Grothendieck topologies and étale cohomology

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1 Grothendieck topologies

1.1 Motivation

Around 1950 the Zariski topology was introduced in algebraic geometry, in order to have a topology that is appropriate for the objects (i.e. varieties) that were studied, unlike the Euclidean topology. In the late 1950s this was generalized to schemes.

In 1949 Weil had proposed conjectures, now named after him, relating properties of algebraic varieties over finite fields to the topological properties of their counterparts over $\mathbb{C}$. These conjectures have been discussed before in class, and the case of curves has been proven.

At some point it was realized that the existence of a "Weil cohomology theory", mimicking the properties of algebraic topology, can solve the Weil conjectures. This observation is probably due to Serre, who attributes it himself to Weil. But in algebraic topology one often uses constant sheaves. Unfortunately the Zariski topology is not adapted to these sheaves as the following proposition shows.

Proposition 1. Let X be an irreducible topological space. Let $\mathcal{F}$ be a constant sheaf on X. Then

(1) $\quad \mathrm{H}^i(X,\mathcal{F}) = 0$

for $i \geq 1$.

Proof. Every nonempty open set U of X is connected, so $\mathcal{F}$ is a flabby sheaf. Therefore all higher cohomology vanishes. □

This applies in particular to (irreducible) schemes with the Zariski topology. So we conclude that to define a topology on a scheme which gives a meaningful cohomology theory for constant sheaves we need to find something different.

When discussing the motivation for the étale topology in section 2.1 another bad property of the Zariski topology is given. But remark that the Zariski topology is already the good topology to calculate the so-called coherent cohomology of quasicoherent sheaves: these cohomology groups will be isomorphic for all the subcanonical topologies discussed in section 1.5.

1.2 Definitions

The notion of a Grothendieck topology is a very natural one (albeit maybe in hindsight). To realize this we consider the basic definitions of sheaf theory. Recall that a presheaf $\mathcal{F}$ on a topological space X is an assignment

(2) $\quad U \mapsto \mathcal{F}(U)$

of sets (or (abelian) groups, rings, modules, ...) to every open set U of the space, together with restriction morphisms $\mathrm{res}_{V,U}$ for $V \subseteq U$. In the functorial language this is nothing but a functor on the category of open sets of X, where morphisms correspond to inclusions.

A sheaf is a separated presheaf satisfying the glueing property, i.e. it is completely determined by its local data. The separatedness implies that for every open cover $U = \bigcup_{i\in I} U_i$ and sections $f, g \in \mathcal{F}(U)$ such that for all i we have $f|_{U_i} = g|_{U_i}$ we have $f = g$ globally. And the glueing condition says that if we are given $U = \bigcup_{i\in I} U_i$

and sections $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then there exists a section f on U restricting to f_i on each U_i.

Again, this can be interpreted in purely categorical terms: intersections are actually fiber products, and the glueing property can be taken as the exactness of the equaliser

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j) \tag{3}$$

if the category of values of $\mathcal{F}$ has products. So there is no need to restrict oneself to topological spaces for sheaf theory: as long as the category shares some properties with the category of open sets of a topological space (or rather: gives information similar to open covers!) one can generalize without any problem.

Definition 2. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ consists of sets of morphisms $\{U_i \to U\}$ which are called *covers* for each object U such that

1. if $V \to U$ is an isomorphism the singleton $\{V \to U\}$ is a cover;
2. if $\{U_i \to U\}$ is a cover, and $V \to U$ is a morphism, then all the fibered products $U_i \times_U V$ exist, and set of induced projections $\{U_i \times_U V \to V\}$ is again a cover;
3. if $\{U_i \to U\}$ is a cover, and for each i we have a cover $\{V_{i,j} \to U_i\}$ then the set of compositions $\{V_{i,j} \to U\}$ is again a cover.

When a category $\mathcal{C}$ is equipped with a Grothendieck topology we call it a *site*.

These axioms don't describe a topology using open sets, but in terms of covers. In the classical notion of a topology one needs to check that the given description of its open sets satisfies some properties with respect to intersections and unions. In the case of a Grothendieck topology on the other hand one considers preservation under base change and composition. Some examples in algebraic geometry come to mind: open immersions, étale morphisms, smooth morphisms, ...

Remark that the terminology (which is the one found in [ALB73, definition 1.1.1]) doesn't completely agree with the terminology in [SGA3$_1$; SGA4$_1$]. Just like different bases for a topological space can induce the same topology, this definition defines what is called a *pretopology*. Different pretopologies can induce the same topology, and hence the associated sheaf theory on the sites is the same.

1.3 First examples

Example 3. Take a topological space X. The category of its open sets, i.e. the objects are open sets and arrows are inclusions, is equipped with a Grothendieck topology. To each open subset U of X we associate the collection of open covers of U. Fibered products of inclusions are intersections.

So the objects act (or in this case: are) "open sets", but the most important thing are the morphisms. These describe "how" the "open set" is "contained" in the space.

This is a so called "small" example. We can also equip the whole category of topological spaces (or schemes) with a Grothendieck topology. To do so we first introduce an important notion.

Definition 4. Let $\{U_i \to U\}$ be a cover in a site in which set-theoretic unions make sense (topological spaces, schemes, ...). It is *jointly surjective* if the set-theoretic union of the images equals U.

Now we can change our focus to algebraic geometry.

Example 5. The *small Zariski site* of a scheme X is the category X_{Zar} which is the full subcategory of Sch/X of objects $U \to X$ that are open immersions equipped with a Grothendieck topology by defining a cover $\{U_i \to X\}$ to be a jointly surjective set of open embeddings.

And we also have a bigger version.

Example 6. The *big Zariski site* $(\mathrm{Sch}/X)_{\mathrm{Zar}}$ of a scheme X is the category Sch/X equipped with a Grothendieck topology by defining a cover $\{U_i \to U\}$ to be a jointly surjective set of open embeddings.

There are also the étale versions.

Example 7. The *small étale site* of a scheme X is the category $X_{\text{ét}}$ which is the full subcategory of Sch/X of objects $U \to X$ that are étale and locally of finite presentation equipped with a Grothendieck topology by defining a cover $\{U_i \to X\}$ to be a jointly surjective set of étale morphisms. If $U \to X$ and $V \to X$ are two objects of $X_{\text{ét}}$ every arrow $U \to V$ in $X_{\text{ét}}$ is necessarily étale.

Remark 8. If one takes $X = \operatorname{Spec} k$ the small étale site yields already interesting properties. In case of the Zariski topology there is only one open set, the space itself. But if k is not separably closed we can consider a separable extension and this yields an étale morphism, which is immediately jointly surjective, so it yields a cover. We can also consider products of separable extensions. Hence studying the small étale site of a point is equivalent to studying (products of) separable extensions and their tensor products, which is exactly what Galois cohomology is about.

Example 9. The *big étale site* $(\mathrm{Sch}/X)_{\text{ét}}$ of a scheme X is the category Sch/X equipped with a Grothendieck topology by defining a cover $\{U_i \to U\}$ to be a jointly surjective set of étale morphisms that are locally of finite presentation.

One might feel uncomfortable with the existence of two different sites, the small one containing the "opens" of X while the big one contains strictly more information. But at least for the Zariski and étale topology there is no difference in the cohomology groups (after we've introduced sheaf theory and sheaf cohomology). Remark also that for some topologies there is no small site (for instance the cdh topology, introduced in section 1.5).

1.4 Sheaves on Grothendieck topologies

A sheaf on a topological space is a contravariant functor from the category of its open sets to some interesting category, satisfying some glueing data. This immediately generalizes to sheaves on sites as follows.

Definition 10. Let $\mathcal{C}$ be a site. Let $\mathcal{F}\colon \mathcal{C}^{\mathrm{opp}} \to \mathrm{Set}$ be a functor, or *presheaf*.

1. We call $\mathcal{F}$ a *separated* presheaf if for every cover $\{U_i \to U\}$ and for all sections $f, g \in \mathcal{F}(U)$ whose pullbacks to $\mathcal{F}(U_i)$ are equal, we have the equality $f = g$.

