The torsor for a formally smooth morphism

Pieter Belmans

March 25, 2015

Abstract

The goal is to outline some ideas in the construction [EGA IV_4, §16.5.14] and the proof of [EGA IV_4, proposition 16.5.7]. Before doing so we also present some properties of formally smooth, formally unramified and formally étale morphisms.

We also explain how the finite presentation in [EGA IV_4, corollaire 16.5.8] induces a mistake in the proof of [EGA IV_4, proposition 17.1.6] and how we can fix it, which is something we learnt from MO10741.

Contents

---

commit: `b52050f` date: `2015-02-03 11:34:11 +0100` `(HEAD, master)`

1 Introduction

Recall that the following definitions [EGA IV_4, définition 17.1.1].

Definition 1. Let $f: X \to Y$ be a morphism of schemes. We say that it is *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if

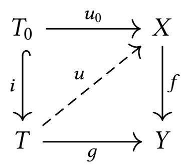
1. for every affine scheme T;
2. for every closed subscheme $i: T_0 \hookrightarrow T$ defined by a nilpotent ideal $\mathcal{J}$ of $\mathcal{O}_T$ (i.e. a nilpotent ideal I of A such that $T = \operatorname{Spec} A$);
3. for every morphism $g: T \to Y$;

the morphism

$$\operatorname{Hom}_Y(T,X) \to \operatorname{Hom}_Y(T_0,X) \tag{1}$$

obtained from $T_0 \to T$ is surjective (resp. injective, resp. bijective).

In other words, there exists (resp. there exists at most one, resp. there exists exactly one) lifting in the diagram

(2)


for a morphism $u_0: T_0 \to X$ making it commutative.

Remark 2. In section 4 it is explained that these conditions are local on the source or target, in particular one can check things on affine open covers of X or Y. On the other hand, in the definition we require T be affine.

The goal of this note is to:

1. explain why we get, in complete generality, a *torsor under* $\mathcal{H}om_{\mathcal{O}_{T_0}}(u_0^*(\Omega^1_{X/Y}),\mathcal{J})$ for the set of liftings (as a sheaf on T), if we drop the condition that T be affine in (2);
2. explain what the role of T *being affine* is in the definition of formal smoothness (resp. formal unramifiedness, resp. formal étaleness);
3. explain why we can check things on an *affine open cover* of X and Y.

In section 2 we explain when this condition can be dropped, when it can't, and what it means to and why we get a "(pseudo)torsor under $\mathcal{H}om_{\mathcal{O}_{T_0}}(u_0^*(\Omega^1_{X/Y}),\mathcal{J})$". From now on we will denote this sheaf by $\mathcal{G}$.

Remark 3. One can see that the condition that $\mathcal{J}$ be nilpotent can be replaced by it being square-zero, by taking a chain of square-zero closed immersions, as in [EGA IV_4, remarques 17.1.2(ii)].

2 Torsors and all that

We now redo the discussion of [EGA IV_4, §16.5.14]. We use the same notation as in (2), but now let T be *any* scheme, not just an affine scheme, and we don't put any assumptions on f. This is because the main result of this section doesn't care about formal smoothness (which is only introduced in [EGA IV_4, §17] anyway).

Remark 4. I.e. we are given the morphisms $f: X \to Y$, $g: T \to Y$, a closed subscheme $i: T_0 \hookrightarrow T$ given by a square-zero ideal ideal $\mathcal{J}$ of $\mathcal{O}_T$ and a morphism $u_0: T_0 \to X$ making the square (2) commutative. We then ask ourselves whether there exists a lift $u: T \to X$.

2.1 Definition

Let's recall the definition of a (pseudo)torsor, because if you are like me you have never actually used it up to now.

Definition 5. Let X be a topological space. Let $\mathcal{G}$ be a sheaf of groups[1] on X. A *pseudotorsor under* $\mathcal{G}$ is a sheaf of sets $\mathcal{F}$ on X together with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that for an $U \subseteq X$ open we have

1. either $\mathcal{F}(U) = \emptyset$;
2. or $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ transitive.

