathbb{P}^1$ and $\mathbb{P}^2$ by checking it for line bundles

May 28: Exceptional collections on quadrics and Grassmannians (Till Werhan)

Discuss the seminal paper [16].

1. Construct Kapranov's exceptional collection for Grassmannians.
2. Discuss spinor bundles, and describe and construct Kapranov's exceptional collection on quadrics.

If time permits, one can discuss other constructions of exceptional collections for Grassmannians, such as [7].

June 18: Exceptional collections on generalised Grassmannians (Pieter Belmans)

To be specified later, but suggestions are:

1. Give an overview of the state-of-the-art on the construction of exceptional collections.
2. Explain the complexities (with a view towards [19]) of using not necessarily irreducible equivariant vector bundles.
3. Discuss Lefschetz collections, and the state-of-the-art of their construction.

June 25: Exceptional collections on generalised Grassmannians II (Matt Young)

3 A conjectural description

July 2: Kuznetsov–Polishchuk I (Catharina Stroppel)

This talk, and the next, give a discussion of the construction in [19].

1. Briefly discuss the vanishing conjecture and phantom subcategories.
2. Explain how sometimes $\operatorname{rep} L$ provides enough equivariant vector bundles, and sometimes one needs $\operatorname{rep} L$ [19, page 511].
3. Introduce exceptional blocks [19, §3].
4. If time permits, discuss the link with strongness, purity and Koszulity [19, §4].

July 9: Kuznetsov–Polishchuk II (Joanna Meinel)
Continuing where the previous talk left off:

1. Explain how to construct exceptional blocks [19, §5].
2. Describe how to construct these in a classical type of your choice [19, §9], only discussing the technical details from §6 and §7 where needed.
3. Mention the results from [11], where fullness of the Kuznetsov–Polishchuk collection is shown (together with an explicit description of the objects).
4. If time permits, outline what should be done to construct exceptional blocks in the exceptional Dynkin types (see question 1.11, remark 5.9, and the construction in §9 in the classical types).

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