2. We call $\mathcal{F}$ a *sheaf* if it is separated and for every cover $\{U_i \to U\}$ and for all sections $f_i \in \mathcal{F}(U_i)$ such that $\mathrm{pr}_1^*(f_i) = \mathrm{pr}_2^*(f_j)$ in $\mathcal{F}(U_i \times_U U_j)$ there exists a section f of $\mathcal{F}(U)$ such that it restricts to f_i on $\mathcal{F}(U_i)$.

Remark that these conditions are equivalent to the exactness of the equaliser diagram in (3), while for separatedness we only need the injectivity of the first arrow.

By applying the forgetful functor we can consider sheaves of groups, rings, etc. And remark that for a general site we don't have equality of the projections $U_i \times_U U_i \to U_i$, unlike the classical topological case where $U_i \cap U_i = U_i$. An example of this can be found in (12) This yields some counterintuitivity in certain sites.

The following definition explains the terminology used in [SGA4$_1$; SGA4$_2$; SGA4$_3$]. It is almost never used to its full extent in the later literature on étale cohomology as this level of generality is not necessary. For completeness' sake we define it.

Definition 11. Let $\mathcal{C}$ be a category. If $\mathcal{C}$ is equivalent to the category of sheaves on some site then it is called a *topos*.

There is moreover a characterisation available in categorical properties, see [SGA4$_1$, théorème IV.1.2]. Remark that we should also be careful about the universes relative to which we are working. The interested (or masochistic) reader is referred to [SGA4$_1$].

1.5 More examples

In this section a non-exhaustive list of Grothendieck topologies on categories of schemes is given, and they are compared. We have already introduced the Zariski and étale topology in section 1.2. We can now introduce a closely related one.

Example 12. The *big smooth site* $(\mathrm{Sch}/X)_{\mathrm{sm}}$ of a scheme X is the category Sch/X equipped with a Grothendieck topology by defining a cover $\{U_i \to U\}$ to be a jointly surjective set of smooth morphisms that are locally of finite presentation.

This topology is close to the étale topology, because its associated topos is the same as the étale topos [Stacks, tag 055S].

The following topologies are both called flat topologies, and they only differ in the finiteness conditions that are used. They are used in for example descent theory and the theory of algebraic stacks.

Example 13. The *big fppf site* $(\mathrm{Sch}/X)_{\mathrm{fppf}}$ of a scheme X is the category Sch/X equipped with a Grothendieck topology by defining a cover $\{U_i \to U\}$ to be a jointly surjective set of flat morphisms that are locally of finite presentation. The abbrevation fppf signifies "fidèlement plat et de présentation finie".

Example 14. The *big fpqc site* $(\mathrm{Sch}/X)_{\mathrm{fpqc}}$ of a scheme X is the category Sch/X equipped with a Grothendieck topology by defining a cover $\{U_i \to U\}$ to be a set of maps such that $\bigsqcup U_i \to U$ is a faithfully flat morphism such that every quasicompact open subset of U is the image of a quasicompact open subset of $\bigsqcup U_i$. The abbreviation fpqc signifies "fidèlement plat et quasi-compacte".

An important type of (pre)sheaves are the representable (pre)sheaves. Given an object C in a category $\mathcal{C}$ we can always consider the morphisms $\mathrm{Hom}_{\mathcal{C}}(-, C)$ into that object. This yields a presheaf of sets (or groups, if C is a group object, etc.), and one could ask himself whether it is a sheaf. We have the following theorem which is at the heart of descent theory.

Theorem 15. Let X be a scheme. A representable functor on Sch/X is a sheaf in the fpqc topology.

So it is also a sheaf in any weaker topology, especially the Zariski topology. The topology such that every representable functor is a sheaf is called *canonical*, any weaker topology is *subcanonical*.

We can compare these (and many other) topologies in the diagram of figure 1. This comparison is based on [1112.5206; Gei06; Voe96; SV96; Stacks; Sch12]. Some remarks on this diagram:

1. The main stem of topologies on the left is discussed in [Stacks, tag 03FE]. The most important ones have been introduced in this text, the others are discussed in [Stacks].
2. We observe that there are many non-subcanonical topologies, i.e. not every representable functor is a sheaf in these topologies. These topologies are mostly used in a motivic and arithmetic context, and the reason for not being subcanonical is that they involve proper morphisms. All these topologies are depicted on the right of the main stem.
3. The h topology originates in Voevodsky's homology of schemes. The qfh topology is the h topology with an extra quasi-finiteness condition.
4. The rather obscure étale h (or eh) topology is what the completely decomposed (or cdh) topology is to the Nisnevich topologies (if one is familiar with them): abstract blow-ups are added as covering maps.
5. For the definitions of the other topologies one should read [1112.5206]. The abbreviations cdp, ℓ'dp and fpsℓ' (where ℓ is a prime) refer to completely decomposed proper, ℓ'-decomposed proper and "fini, plat, surjectif et premier à ℓ'". Gabber's ℓ'-topology is then taken as the topology generated by the Nisnevich and ℓ'dp covers.
6. The naive fpqc topology is a Grothendieck topology defined by taking fpqc morphisms without the finiteness conditions, but it is not subcanonical.

1.6 Comparison of cohomology

With such a plethora of topologies it is important to have cohomological comparison results, i.e. we would like to know under which conditions the cohomology groups turn out to be isomorphic. One can then choose the best topology to compute the desired cohomology groups. Three examples are discussed here, a myriad others exist. The first result is that the Zariski topology is already good enough to compute coherent cohomology [Mil80, proposition 3.7].

Theorem 16. Let X be a scheme. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Denote by $\mathcal{F}_{\mathrm{fpqc}}$ its associated sheaf on X_{fpqc}. Then we have the isomorphisms

$$\mathrm{H}^i(X_{\mathrm{Zar}}, \mathcal{F}) \cong \mathrm{H}^i(X_{\mathrm{fpqc}}, (\mathcal{F}_{\mathrm{fpqc}})) \tag{5}$$

for all $i \geq 0$.

We can also prove the analogous comparison with the étale topology:

$$\mathrm{H}^i(X_{\mathrm{Zar}}, \mathcal{F}) \cong \mathrm{H}^i(X_{\mathrm{ét}}, (\mathcal{F}_{\mathrm{ét}})) \tag{6}$$

for all $i \geq 0$.

If on the other hand we consider smooth group schemes, i.e. the sheaves represented by group objects in Sch/X, we obtain the following comparison result, saying that we need to look at the étale topology (or finer) to get the true cohomology groups [Mil80, theorem 3.9].

(4)

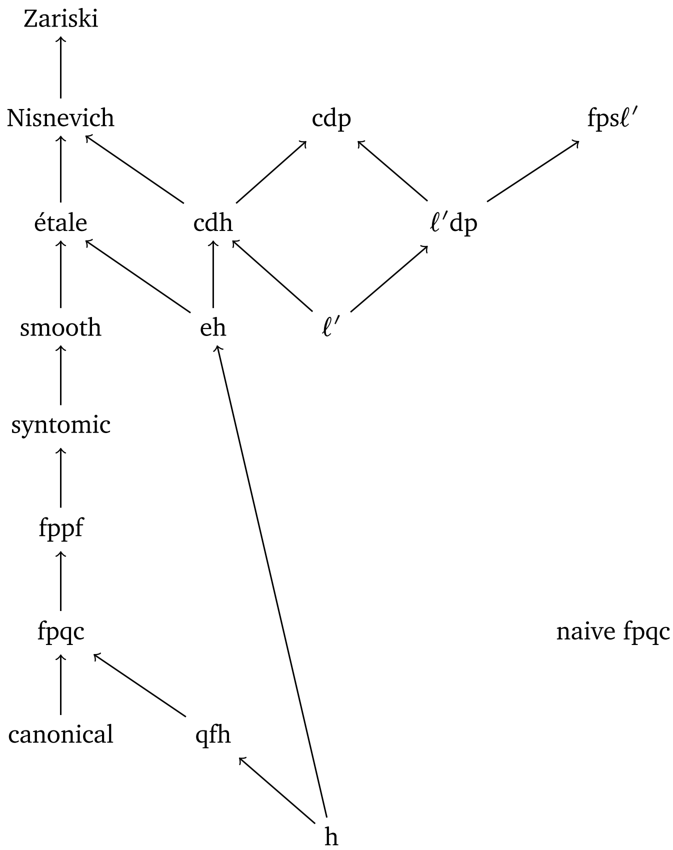


Figure 1: A comparison of Grothendieck topologies on the category of schemes

Theorem 17. Let X be a scheme. Let G be a smooth, quasiprojective, commutative group scheme over X. Then we have the isomorphisms

$$\mathrm{H}^i(X_{\text{ét}}, G) \cong \mathrm{H}^i(X_{\text{fpqc}}, G) \tag{7}$$

for all $i \geq 0$.