If there exists a covering $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}(U_i) \neq \emptyset$ for all $i \in I$ then $\mathcal{F}$ is called a *torsor under* $\mathcal{G}$.

The best intuition for a torsor that I've seen is the following quote, from John Baez:

> A torsor is like a group that has forgotten its identity.

So for any $\mathcal{G}$ we always get the trivial torsor, which is $\mathcal{G}$ itself.

2.2 The sheaf of sets

Now that we know what a (pseudo)torsor is, we can introduce the sheaf of sets that we will use. Observe that $|T| = |T_0|$, the closed immersion defined by $\mathcal{J}$ only changes the structure sheaf and not the underlying topological spaces.

Definition 6. Let $\mathcal{P}$ be the sheaf of sets on T defined by

$$\Gamma(U, \mathcal{P}) := \left\{ u: U \to X \mid u_0|_{U_0} = u \circ (j|_{U_0}), U_0 := i^{-1}(U) \right\} \tag{3}$$

for any open U of T.

That it is a sheaf follows from the fact that morphisms of schemes are defined locally. It is also a sheaf on T_0, as the underlying topological spaces agree.

The sheaf of groups under which it will be a torsor is

$$\mathcal{G} := \mathcal{H}om_{\mathcal{O}_{T_0}}(u_0^*(\Omega^1_{X/Y}), \mathcal{J}). \tag{4}$$

[1] Not necessarily commutative, but in our case it will be.

This is a $\mathcal{O}_{T_0}$-module.

We are trying to understand *when* there is a morphism $u\colon T \to X$ such that $u_0 = u \circ i$, in other words we wish to understand [EGA IV$_4$, proposition 16.5.17], whose statement is repeated in proposition 9. So we have to construct the desired action of $\mathcal{G}$ on $\mathcal{P}$. We do this affine-locally.

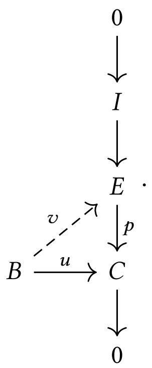
2.3 The affine situation

We first give (and prove) the affine situation, it corresponds to [EGA IV$_1$, corollary 0_{IV}.20.1.3]. The proof is a straight-forward check, but it's not something that I can check in all gory details on the spot so I decided to write it down. For the definition of $\mathrm{Der}_A(B,I)$ one is referred to loc. cit.

Lemma 7. Let A be a ring. Let $p\colon E \to C$ and $u\colon B \to C$ be morphisms of A-algebras, such that $I := \ker(p)$ has $I^2 = 0$. Then the set of morphisms $v\colon B \to E$ of A-algebras such that $u = p \circ v$ is either empty or a torsor for $\mathrm{Der}_A(B,I)$.

Proof. I.e. the situation asks for liftings as in the diagram

(5)



We have to prove that if we take two liftings v_1 and v_2 their difference is an A-derivation of B into I, and vice versa that if we take a lifting v and a derivation Θ their sum is another lifting in the diagram.

1. Let v_1 and v_2 be two liftings for (5). Then we get $\theta := v_1 - v_2\colon B \to E$. But these maps are liftings of u, hence they agree when composed with p, so θ is actually a map $B \to I$ (considered as A-modules). We just have to check that it is a A-derivation of B into I. Everything except the Leibniz rule is immediate. Observe that for any $b, b' \in B$

   (6) $$v_1(bb') - v_1(b)v_1(b') = v_2(bb') - v_2(b)v_2(b')$$

   as the v_i are originally morphisms of A-algebras and both sides equal 0. Rearranging we get

   (7) $$v_1(bb') - v_2(bb') = v_1(b)v_1(b') - v_2(b)v_2(b')$$

where the left-hand side equals $\Theta(bb')$. To show that the right-hand side equals $b\Theta(b')+b'\Theta(b)$, observe that we can rewrite

$$\begin{aligned} &b\Theta(b') + b'\Theta(b) \\ &\quad = b(v_1(b') - v_2(b')) + b'(v_1(b) - v_2(b')) \\ &\quad = v_1(b)\,(v_1(b') - v_2(b')) + v_2(b')\,(v_1(b) - v_2(b)) \\ &\quad = v_1(b)v_1(b') - v_2(b)v_2(b') \end{aligned} \tag{8}$$

where we have used that E has two B-module structures via v_1 and v_2 that agree for elements of I.

