Generic non-semisimplicity of small quantum cohomology of Kronecker moduli

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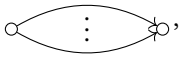
This is the standalone version of Appendix A in (a future version of) Sergey Galkin, Naichung Conan Leung, Changzheng Li, and Rui Xiong: *A-D-E diagrams, Hodge–Tate hyperplane sections and semisimple quantum cohomology* [6].

Abstract

In this short note we explain how the semisimplicity obstruction by Galkin–Leung–Li–Xiong shows that certain Kronecker quiver moduli have non-semisimple small quantum cohomology.

Quiver moduli are moduli spaces of (semi)stable representations of a quiver, and they admit many strong results, see [10] for a survey. We will focus on the case where the quiver is the m-Kronecker quiver with m arrows:

(1)  ,

where we will throughout assume that $m \geq 3$. We will write (d, e) for the dimension vector, where $d, e \geq 1$. We will assume that $\gcd(d, e) = 1$. As there are only 2 vertices, there is a unique non-trivial stability function, for which stability coincides with semistability. We denote the moduli space of (semi)stable representations of the m-Kronecker quiver with dimension vector (d, e) by $\mathrm{M}^m_{(d,e)}$.

The important properties for us are that

1. $\mathrm{M}^m_{(1,e)} \cong \mathrm{Gr}(e, m)$ and $\mathrm{M}^m_{(d,1)} \cong \mathrm{Gr}(d, m)$;
2. $\mathrm{M}^m_{(d,e)}$ is Hodge–Tate [8, Theorem 3];
3. $\mathrm{M}^m_{(d,e)}$ is a smooth projective rational Fano variety of dimension $mde - d^2 - e^2 + 1$, Picard rank 1, and index m, see [5, Corollary 5.2].

In addition to generic semisimplicity of small quantum cohomology of Kronecker moduli which happen to be Grassmannians, we have the following important result [9, Theorem 5.23], which follows from an explicit presentation of the small quantum cohomology.

Proposition 1 (Meng)**.** *The small quantum cohomology* $\mathrm{QH}(\mathrm{M}^3_{(2,3)})$ *is generically semisimple.*

Given that a version of Schofield's conjecture predicts that the derived category of a quiver moduli space (and thus $\mathrm{M}^m_{(d,e)}$ in particular) admits a full exceptional collection [2], and Dubrovin's conjecture then predicts that its big quantum cohomology is generically semisimple, one might (optimistically) wonder whether the small quantum cohomology is already semisimple, as is the case in all known examples. However, the following shows that this is *not* the case.

Theorem 2. *Assume that* $m \geq 5$ *is odd. Then the small quantum cohomology* $\mathrm{QH}(\mathrm{M}^m_{(2,m)})$ *is not generically semisimple.*

We will obtain this as a consequence of [6, Theorem 1.2], and thus we can conclude that this result is also strong enough to show that certain Kronecker moduli spaces have generically non-semisimple small quantum cohomology, similar to the application in [6, Theorem 2.3] for exceptional generalized Grassmannians.

Two lemmas Given a polynomial with integer coefficients

$$P(q) = \sum_{i=0}^{N} a_i q^i \in \mathbb{Z}[q] \tag{2}$$

and an integer $n \geq 1$, we define

$$\tilde{b}_j := \sum_{k \equiv j \bmod n} a_k \tag{3}$$

for $j \in \mathbb{Z}/n\mathbb{Z}$. Later on, the polynomial will be the even Poincaré polynomial of a smooth projective variety, and the $\tilde{b}_j$ will be the index-periodic even Betti numbers.

Lemma 3. *Assume that $\tilde{b}_j \leq \tilde{b}_{jk}$ for all $j, k \in \mathbb{Z}/n\mathbb{Z}$. Then $P(\mathrm{e}^{2\pi \mathrm{i}/n}) = \sum_{d|n} \mu(n/d)\tilde{b}_d$, in particular, it is an integer.*

Proof. By assumption we have that $\tilde{b}_j = \tilde{b}_{jk}$ if $k \in (\mathbb{Z}/n\mathbb{Z})^\times$. The orbits of $(\mathbb{Z}/n\mathbb{Z})^\times$ on $\mathbb{Z}/n\mathbb{Z}$ correspond to the divisors of n, thus we can write

$$P(\mathrm{e}^{2\pi \mathrm{i}/n}) = \sum_{j=1}^{n} \tilde{b}_j \mathrm{e}^{2\pi \mathrm{i} j/n} = \sum_{d|n} \tilde{b}_d \sum_{(j,n)=d} \mathrm{e}^{2\pi \mathrm{i} j/n} = \sum_{d|n} \tilde{b}_d \sum_{(j,n/d)=1} \mathrm{e}^{2\pi \mathrm{i} j/(n/d)} = \sum_{d|n} \mu(n/d)\tilde{b}_d \tag{4}$$

where the last step uses standard properties of Ramanujam sums [7, Theorem 271], and the latter expression is indeed an integer. □

We also recall the q-Lucas theorem on evaluations of Gaussian binomial coefficients $\begin{bmatrix} a \\ b \end{bmatrix}_q$ at roots of unity [4, Proposition 2.2].