The most important result in our case, which will motivate our choice of the étale topology, is the following [Mil80, theorem 3.12].

Theorem 18. Let X be a smooth scheme over $\mathbb{C}$. Let M be a finite abelian group. Then we have the isomorphisms

$$\mathrm{H}^i(X(\mathbb{C}), \underline{M}) \cong \mathrm{H}^i(X_{\text{ét}}, \underline{M}) \tag{8}$$

for all $i \geq 0$.

The necessity of the finiteness will be discussed later. So we can conclude that the “algebraic” definition of the étale topology suffices to obtain the same results as with the finer analytic topology (although a real comparison doesn’t really make sense).

2 Étale cohomology

2.1 Motivation

We have seen that the Zariski cohomology groups of a constant sheaf are not quite what we want them to be. And there is a second issue with the Zariski topology, when one tries to mimick techniques from topology or analysis: the inverse function theorem fails. By fixing this failure of the Zariski topology we obtain without further ado the étale topology.

Consider the map $\mathbb{A}_k^1 \to \mathbb{A}_k^1 : x \mapsto x^n$. If $n \nmid \operatorname{char} k$ this morphism is étale everywhere except at the origin. But in the Zariski topology it is nowhere a local isomorphism: we'd need nonempty open subsets U and V of $\mathbb{A}_k^1$ such that the restriction of the map yields an isomorphism between V and U. As our objects are affine, we can consider the algebraic side of the picture: the map $\mathbb{A}_k^1 \to \mathbb{A}_k^1$ is written as $k[t] \to k[t] : t \mapsto t^n$. As the function fields of the affine line and the open sets U, V are all $k(t)$, and the induced morphism is $k(t) \to k(t) : t \mapsto t^n$, we cannot get an isomorphism.

If we consider the analytic situation, for $n = 2$ and $k = \mathbb{C}$, we need to construct an inverse function $x \mapsto \sqrt{x}$ on some open set. Complex analysis tells us that by choosing a nice curve from 0 to ∞ we can construct this inverse function. But the open set that is the complement of this curve is not open in the Zariski topology, only in the analytic topology. So the idea that is already present in the definition of an étale morphism is that we "add" those inverse functions, such that we get finite covers étale locally.

2.2 The étale topology

Using the following proposition one can define the étale topology and compare it to the Zariski topology, as was implicitly done in sections 1.2 and 1.5.

Proposition 19.

1. Let $f : X \to X'$ and $g : Y \to Y'$ be étale morphisms over a base scheme S. Then the base change

   (9) $\quad f \times_S g : X \times_S Y \to X' \times_S Y'$

   is also étale.

2. Let $f : X \to Y$ and $g : Y \to Z$ be étale morphisms. Then the composition

   (10) $\quad g \circ f : X \to Z$

   is also étale.

3. An open immersion is an étale morphism.

Hence we can define both a small étale site $X_{\text{ét}}$ and its big counterpart $(\mathrm{Sch}/X)_{\text{ét}}$ for any scheme X, as was done in examples 7 and 9.

Having defined a topology and a category of sheaves, we can immediately apply the techniques of sheaf cohomology as developed for classical topologies. We will simply call sheaf cohomology with respect to the étale topology by the obvious "étale cohomology". This étale cohomology and its applications concerning the Weil conjectures are discussed in the next section.

With some extra care because we are using Grothendieck topologies, one can define direct and inverse images of sheaves on the étale site along a morphism $f : X \to Y$. As in the case of the Zariski topology we obtain an adjoint pair (f^*, f_*). As in the classical case the inverse image functor f^* is exact, while f_* is in general only left exact. So we can derive f_* to obtain higher direct images $\mathbf{R}^q f_*$. If f is a finite morphism then f_* is also right exact, and the higher direct images vanish.

To compute the higher direct images in case of a composition of morphisms there exists a technical tool called a *spectral sequence*. It exists in the full generality of topological spaces. To discuss spectral sequences here would be impossible, but the main result is that we can approximate the higher direct image of a composition using the composition of the higher direct images: let $f : X \to Y$ and $g : Y \to Z$ be continuous morphisms of topological spaces and $\mathcal{F}$ a sheaf of abelian groups on X, then we have

(11) $$\mathrm{E}_2^{p,q} = \mathbf{R}^p g_* \circ \mathbf{R}^q f_*(\mathcal{F}) \Rightarrow \mathrm{E}_\infty^{p,q} = \mathbf{R}^{p+q}(g \circ f)_*(\mathcal{F}).$$

2.3 Galois cohomology

As discussed in remark 8 Galois cohomology can be considered as a special case of étale cohomology. In this section this is elaborated.

Let K/k be a Galois extension of fields. Then we have that the tensor product of fields gives

(12) $$K \otimes_k K \cong \prod_{g \in \mathrm{Gal}(K/k)} K$$

or in more geometric terms

(13) $$\operatorname{Spec} K \times_k \operatorname{Spec} K \cong \coprod_{g \in \mathrm{Gal}(K/k)} \operatorname{Spec} K.$$

To give the link with Galois cohomology, the isomorphism (12) should be made more explicit. It is actually the morphism

(14) $$x \otimes y \mapsto (xg(y))_{g \in \mathrm{Gal}(K/k)}$$

hence the information contained in the Galois group is made visible.

In Galois cohomology the main object of interest is $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, the absolute Galois group. It is obtained by taking the limit over the finite separable extensions. These extensions are exactly the étale covers, and the decomposition from (13) yields an interpretation of étale sheaves in terms of Galois modules [Con].

Moreover, by iterating this construction for a finite extension K/k we get

(15) $$K \otimes_k K \otimes_k K \cong \prod_{g_1, g_2 \in \mathrm{Gal}(K/k)} \operatorname{Spec} K$$

and so on. These objects are actually the "intersections" from the theory of Čech cohomology for Grothendieck topologies [Mil80, Section III.2]. If one is familiar with the construction of the bar resolution from group cohomology, it is clear that the Čech complex and the standard complex agree. So this proves the statement that Galois cohomology is just étale cohomology for $X = \operatorname{Spec} k$.

2.4 Weil cohomologies

As discussed before we are looking for a *Weil cohomology theory*, which is a contravariant functor $\mathrm{H}^\bullet$ from smooth projective varieties over a base field k to graded algebras over a coefficient field K of characteristic zero satisfying a list of properties. By the very existence of this cohomology theory one can prove a large part of the Weil conjectures by its formal properties. But this does not apply to every aspect of the Weil conjectures, most notably the Riemann hypothesis.