2. Conversely, let v_1 be a lifting for (5) and Θ an A-derivation of B into I. Then we get $v_2 := v_1 + \Theta\colon B \to E$. Everything except multiplicativity is immediate (use that Θ sends the image of A under the structure map $A \to B$ to zero). Observe that for $b, b' \in B$ we have

(9)

$$\begin{aligned} &v_2(bb') - v_2(b)v_2(b') \\ &\quad = v_1(bb') + \Theta(bb') - (v_1(b) + \Theta(b))(v_1(b') + \Theta(b')) \\ &\quad = v_1(bb) - v_1(b)v_1(b') + \Theta(bb') - v_1(b)\Theta(b') - v_1(b')\Theta(b) - \Theta(b)\Theta(b') \\ &\quad = 0 \end{aligned}$$

proving that the map v_2 is multiplicative.

One also has to check that everything restricts nicely, which is done in [EGA IV_1, §0_{IV}.20.5] (see also remark 8). □

2.4 The global situation

Now consider the commutative diagram (2), with assumptions as in remark 4. Consider an affine open $U = \operatorname{Spec} C$ of T, and denote $U_0 := i^{-1}(U) = \operatorname{Spec} C/I$ which is again affine, where $I = \Gamma(U, \mathcal{J})$ is a square-zero ideal of C.

Assume that U is sufficiently small such that

1. $u_0(U_0) \subseteq V = \operatorname{Spec} B \subseteq X$;
2. $g(U) = f(u_0(U_0)) \subseteq W = \operatorname{Spec} A \subseteq Y$.

This reduces the global situation to the situation discussed in section 2.3. By lemma 7 we know that $\mathcal{P}(U_0)$ is a torsor for $\operatorname{Der}_A(B, I)$.

We have to check that this action is independent of our choices, but as indicated in the proof of lemma 7 we are good.

Remark 8. If we denote ψ the map $B \to C/I$ obtained from $u_0|U_0\colon U_0 \to V$, then I comes equipped with the structure of a B-module. By the universal property of $\Omega^1_{B/A}$ this induces an isomorphism

$$\operatorname{Hom}_B(\Omega^1_{B/A}, I) \to \operatorname{Der}_A(B, I) \tag{10}$$

by precomposing $v\colon \Omega^1_{B/A} \to I$ with $\mathrm{d}_{B/A}$. This is either well-known, or you can read [EGA IV_1, theorem 0_{IV}.20.4.8.2(ii)].

Moreover, as I is square-zero, hence it comes also equipped with the structure of a C/I-module, every B-morphism $v\colon \Omega^1_{B/A} \to I$ can be considered as a C/I-morphism $\Omega^1_{B/A} \otimes_B (C/I) \to I$. But this globalises (as $\mathcal{J}$ is square-zero), and we have that the sheaf $\mathcal{G}$ as introduced in (2) can be considered as a quasicoherent $\mathcal{O}_{Y_0}$-module, such that

$$\mathrm{Der}_A(B,I) \cong \Gamma(U_0, \mathcal{G}). \tag{11}$$

The previous discussion proves the following result.

Proposition 9. Let $f: X \to Y$ and $g\colon T \to Y$ be morphisms. Let $i\colon T_0 \hookrightarrow T$ be a closed subscheme defined by a quasicoherent ideal sheaf $\mathcal{J}$ such that $\mathcal{J}^2 = 0$.