Lemma 4. *For $a, b \in \mathbb{N}$ and $0 \leq r, s < n$, we have*

$$\begin{bmatrix} an + r \\ bn + s \end{bmatrix}_{\mathrm{e}^{2\pi \mathrm{i}/n}} = \binom{a}{b} \cdot \begin{bmatrix} r \\ s \end{bmatrix}_{\mathrm{e}^{2\pi \mathrm{i}/n}}. \tag{5}$$

Poincaré polynomials of Kronecker moduli Now we consider the Kronecker moduli spaces $\mathrm{M}^m_{(2,e)}$ for e odd and $m, e \geq 3$. There is a duality

$$\mathrm{M}^m_{(2,e)} \cong \mathrm{M}^m_{(2,2m-e)}, \tag{6}$$

(with both being empty when $e > 2m$) see, e.g., [1, Corollary 4.1], thus we can assume $e \leq m$.

Their Poincaré polynomials admit a closed expression using a resolved Harder-Narasimhan type recursion [11, Section 7].

Lemma 5. *Let $e, m \geq 3$, and assume that $e \leq m$, and e is odd. The Poincaré polynomial $\mathrm{P}^m_{(2,e)}(q)$ of $\mathrm{M}^m_{(2,e)}$ is given by*

$$\mathrm{P}^m_{(2,e)}(q) = \frac{1}{q(q-1)} \left(\frac{1}{q+1} \begin{bmatrix} 2m \\ e \end{bmatrix}_q - \sum_{k=0}^{(e-1)/2} q^{(m-e+k)k} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} m \\ e-k \end{bmatrix}_q \right). \tag{7}$$

We can now derive the main observation.

Lemma 6. *Let $e, m \geq 3$, and assume that $e \leq m$, and e is odd. Then $\mathrm{P}^m_{(2,e)}(\mathrm{e}^{2\pi\mathrm{i}/m})$ is strictly imaginary if and only if $e = m \geq 5$.*

Proof. If $e < m$, then all Gaussian binomial coefficients in Lemma 5 evaluate to zero at $q = \mathrm{e}^{2\pi\mathrm{i}/m}$ by Lemma 4, because we always have $r = 0$ and $s = e \geq 1$, and thus $\mathrm{P}^m_{(2,e)}(\mathrm{e}^{2\pi\mathrm{i}/m}) = 0$.

If $e = m$, we find

$$\begin{aligned} \mathrm{P}^m_{(2,m)}(\mathrm{e}^{2\pi\mathrm{i}/m}) &= \frac{1}{\mathrm{e}^{2\pi\mathrm{i}/m}(\mathrm{e}^{2\pi\mathrm{i}/m}-1)}\left(\frac{1}{\mathrm{e}^{2\pi\mathrm{i}/m}+1}\cdot 2 - 1\right) \\ &= -\frac{1}{\mathrm{e}^{2\pi\mathrm{i}/m}(\mathrm{e}^{2\pi\mathrm{i}/m}+1)}, \end{aligned} \tag{8}$$

as for $k = 0$ in the summation in Lemma 5 we have $\begin{bmatrix} m \\ 0 \end{bmatrix}_{\mathrm{e}^{2\pi\mathrm{i}/m}} = \begin{bmatrix} m \\ m \end{bmatrix}_{\mathrm{e}^{2\pi\mathrm{i}/m}} = 1$ by definition, whilst all other Gaussian binomial coefficients in the sum evaluate to zero at $q = \mathrm{e}^{2\pi\mathrm{i}/m}$ by Lemma 4 because for $k \geq 1$ we will have $r = 0$ and $s \geq 1$, whereas $\begin{bmatrix} 2m \\ m \end{bmatrix}_{\mathrm{e}^{2\pi\mathrm{i}/m}} = \binom{2}{1}\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathrm{e}^{2\pi\mathrm{i}/m}} = 2$ by another application of Lemma 4. This is strictly imaginary for $m \geq 5$. □

Proof of Theorem 2. By Lemma 6 we have that $\mathrm{P}^m_{(2,m)}(\mathrm{e}^{2\pi\mathrm{i}/m})$ is strictly imaginary (and thus nonzero). By Lemma 3, this invalidates the criterion of [6, Theorem 1.2], and thus the small quantum cohomology of $\mathrm{M}^m_{(2,m)}$ is not generically semisimple. □

Let us illustrate what happens in the smallest example, where $m = 5$.

Example 7. The Kronecker moduli space $\mathrm{M}^5_{(2,5)}$ is a 22-dimensional smooth projective Fano variety, of index 5. Using the algorithm for Betti numbers of [11, Corollary 6.9], and implemented in [3], or by working out the special case from Lemma 5, we obtain that the even Betti numbers are

$$1, 1, 3, 4, 8, 11, 17, 22, 30, 35, 41, 41, 41, 35, 30, 22, 17, 11, 8, 4, 3, 1, 1. \tag{9}$$

Hence, the index-periodic even Betti numbers $\tilde{\mathrm{b}}_0, \ldots, \tilde{\mathrm{b}}_4$ are

$$78, 77, 78, 77, 77. \tag{10}$$

We see that $\tilde{\mathrm{b}}_2 > \tilde{\mathrm{b}}_4$, hence by [6, Theorem 1.2] we obtain that $\mathrm{QH}(\mathrm{M}^5_{(2,5)})$ cannot be generically semisimple.

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