The list of properties is

finite-dimensionality each $\mathrm{H}^i(X)$ is a finite-dimensional K-vectorspace;

vanishing if X is n-dimensional then $\mathrm{H}^i(X) = 0$ if $i \notin \{0, \ldots, 2n\}$;

orientation the top cohomology $\mathrm{H}^{2n}(X)$ is isomorphic to K;

Poincaré duality we have a non-degenerate pairing

(16) $\mathrm{H}^i(X) \times \mathrm{H}^{2n-i}(X) \to \mathrm{H}^{2n}(X) \cong K$;

Künneth isomorphism we have a canonical isomorphism

(17) $\mathrm{H}^\bullet(X) \otimes_K \mathrm{H}^\bullet(Y) \to \mathrm{H}^\bullet(X \times_k Y)$;

cycle map we have a morphism

(18) $\gamma_X \colon \mathrm{Z}^i(X) \to \mathrm{H}^{2i}(X)$

interpreting algebraic cycles of codimension i in the $2i$th cohomology group such that this is compatible with the Künneth isomorphism and if X is a single point the map γ_X is $\mathbb{Z} \hookrightarrow K$.

The following two properties are not a part of the basic list of axioms, but are satisfied in the case of étale cohomology. They relate to the Riemann hypothesis part of the Weil conjectures.

weak Lefschetz for every smooth hyperplane section $j \colon W \subset X$ are the pullback maps $j^* \colon \mathrm{H}^i(X) \to \mathrm{H}^i(W)$ isomorphisms for $i \leq n-2$ and a monomorphism for $i = n-1$;

hard Lefschetz for every hyperplane section $j \colon W \subset X$ we denote $w = \gamma_X(X)$ in $\mathrm{H}^2(X)$, then the *hard Lefschetz operator* is defined by

(19) $\mathrm{L} \colon \mathrm{H}^i(X) \to \mathrm{H}^{i+2}(X) : x \mapsto x \cdot w$

and the iterated maps

(20) $\mathrm{L}^i \colon \mathrm{H}^{n-i}(X) \to \mathrm{H}^{n+i}(X)$

is an isomorphism for $i = 1, \ldots, n$.

Up to now, four “classical” Weil cohomologies have been established. These are

ℓ-adic cohomology where k is arbitrary and $K = \mathbb{Q}_\ell$ for $\ell \neq \operatorname{char} k$, using the étale topology;

singular cohomology where $k = K = \mathbb{C}$, using the analytic topology;

(algebraic) de Rham cohomology where $\operatorname{char} k = 0$ and $K = k$, using the Zariski topology;

crystalline cohomology where k is a perfect field of characteristic p and K is the fraction field of the ring of Witt vectors of k, using divided power thickenings of the Zariski topology.

We can now elaborate on the relation between étale cohomology and ℓ-adic cohomology. As discussed before, the first is nothing but the calculation of sheaf cohomology in the étale topology, without further ado. For quasicoherent sheaves the cohomology groups agree with the Zariski (or coherent) cohomology groups. But the computation is not necessarily easier in the étale topology, so the benefits of using the étale topology lie elsewhere: the classes of torsion or constructible sheaves (these notions will be defined below). The reason why we use this type of sheaves can be motivated using the Tate modules, as discussed in the previous lecture, or by the example below.

So ℓ-adic cohomology is the Weil cohomology theory one obtains by using the étale topology and the constructible sheaves. As the overview suggests: the field of coefficients for ℓ-adic cohomology is $\mathbb{Q}_\ell$ where ℓ is different from the characteristic of the base field of the varieties. But it is important to remark that ℓ-adic cohomology is *not* just étale cohomology for the constant sheaf $\mathbb{Q}_\ell$!

It is of course possible to calculate the cohomology groups of the constant sheaf $\mathbb{Q}_\ell$. But they are not what we are looking for: the étale topology behaves good for constant torsion sheaves, but not for non-torsion sheaves such as $\mathbb{Q}_\ell$ or $\mathbb{Z}$. Recall that we can obtain the étale fundamental group $\pi_1(X, x)$, by using the construction with covering spaces [Mil12, section I.3]. We then want the first cohomology group of a constant sheaf to be related to the π_1 of that scheme, but this fails for nontorsion sheaves as the following example shows.

Example 20. Let X be a normal scheme then have

$$\mathrm{H}^1(X_{\text{ét}}, \mathbb{Z}) \cong \mathrm{Hom}_{\text{cont}}(\pi_1(X, x), \mathbb{Z}) = 0 \tag{21}$$

where $\mathbb{Z}$ has the discrete topology. The cohomology group is actually zero, because a continuous map $f : \pi_1(X, x) \to \mathbb{Z}$ yields an open subgroup $f^{-1}(0)$. But as $\pi_1(X, x)$ is a compact topological group such a subgroup is necessarily of finite index. But this implies that $f^{-1}(0) = \pi_1(X, x)$ as $\mathbb{Z}$ has no finite subgroups.

So using the constant sheaf with values in $\mathbb{Q}_l$ is not going to work. On the other hand we observe that for the torsion sheaves $\mathbb{Z}/\ell^n\mathbb{Z}$ we have

$$\begin{aligned} \varprojlim \mathrm{H}^1(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) &\cong \varprojlim \mathrm{Hom}_{\text{cont}}(\pi_1(X, x), \mathbb{Z}/\ell^n\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\text{cont}}(\pi_1(X, x), \mathbb{Z}_\ell) \end{aligned} \tag{22}$$

where $\mathbb{Z}_\ell$ is equipped with the ℓ-adic topology. After tensoring this with $\mathbb{Q}_\ell$ we obtain an object that has coefficients in a field of characteristic zero, as desired. In other words, we define

$$\mathrm{H}^i(X, \ell) := \varprojlim \mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \tag{23}$$

to be the *ℓ-adic cohomology groups*.

Now one of course has to prove that this construction yields a good Weil cohomology theory. And one has to work over a separably closed field k, which is possible by base changing the varieties over k to the separable closure k^{sep}.

2.5 Constructible sheaves

Zariski cohomology is all about (quasi)coherent sheaves, where we try to describe the properties of schemes using their categories of quasicoherent sheaves. When trying to mimick algebraic topology, where one often uses constant sheaves, we have seen that the Zariski topology has too few open sets. Hence we have introduced the étale topology.

So we are interested in (locally) constant sheaves, but this subcategory of sheaves is not well-behaved, for instance pushforward along proper morphisms, or even closed immersions, fails.

Example 21. Let $(A, \mathfrak{m})$ be a local ring. Consider the scheme $X = \operatorname{Spec} A$, and let $i\colon \operatorname{Spec} k(\mathfrak{m}) \to X$ be the immersion of the closed point. Consider the constant sheaf $\mathbb{Z}$ on $\operatorname{Spec} k(\mathfrak{m})_{\text{ét}}$. Then $i_*(\mathbb{Z})$ is not locally constant, because it only has a non-zero stalk in $\mathfrak{m}$. But a locally constant sheaf on $X_{\text{ét}}$ is either zero, or non-trivial at all stalks.

We will consider the smallest subcategory of sheaves that contains the (finite) constant sheaves and that satisfies the preservation property along proper morphisms. These will be the constructible sheaves.

They form an abelian category of sheaves, and in case you're familiar with the representability arguments from the theory of algebraic stacks: the constructible sheaves are exactly the sheaves represented by an étale algebraic space of finite type.

Definition 22. Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{\text{ét}}$. It is *finite* if $\mathcal{F}(U)$ is finite for all quasicompact U. It has *finite stalks* if $\mathcal{F}_x$ is finite for all geometric points of x in X.

Neither of these conditions implies the other. But if $\mathcal{F}$ is locally constant then having finite stalks implies that it is finite.

Definition 23. Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{\text{ét}}$. It is *locally constant* if there exists an étale cover $\{U_i \to X\}_{i \in I}$ such that the restriction $\mathcal{F}|_{U_i}$ is a constant sheaf for all $i \in I$.

We now give a definition of constructible sheaves in terms of the properties we want this subcategory to have. After that we give a characterisation in terms of the previous definitions.

Definition 24. Let $\mathcal{F}$ be a sheaf on $(\mathrm{Sch}/X)_{\text{ét}}$. Denote $f : (\mathrm{Sch}/X)_{\text{ét}} \to X_{\text{ét}}$, the change of sites that is the identity on the objects. We call $\mathcal{F}$ *locally constructible* if $\mathcal{F} \cong f^* \circ f_*(\mathcal{F})$. It is moreover *constructible* if it is locally constructible and its espace étalé is of finite type.