Let $u_0\colon T_0 \to X$ be any morphism making (2) commute. Then there exists on $\mathcal{P}$ the structure of a pseudotorsor under the $\mathcal{O}_{T_0}$-module $\mathcal{G} \coloneqq \mathcal{H}om_{\mathcal{O}_{T_0}}(u_0^*(\Omega^1_{X/Y}), \mathcal{J})$.

3 Torsors in the formally smooth case

We can now tie together the discussion on (pseudo)torsors and the notion of formally smooth morphisms. Recall that for a sheaf of groups as in the definition of a (pseudo)torsor, the set of isomorphism classes of torsors under $\mathcal{G}$ is given by $\mathrm{H}^1(X, \mathcal{G})$, whose correspondence can be found in e.g. [EGA IV_4, §16.5.15].

Remark 10. One concludes that, in the situation of remark 4, and moreover $f: X \to Y$ formally smooth, one has that the (pseudo)torsor obtained in proposition 9 is actually a torsor, because we take for the open cover any affine open cover of T.

I.e. we have that there are always local liftings on T, but they don't necessarily glue together to a global lifting $T \to X$.

Remark 11. Observe that the notion of torsors for formally smooth morphisms is important for the *infinitesimal lifting criterion*, i.e. the purely functorial description of smooth morphisms as morphisms which are formally smooth and locally of finite presentation. Its proof, one uses that torsors on X under some group $\mathcal{G}$ are classified by $\mathrm{H}^1(X, \mathcal{G})$. One then wishes to show that under our conditions the $\mathcal{G}$ of our choice is a quasi-coherent module, so it doesn't have cohomology on affine schemes.

Remark 12. As explained after the definition of formal smoothness in [Stacks, tag 02GZ] we *can* drop the condition that T be affine in the definition of formal étaleness and formal unramifiedness, as done in [Stacks, tags 04F1, 04FD].

4 Being formally smooth *is* local on the source or target

Recall [EGA IV$_4$, proposition 17.1.6], which goes as follows.

Proposition 13. Let $f: X \to Y$ be a morphism of schemes. Then

1. Let $(U_\alpha)_\alpha$ be an open cover of X, and denote $i: U_\alpha \hookrightarrow X$ the canonical injection.

   Then f is formally smooth (resp. formally unramified, resp. formally étale) if and only if each of the $f \circ i_\alpha$ is formally smooth (resp. formally unramified, resp. formally étale).

2. Let $(V_\lambda)_\lambda$ be an open cover of Y.

   Then f is formally smooth (resp. formally unramified, resp. formally étale) if and only of each of the $f|_{f^{-1}(V_\lambda)}: f^{-1}(V_\lambda) \to V_\lambda$ is formally smooth (resp. formally unramified, resp. formally étale).

To summarise: being formally smooth (resp. formally unramified, resp. formally étale)

1. is local on the source;
2. is local on the target.

But observe that the proof, as written in loc. cit. is actually wrong. This is explained in MO10731. This also affects the discussion in remark 11 as the proof goes along the same lines, but the Stacks project incorporates the results of Raynaud–Gruson so it should be fine.

The references to these results in the Stacks project are given in table 1. Unfortunately they are only stated in the required form for algebraic spaces, but one can replace étale by Zariski open immersion to get the desired statement.

	local on the source and target
formally smooth	tag 061K
formally unramified	tag 04G8
formally étale	tag 04GD

Table 1: Localness of formal smoothness (resp. formal unramifiedness, resp. formal étaleness) in the Stacks project

References

[EGA IV_1] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Première partie*. French. 20. Publications Mathématiques de l'IHÉS, 1964, pp. 5–259.

[EGA IV_4] Jean Dieudonné and Alexander Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie*. French. 32. Publications Mathématiques de l'IHÉS, 1967, pp. 5–361.

[Stacks] *The Stacks Project*. 2015. URL: `http://stacks.math.columbia.edu`.