An espace étalé is (in a topological context) a way of representing a sheaf on a topological space X as another topological space, together with a morphism to X. One can do the same thing for a sheaf on the étale topology, but the representing object is now an algebraic space (not a scheme) [Mil80, theorem V.1.5].

So a locally constructible sheaf is a sheaf on the big étale site that is completely defined by its restriction on the small étale site. By adding a finiteness assumption we get constructibility. We can characterise them in two ways.

Proposition 25. Let $\mathcal{F}$ be sheaf on $X_{\text{ét}}$. It is constructible if one of the following equivalent conditions A or B is satisfied

A. locally constant in a closed subscheme, with finite stalks

1. every irreducible closed subscheme Z of X contains a (nonempty) open subscheme U of Z such that the restriction $\mathcal{F}|_U$ is locally constant;
2. it has finite stalks.

B. stratification every quasicompact open $U \subseteq X$ is a finite union $U = \bigcup_i Z_i$ of locally closed subschemes Z_i such that $\mathcal{F}|_{Z_i}$ is finite and there exists a finite étale morphism $Z_i' \to Z_i$ such that $\mathcal{F}|_{Z_i'}$ is constant.

2.6 Glueing

The following section discusses a collection of six functors. It is used to cut up the category of sheaves on the (small) étale site and paste it together again, and uses the situation depicted in (24). In section 3.6 another set of six functors is introduced, and they are called "Grothendieck's six operations". These exist in other contexts, one of them is discussed in section 3.6.

Let X be a scheme, U an open subscheme of X and Z a closed subscheme of X such that $X = Z \sqcup U$ on the level of the underlying sets. This yields the situation

(24) $$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

Take $\mathcal{F}$ on $X_{\text{ét}}$. Then we have sheaves $i^*(\mathcal{F})$ and $j^*(\mathcal{F})$ on Z and U. By adjointness we get a morphism $\mathcal{F} \to j_* \circ j^*(\mathcal{F})$, and applying i^* yields

(25) $$\phi_{\mathcal{F}}\colon i^*(\mathcal{F}) \to i^* \circ j_* \circ j^*(\mathcal{F})$$

on Z. So for every sheaf $\mathcal{F}$ on $X_{\text{ét}}$ we have an induced triple

(26) $$(i^*(\mathcal{F}), j^*(\mathcal{F}), \phi_{\mathcal{F}}).$$

Now we define $\mathbf{T}(X)$ to be the category whose objects are triples $(\mathcal{F}_1, \mathcal{F}_2, \phi)$ where $\mathcal{F}_1 \in \mathrm{Sh}(Z_{\text{ét}})$, $\mathcal{F}_2 \in \mathrm{Sh}(U_{\text{ét}})$ and $\phi\colon \mathcal{F}_1 \to i^* \circ j_*(\mathcal{F}_2)$, and its morphisms are pairs $(\psi_1\colon \mathcal{F}_1 \to \mathcal{F}_1', \psi_2\colon \mathcal{F}_2 \to \mathcal{F}_2')$ such that

(27) $$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{\phi} & i^* \circ j_*(\mathcal{F}_2) \\ {\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle i^*\circ j_*(\psi_2)} \\ \mathcal{F}_1' & \xrightarrow[\phi']{} & i^* \circ j_*(\mathcal{F}_2') \end{array}$$

commutes. This category $\mathbf{T}(X)$ is equivalent to $\mathrm{Sh}(X_{\text{ét}})$. Moreover, exactness of a sequence is checked by checking exactness of the induced sequences in $\mathrm{Sh}(Z_{\text{ét}})$ and $\mathrm{Sh}(U_{\text{ét}})$.

We are now in the position to define the six functors. These are

(28) $$\mathrm{Sh}(Z_{\text{ét}}) \underset{i^!}{\overset{i^*}{\leftrightarrows}} \xrightarrow{i_*} \mathrm{Sh}(X_{\text{ét}}) \underset{j_*}{\overset{j_!}{\leftrightarrows}} \xrightarrow{j^*} \mathrm{Sh}(U_{\text{ét}}).$$

The functors i_*, i^*, j_* and j^* are the usual adjoint functors (in a Grothendieck topology context). We also have "exceptional" functors: $j_!$ is the *extension by zero*, and $i^!$ is the subsheaf of *sections with support in Z*.

Given the equivalence of the categories $\mathbf{T}(X)$ and $\mathrm{Sh}(X_{\text{ét}})$ we can describe these as

$$(29) \qquad \begin{array}{llll} i^*: & \mathcal{F}_1 \leftarrow\!\shortmid (\mathcal{F}_1, \mathcal{F}_2, \phi) & j_!: & (0, \mathcal{F}_2, 0) \leftarrow\!\shortmid \mathcal{F}_2 \\ i_*: & \mathcal{F}_1 \mapsto (\mathcal{F}_1, 0, 0) & j^*: & (\mathcal{F}_1, \mathcal{F}_2, \phi) \mapsto \mathcal{F}_2 \\ i^!: & \ker\phi \leftarrow\!\shortmid (\mathcal{F}_1, \mathcal{F}_2, \phi) & j_*: & (i^* \circ j_*(\mathcal{F}_2), \mathcal{F}_2, \mathrm{id}) \leftarrow\!\shortmid \mathcal{F}_2. \end{array}$$

We have the following easy results for these six functors [Mil80, proposition 3.14].

Proposition 26. Each functor is left adjoint to the one below it. The functors i_*, i^*, $j_!$ and j^* are exact while the functors j_* and $i^!$ are left exact. The compositions $i^* \circ j_!$, $i^! \circ j_!$, $i^! \circ j_*$ and $j^* \circ i_*$ are zero. The functors i_* and j_* are fully faithful.

We have the exact sequences

$$(30) \quad 0 \to j_! \circ j^*(\mathcal{F}) \to \mathcal{F} \to i_* \circ i^*(\mathcal{F}) \to 0$$

and

$$(31) \quad 0 \to i_* \circ i^!(\mathcal{F}) \to \mathcal{F} \to j_* \circ j^*(\mathcal{F}).$$

3 Results in étale cohomology

3.1 Proper base change

From this point on I will give a quick overview of some the properties of étale cohomology in Chapter 6 of [Mil80], backed by the expositions in [SGA4_3] and [SGA4½] where appropriate.

The first result is the proper base change theorem, and its related corollaries. First we give the technical result that lies at the heart of the actual theorems.

Theorem 27. Let $\pi\colon Y \to X$ be a proper morphism. Let $\mathcal{F}$ be a constructible sheaf on $(\mathrm{Sch}/Y)_{\text{ét}}$. Then $\mathbf{R}^i\pi_*(\mathcal{F})$ is a constructible sheaf on $(\mathrm{Sch}/X)_{\text{ét}}$ for all $i \geq 0$.

The proof of this theorem in [SGA4_3] is difficult and long, it has been simplified by Artin in [ALB73] using representability arguments and algebraic spaces.

Before giving the actual proper base change theorem, we discuss how the general base change morphism is obtained. Let

$$(32)\quad \begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow[f]{} & X \end{array}$$

be a pullback diagram. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. We have the morphism

$$(33)\quad \mathcal{F} \to f'_* \circ f'^*(\mathcal{F})$$

induced by adjunction. The Grothendieck spectral sequence for a sheaf on $X_{\text{ét}}$ and morphisms $f : X \to Y$ and $g : Y \to Z$ is given by

$$(34)\quad \mathrm{E}_2^{p,q} = \mathbf{R}^p g_* \circ \mathbf{R}^q f_*(\mathcal{F}) \Rightarrow \mathrm{E}_\infty^{p,q} = \mathbf{R}^{p+q}(g \circ f)_*(\mathcal{F})$$

which yields the edge morphisms

$$(35)\quad \mathbf{R}^p g_* \circ f_*(\mathcal{F}) \to \mathbf{R}^p(g \circ f)_*(\mathcal{F})$$

and

$$(36)\quad \mathbf{R}^p(g \circ f)_*(\mathcal{F}) \to g_* \circ \mathbf{R}^p f_*(\mathcal{F}).$$

Then consider the composition

$$(37)\quad \begin{aligned} \mathbf{R}^p\pi_*(\mathcal{F}) &\to \mathbf{R}^p\pi_* \circ f'_* \circ f'^*(\mathcal{F}) && (33)\\ &\to \mathbf{R}^p(\pi \circ f')_* \circ f'^*(\mathcal{F}) && (35)\\ &= \mathbf{R}^p(f \circ \pi')_* \circ f'^*(\mathcal{F}) && (32)\\ &\to f_* \circ \mathbf{R}^p\pi'_* \circ f'^*(\mathcal{F}) && (36) \end{aligned}$$

and so by adjunction we obtain the *base change morphism*

$$(38)\quad f^* \circ \mathbf{R}^p\pi_*(\mathcal{F}) \to \mathbf{R}^p\pi'_* \circ f'^*(\mathcal{F}).$$

We are interested in conditions that ensure this morphism is an isomorphism, either by restricting the morphisms, the type of sheaves, the schemes or a combination of these. We can now give a first example of such a base change theorem.

Corollary 28 (Proper base change)**.** Let

$$(39)\qquad \begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow[f]{} & X \end{array}$$

be a pullback diagram, where π is a proper morphism. Let $\mathcal{F}$ be a torsion sheaf on $Y_{\text{ét}}$. Then the base change morphism

$$(40)\quad f^* \circ \mathbf{R}^i \pi_*(\mathcal{F}) \to \mathbf{R}^i \pi'_* \circ f'^*(\mathcal{F})$$

is an isomorphism for all $i \geq 0$.

If we apply the proper base change theorem to a fiber, i.e. one of the derived functors reduces to cohomology groups, we get a special corollary that we will state for later use.

Corollary 29. Let $\pi\colon Y \to X$ be a proper morphism. Let x be a geometric point of X. Let $Y_x \to x$ be the (geometric) fiber. Let $\mathcal{F}$ be a torsion sheaf on $Y_{\text{ét}}$. Then we have a canonical isomorphism

$$(41)\quad (\mathbf{R}^i \pi_*(\mathcal{F}))_x \to \mathrm{H}^i(Y_x, \mathcal{F}|_{Y_x})$$

for all $i \geq 0$.

Corollary 30 (Invariance under change of base field)**.** Let $k \subseteq K$ be separably closed fields. Let X be a scheme that is proper over $\operatorname{Spec} k$. Let $\mathcal{F}$ be a torsion sheaf on $X_{\text{ét}}$. Then

$$(42)\quad \mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(X \otimes_k K, \mathcal{F}|_{X \otimes_k K})$$

for all $i \geq 0$.

Corollary 31 (Finiteness)**.** Let X be proper over a field k. Let $\mathcal{F}$ be a constructible sheaf on $X_{\text{ét}}$. Then $\mathrm{H}^i(X, \mathcal{F})$ is a finite-dimensional k-vectorspace for all $i \geq 0$.

3.2 Higher direct images with proper support

Definition 32. Let X be a separated scheme of finite type over a field. Let $\mathcal{F}$ be a sheaf on X. The *group of sections with proper support* of $\mathcal{F}$ is

$$(43)\quad \Gamma_{\mathrm{c}}(X, \mathcal{F}) := \bigcup_Z \ker(\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus Z, \mathcal{F}))$$

where Z runs through the complete subvarieties of X.

This functor is left exact, but not right exact. If we were to naively consider its derived functors we would not get interesting results. This is related to the fact that on an affine scheme there aren't many sections with proper support.

To define the correct version of "derived" functor (which is a misleading name, it is a functor living on the level of derived categories, but it is not a derived functor as such) in a meaningful way we have to consider an open embedding $j\colon X \hookrightarrow \overline{X}$ in a complete variety, a so called *compactification*. By a theorem of Nagata a compactification always exists for a separated scheme of finite type over a field. This yields the following definition.

Definition 33. Let X be a separated scheme of finite type over a field. Let $\mathcal{F}$ be a sheaf on X. The *cohomology groups with proper support* are

$$\mathrm{H}^p_c(X,\mathcal{F}) := \mathrm{H}^p\left(\overline{X}, j_!(\mathcal{F})\right) \tag{44}$$

where $j_!$ is the extension by zero functor.

If we consider a single complete subvariety this is well-known in a general topological context [Har77, exercise III.2.3], where we have the exact sequence

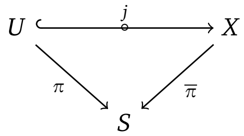
$$\begin{aligned} 0 \to \mathrm{H}^0_Z(X,\mathcal{F}) \to \mathrm{H}^0(X,\mathcal{F}) \to \mathrm{H}^0(X\setminus Z,\mathcal{F}) \to \dots \\ \dots \to \mathrm{H}^1_Z(X,\mathcal{F}) \to \mathrm{H}^1(X,\mathcal{F}) \to \mathrm{H}^1(X\setminus Z,\mathcal{F}|) \to \dots \end{aligned} \tag{45}$$

relating the cohomology with supports to the cohomology on X and $X \setminus Z$. This can moreover be viewed in light of the glueing techniques from section 2.6.

The definition actually depends on the choice of the compactification, but if $\mathcal{F}$ is a torsion sheaf we can prove it is independent after all. To do so we introduce a relative notion.

Definition 34. Let $\pi\colon U \to S$ be a morphism. It is *compactifiable* if it can be embedded in a diagram

(46)



where j is an open immersion and $\overline{\pi}$ is a proper morphism.

The associated derived functors, which are called *higher direct images with proper support* are defined by setting

$$\mathbf{R}^i_c\pi_*(\mathcal{F}) := \mathbf{R}^i\overline{\pi}_* \circ j_!(\mathcal{F}). \tag{47}$$

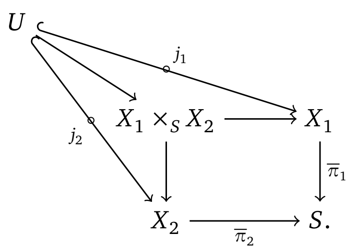
We are using the notation from [Mil80] here, in [SGA4$_3$] this is denoted $\mathbf{R}^i\pi_!$. The reason this notation is used in [Mil80] is to stress the fact that we're not deriving the functor $\pi_!$ here.

We can now state (and prove) the invariance under the choice of compactification.

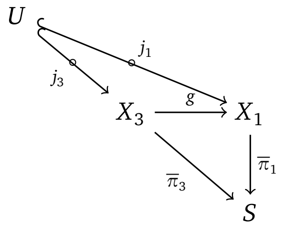
Theorem 35. Let $\pi\colon U \to S$ be a compactifiable morphism. Let $\mathcal{F}$ be a torsion sheaf on U. The sheaves $\mathbf{R}^i_c\pi_*(\mathcal{F})$ do not depend on the choice of the compactification $j\colon U \hookrightarrow X$.

Proof. Let $j_1\colon U \hookrightarrow X_1$ and $j_2\colon U \hookrightarrow X_2$ be two compactifications. Consider the diagram

(48)



If we denote X_3 the closure of the image of U in $X_1 \times_S X_2$ we get a third compactification $j_3 : U \to X_3$ which fits into the diagram. We obtain a proper map $g : X_3 \to X_1$ such that

(49) 

commutes. Hence by the Grothendieck spectral sequence applied to derived functors induced by the composition $\overline{\pi}_3 = \overline{\pi}_1 \circ g$ we obtain the spectral sequence

$$\mathbf{R}^n \overline{\pi}_{1,*} \circ \mathbf{R}^m g \circ j_{3,!}(\mathcal{F}) \Rightarrow \mathbf{R}^{n+m} \overline{\pi}_{3,*} \circ j_{3,!}(\mathcal{F}). \tag{50}$$

By (41) in corollary 29 the stalks of the sheaf $\mathbf{R}^m g_* \circ j_{3,!}(\mathcal{F})$ can be determined by the cohomology groups for the sheaves $j_{3,!}(\mathcal{F})|_{X_{3,x}}$ on the fibers of the morphism g. But as the fiber $X_{3,x}$ either consists of a single point if $x \in U$, and $j_{3,!}(\mathcal{F})|_{X_{3,x}} = 0$ in the other case, we obtain for $m = 0$

$$g_* \circ j_{3,!}(\mathcal{F}) \cong j_{1,!}(\mathcal{F}) \tag{51}$$

and

$$\mathbf{R}^m g_* \circ j_{3,!}(\mathcal{F}) = 0 \tag{52}$$

for $m \geq 1$. Hence the spectral sequence (50) just computes

$$\mathbf{R}^n \overline{\pi}_{1,*} \circ j_{1,!}(\mathcal{F}) \cong \mathbf{R}^n \overline{\pi}_{3,*} \circ j_{3,!}(\mathcal{F}) \tag{53}$$

and this is exactly the definition of the higher direct images with proper support. □

3.3 Smooth base change

We now want our torsion to be prime to the characteristics of the residue fields of the points in the scheme we are considering. To do so we define

Definition 36. Let X be a scheme. Denote

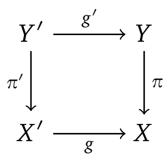
$$\operatorname{char} X := \{p \mid \exists x \in X : p = \operatorname{char} k(x)\}. \tag{54}$$

Equivalently we could say that this is the image of the canonical morphism to the final object $X \to \operatorname{Spec} \mathbb{Z}$.

Let $\mathcal{F}$ be a sheaf on X. Then the torsion of $\mathcal{F}$ is *prime to* $\operatorname{char} X$ if the multiplication morphism $p : \mathcal{F} \to \mathcal{F}$ is injective for all $p \in \operatorname{char} X \setminus \{0\}$.

As the title of this section suggests we also have a smooth base change theorem, which is the following.

Theorem 37 (Smooth base change theorem)**.** Let

(55) 

be a pullback diagram, where g is a smooth and π a quasicompact morphism. Let $\mathcal{F}$ be a torsion sheaf on $Y_{\text{ét}}$ such that its torsion is prime to $\operatorname{char} X$. Then the base change morphism

$$g^* \circ \mathbf{R}^i \pi_*(\mathcal{F}) \to \mathbf{R}^i \pi'_* \circ g'^*(\mathcal{F}) \tag{56}$$

is an isomorphism for all $i \geq 0$.

We have proved a similar theorem in section 3.1. But the hypotheses are different in that case. For the proper base change we have the (strong) properness assumption on the morphism π, while there is no condition on the torsion of the sheaf. For the smooth case we can on the other hand use a limit argument to obtain the result for arbitrary schemes over a separably closed field, but the torsion of the sheaf should be prime to the characteristic of k.

Using smooth and proper base change we can obtain the following result.

Corollary 38 (Smooth specialisation of cohomology groups)**.** Let $\pi\colon Y \to X$ be a proper and smooth morphism. Let $\mathcal{F}$ be a constructible and locally constant sheaf on $Y_{\text{ét}}$ whose torsion is prime to $\operatorname{char} X$. Then for all $i \geq 0$ the sheaves $\mathbf{R}^i \pi_*(\mathcal{F})$ are again both constructible and locally constant. If X is moreover connected, then the cohomology groups from (41) are all isomorphic.

Another result from the smooth base change theorem, which should be interpreted in terms of motives and their $\mathbb{A}^1$-homotopy invariance is the so called acyclicity of taking the affine space over a scheme. First some definitions.

Definition 39. Let $g: Y \to X$ be a morphism. Let $n \geq -1$ be an integer. Then g is *n-acyclic* if for all $X' \to X$ étale of finite type and for all torsion sheaves $\mathcal{F}$ on X' whose torsion is prime to $\operatorname{char} X$, the morphism

$$\mathrm{H}^i(X', \mathcal{F}) \to H^i(Y \times_X X', \mathcal{F}|_{Y \times_X X'}) \tag{57}$$

is an isomorphism for $i = 0, \ldots, n$ and injective for $i = n + 1$. If it is n-acyclic for all $n \geq -1$ it is called *acyclic*.

We then have the following theorem.

Theorem 40. Let X be any scheme. The canonical morphism $g: \mathbb{A}^n_X \to X$ is acyclic.

This means that we have an isomorphism

$$\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(\mathbb{A}^n_X, g^*(\mathcal{F})). \tag{58}$$

$$\begin{array}{ccc} \mathbb{A}^n_{X'} & \xrightarrow{g'} & X' \\ \downarrow & & \downarrow \\ \mathbb{A}^n_X & \xrightarrow[g]{} & X. \end{array} \tag{59}$$

Let $\mathcal{F}$ be a torsion sheaf on X' such that its torsion is prime to $\operatorname{char}(X)$. Then we can phrase the acyclicity as either the isomorphism of the cohomology groups

$$\mathrm{H}^i(X', \mathcal{F}) \cong \mathrm{H}^i(\mathbb{A}^n_{X'}, g'^*(\mathcal{F})) \tag{60}$$

or the fact that $\mathcal{F} \cong g'_* \circ g'^*(\mathcal{F})$ and $\mathbf{R}^i g'_* \circ g'^*(\mathcal{F}) = 0$ for $i \geq 1$. This yields both a motivic interpretation, and a means to prove the results from the next section.

3.4 Purity

First we define the twisting operation as known for coherent cohomology in the case of $\mathbb{Z}/n\mathbb{Z}$-modules and étale cohomology. We will denote by $(-c)$ the twisting by $-c$. First we define for a ring A by $\mu_n(A)$ its group of nth roots of unity. Then we define the Tate twists to be

$$(61)\quad \mu_n(A)^{\otimes r} := \begin{cases} \mu_n(A) \otimes \ldots \otimes \mu_n(A) & r \geq 1 \\ \mathbb{Z}/n\mathbb{Z} & r = 0 \\ \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(\mu_n(A)^{\otimes -r}, \mathbb{Z}/n\mathbb{Z}) & r \leq -1. \end{cases}$$

Then we can define the sheaf $\mathbb{Z}/n\mathbb{Z}(r)$ on $X_{\text{ét}}$ by taking

$$(62)\quad \Gamma(U, \mathbb{Z}/n\mathbb{Z}(r)) := \mu_n(\Gamma(U, \mathcal{O}_U))^{\otimes r}.$$

Finally we can define the twisting operation for every sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules by

$$(63)\quad \mathcal{F}(r) := \mathcal{F} \otimes \mathbb{Z}/n\mathbb{Z}(r).$$

Now consider the scenario

$$(64)\quad \begin{array}{ccccc} Z & \overset{i}{\hookrightarrow} & X & \overset{j}{\hookleftarrow} & U \\ & {}_{h}\searrow & \downarrow{\scriptstyle f} & \swarrow_{g} & \\ & & S & & \end{array}$$

where i is a closed immersion, j is an open immersion, $X = U \sqcup Z$ on the level of the underlying sets and f, g and h are smooth. This situation is familiar from section 2.6. Then the pair (Z, X) is called a *smooth S-pair*. If for all $s \in S$ the fiber Z_s has codimension c then we say (Z, X) has *codimension c*.

In this set-up one can prove a cohomological purity result, i.e. the cohomology groups are concentrated in degree $2c$ for the cohomology with proper support and sufficiently nice sheaves. This result is valid in a relative situation, but we will restrict ourselves to $S = \mathrm{Spec}\, k$.

Theorem 41 (Cohomological purity)**.** Let (Z, X) be a smooth S-pair of codimension c, where $S = \mathrm{Spec}\, k$. Let $\mathcal{F}$ be a locally constant torsion sheaf on $X_{\text{ét}}$ with torsion prime to $\mathrm{char}\, X$. Then

$$(65)\quad \mathrm{H}^q_Z(X, \mathcal{F}) \cong \mathrm{H}^{q-2c}(Z, \mathcal{F}(-c))$$

for all $q \geq 0$.

This result yields the following corollary, which relates the cohomology of X to the cohomologies of Z and U, mimicking the Gysin exact sequence from algebraic topology which is a special case of the Serre spectral sequence.

Corollary 42 (Gysin exact sequence)**.** Let (Z, X) be a smooth S-pair of codimension c, where $S = \mathrm{Spec}\, k$ for k any field[1]. Let $\mathcal{F}$ be a locally constant sheaf on $X_{\text{ét}}$

[1] In [Mil80] only the separably closed case is proved but it holds in general.

of $\mathbb{Z}/n\mathbb{Z}$-modules, where n is prime to $\mathrm{char}(X)$. We have the exact sequence

$$\begin{aligned}
&0 \to \mathrm{H}^{2c-1}(X,\mathcal{F}) \to \mathrm{H}^{2c-1}(U,\mathcal{F}|_U) \to \mathrm{H}^0(Z,i^*(\mathcal{F})(-c)) \xrightarrow{i_*} \dots \\
&\dots \to \mathrm{H}^{2c}(X,\mathcal{F}) \to \mathrm{H}^{2c}(U,\mathcal{F}|_U) \to \mathrm{H}^1(Z,i^*(\mathcal{F})(-c)) \xrightarrow{i_*} \dots \\
&\vdots \\
&\dots \to \mathrm{H}^{2\dim X-1}(X,\mathcal{F}) \to \mathrm{H}^{2\dim X-1}(U,\mathcal{F}|_U) \to \mathrm{H}^{2(\dim X-c)}(Z,i^*(\mathcal{F})(-c)) \xrightarrow{i_*} \dots \\
&\dots \to \mathrm{H}^{2\dim X}(X,\mathcal{F}) \to \mathrm{H}^{2\dim X}(U,\mathcal{F}|_U) \to 0
\end{aligned} \tag{66}$$

where the maps

$$i_*\colon \mathrm{H}^j(Z,i^*(\mathcal{F})(-c)) \to \mathrm{H}^{j+2c}(X,\mathcal{F}) \tag{67}$$

are the *Gysin maps*, for $j = 0,\dots,2(\dim X - c)$.

3.5 Poincaré duality

From this point on we need the full power of derived categories to state and prove our results. The derived functor version of the statement simply isn't true. This aspect isn't treated in a good way in [Mil80], one needs to resort to [$\mathrm{SGA4}_3$] to really see what's going on.

To state the theorems we need to introduce the exceptional inverse image functor $f^!$ for every morphism $f: X \to Y$ of schemes. So this is more general than the situation from section 2.6. This functor is again motivated by algebraic topology. Its definition is rather involved. We will discuss the topological case [Ive86].

Definition 43. Let $f: X \to Y$ be a morphism of topological spaces. The *direct image with proper support* $f_!$ is a functor $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ defined by setting

$$f_!(\mathcal{F})(U) := \left\{ s \in \mathcal{F}(f^{-1}(U)) \mid f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U \text{ is a proper morphism} \right\}. \tag{68}$$

We observe that f is itself a proper morphism, then $f_! = f_*$. In general we only obtain a subsheaf of the direct image sheaf.

We now apply the same idea as in the construction of the cohomoloy groups with proper support. There is again a naive construction of a derived functor for $f_!$ but we need to circumvent this as it would yield the wrong results. Nevertheless, it will be possible to construct $\mathbf{R}f_!$, the *higher direct image with proper support*. One now tries to construct a right adjoint to this functor, which is called the *exceptional inverse image*. It is either denoted $\mathbf{R}f^!$ or $f^!$. Both notations have their downsides: the first one suggests that it is the derived functor of some (nonexisting) functor, while the second doesn't indicate we are working in a derived context. We will stick to the second notation here. For its complete definition and properties, see [Ive86; $\mathrm{SGA4}_3$].

With the exceptional inverse image functor being defined we can give the "abstract" version of Poincaré duality, which is [$\mathrm{SGA4}_3$, théorème XVIII.3.2.5].

Theorem 44 (Poincaré duality)**.** Let $f: Y \to X$ be a smooth and compactifiable morphism. Denote d the (locally constant) function on Y that assigns the relative dimension of f. Let $n \geq 1$ be a function that is prime to $\mathrm{char} X$. Let $\mathcal{F}$ be a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X. Then the functor

$$\mathbf{R}f_!\colon \mathbf{D}(Y) \to \mathbf{D}(X) \tag{69}$$

admits a right adjoint

(70) $f^!\colon \mathbf{D}(X) \to \mathbf{D}(Y)$

which is given by

(71) $\mathcal{F}^\bullet \mapsto f^*(\mathcal{F}^\bullet(d)[2d])$

for $\mathcal{F}^\bullet$ in $\mathbf{D}(X)$. There is moreover a trace morphism

(72) $\mathrm{Tr}_f : \mathbf{R}f_! \circ f^!(\mathcal{F}^\bullet) = \mathbf{R}f_! \circ f^*(\mathcal{F}^\bullet(d)[2d]) \mapsto \mathcal{F}^\bullet$

with the adjunction isomorphism

(73) $\mathrm{Hom}_{\mathbf{D}(Y)}(\mathcal{G}^\bullet, f^!(\mathcal{F}^\bullet)) \cong \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{R}f_!(\mathcal{G}^\bullet), \mathcal{F}^\bullet)$

for $\mathcal{G}^\bullet$ in $\mathbf{D}(Y)$.

To state the more down-to-earth version of this duality (where we actually see the duality of certain cohomology groups) we denote the $\mathbb{Z}/n\mathbb{Z}$-dual of a locally constant and constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules by

(74) $\mathcal{F}^\wedge := \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathbb{Z}/n\mathbb{Z})$.

Corollary 45. Let k be an algebraically closed field k. Let X be a smooth and compactifiable scheme over k of pure relative dimension d. Let n be prime to $\mathrm{char}\, k$. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. Then we have a canonical isomorphism

(75) $\mathrm{H}^{2d-i}(X, \mathcal{F}^\wedge(d)) \cong \mathrm{H}^i_{\mathrm{c}}(X, \mathcal{F})^\wedge$.

3.6 Grothendieck's six operations

The adjoint functors $\mathbf{R}f_!$ and $f^!$ introduced in the previous section fit into a more general framework: Grothendieck's six operations. Next to the adjoint pair of the previous section we have the adjoint functors $\mathbf{R}f_*$ and $\mathbf{L}f^*$. When we extend these with the adjoint pair consisting of the internal Hom, denoted $\mathbf{R}\mathcal{H}\mathrm{om}$ and the internal tensor product $- \otimes^{\mathbf{L}} -$ we get an interesting "calculus" on the derived categories of sheaves on schemes, where there are compatibilities between all the operations

This calculus provides the technical means to prove (amongst others) duality results, generalizing Poincaré duality. When applied to the case of schemes we get (coherent) Grothendieck duality, when applied to locally compact topological spaces we get Verdier duality.

3.7 A few words on the literature

The main resource for étale cohomology is [SGA4$_1$; SGA4$_2$; SGA4$_3$]. Unfortunately the level of generality can be a problem. The role that Hartshorne's book on algebraic geometry plays is more or less taken by [Mil80]. It has its downsides, just like Hartshorne's book, but it provides the backbone for a study in étale cohomology.

Another interesting book is [SGA4½] which can be considered as a guide to the SGA's. Similarly, [Tam94] serves as a (slightly more arithmetical) guide to the literature. For motivations and overviews there is also [Mil12]. A recent book is [Fu11], which can be considered as an abridged English version of the SGA's. A more general reference on Grothendieck topologies, sites, sheaf cohomology, etc. is [Stacks].

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