The derived Torelli theorem

Pieter Belmans

Abstract

These are the notes for my lecture for the 2012–2013 course *Surfaces K3 et théorie des déformations* at Université Paris-Sud 11, and provide some background to the actual lecture. The goal is to discuss Fourier-Mukai transforms, and apply the obtained techniques to K3 surfaces. The main result that will be discussed is a version of the global Torelli theorem in the context of derived categories, the so called *derived Torelli theorem*, which is theorem 40. The main reference for these notes (and everything else that is related to Fourier-Mukai transforms) is the excellent book [Huy06]. It is followed closely, to add a bit of originality some of the calculations left as an exercise are done explicitly.

Contents

1 Derived categories in algebraic geometry

The subject of these notes is the notion of derived equivalence (see definition 1), applied to the case of K3 surfaces (see §3). So we should make clear what derived equivalence means, and for this we need to introduce derived categories, which are introduced in §1.1. And we will use Fourier-Mukai transforms (see §2), which requires some knowledge on the functors that can be used in the context of derived categories. This is done in §1.2.

If one is unfamiliar with derived categories, I am afraid this will not be the most gentle introduction. I will only try to give some motivation to the construction, while ignoring most of the technical issues. There are plenty of good references: [CSAM29; Kat06; GM10]. These notes are primarily based on [Huy06], which also contains a quick overview of all the required theory for the applications we have in mind here.

1.1 Derived categories

The main idea behind derived categories is to make working with derived functors more natural. Recall that given a left (or right) exact functor one can determine its derived functors, which form a family of functors. The goal is to capture all of these in one single *total derived functor*. So instead of working with the family $(\mathbf{R}^n F)_{n\in\mathbb{N}}$ one wants to construct a functor $\mathbf{R}F$ replacing the whole family.

To calculate (co)homology one uses injective (or projective, or flat, or flabby, or ...) resolutions. So instead of using a single object, it is natural to consider a whole (co)chain complex of objects. That is why, instead of using an abelian category $\mathcal{A}$ (take for example $\mathcal{A} = \mathrm{Coh}_X$ the abelian category of coherent sheaves on a scheme), one uses $\mathrm{Ch}(\mathcal{A})$: the category of (co)chain complexes over $\mathcal{A}$.

Because the calculation of (co)homology is invariant up to homotopy equivalence, we construct the category $\mathbf{K}(\mathcal{A})$ by identifying morphisms in $\mathrm{Ch}(\mathcal{A})$ which are homotopy equivalent.

The final step is the most technical one, and consists of inverting quasi-isomorphisms to obtain the *derived category*. Recall that a quasi-isomorphism is a morphism which induces isomorphisms in the (co)homology, i.e. if $f : A^\bullet \to B^\bullet$ is a morphism such that $\mathrm{H}^n(A^\bullet) \cong \mathrm{H}^n(B^\bullet)$ for all n then we would like $A^\bullet$ and $B^\bullet$ to be isomorphic in our desired derived category. This way, an object becomes isomorphic to its resolution. The way to obtain this is analogous to the localisation of a ring: we formally add inverses. That this construction works as intended follows from the Gabriel-Zisman theorem.

To summarise, the construction goes through the following steps

1. pick an abelian category $\mathcal{A}$ (Coh_X, Qcoh_X or just A-Mod if you like);
2. consider the abelian category of (co)chain complexes over $\mathcal{A}$;
3. construct the category of (co)chain complexes $\mathbf{K}(\mathcal{A})$ over $\mathcal{A}$ by identifying the homotopy equivalences in $\mathrm{Ch}(\mathcal{A})$;
4. construct the derived category $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ by inverting the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$.

Instead of considering all (co)chain complexes, we can also consider complexes which satisfy a certain boundedness assumption. We could ask for the complexes to be

1. bounded on both sides (denoted $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$),
2. bounded below or above (denoted $\mathbf{D}^{+}(\mathcal{A})$ resp. $\mathbf{D}^{-}(\mathcal{A})$),
3. concentrated in positive or negative degrees (denoted $\mathbf{D}^{\geq 0}(\mathcal{A})$ resp. $\mathbf{D}^{\leq 0}(\mathcal{A})$).

In this text, we will only care about the bounded variant. It is then important to make sure that all the constructions from algebraic geometry preserve this boundedness assumption, which is why we will put some conditions on the schemes that are used (see remark 11).

We can now say what a derived equivalence is. Given two schemes X and Y one can ask whether they are isomorphic. But associated to X and Y are the categories Coh_X (resp. Qcoh_X) and Coh_Y (resp. Qcoh_Y). One could then ask whether these categories are equivalent, which will certainly be the case if X and Y are isomorphic, but a priori there could be more general situations in which non-isomorphic schemes are derived equivalent. For a result in this direction, see proposition 31. And now that we also have derived categories, we can define our main property of interest in analogy with the foregoing discussion.

Definition 1. Let X and Y be schemes over a field k. We say X and Y are *derived equivalent* if there exists a k-linear exact equivalence between $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_X)$ and $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_Y)$.

For this definition to be of use the categories of coherent sheaves should be sufficiently well-behaved, so later on we will put some requirements on X and Y.

Remark 2. Before we introduce total derived functors in the next section, a few words on the philosophy of derived categories. In this context they should be understood as mainly a *computational tool*. Without them, certain arguments would be far more involved, while others are even impossible. A first example of the strength of derived categories is that computing the Ext of two objects in an abelian category $\mathcal{A}$ is the same as computing the Hom of these objects considered in $\mathbf{D}(\mathcal{A})$ (one being shifted). I.e. we have the isomorphism

$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A,B[n]). \tag{1}$$

For more examples of how derived categories make true what you always hoped about derived functors see [Huy06, §3]. Some of the nice properties we will need here are moreover given in §1.2.

1.2 Total derived functors

The previous discussion was rather abstract. We now specialise this to the situation we will need here: we are interested in sheaf cohomology, direct and inverse images and tensor products. By using the total derived functors of these functors we obtain *exact* functors between the derived categories. Again, it is unfortunately impossible to go into details here, [Huy06, §3.3] is on the other hand an extremely lucid exposition which deserves a reading.

Definition 3. Let $f : X \to Y$ be a morphism of noetherian schemes. The *higher direct image* $\mathbf{R}f_*$ is the total derived functor of the left exact direct image functor f_*. By [Huy06, theorem 3.23] we have that it restricts to a functor

$$\mathbf{R}f_*\colon \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_X) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_Y) \tag{2}$$

if f is a projective (or proper) morphism.

Remark 4. As a special case we have the functor $\Gamma = f_*$ for $f : X \to \operatorname{Spec} k$ projective (or proper). Then we obtain cohomology groups by the total derived functor

$$\mathbf{R}\Gamma\colon \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_X) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_{\operatorname{Spec} k}) = \mathbf{D}^{\mathrm{b}}(\mathrm{fd}\,\mathrm{Vec}_k) \tag{3}$$

where we see Serre's theorem on the finite-dimensionality of the cohomology groups pop up.

Remark 5. In the introduction we stated that derived categories should be interpreted as a computational tool. Recall that the compute the composition of derived functors one has to resort to spectral sequences, which are not the most pleasant thing to work with. But in case of the higher direct image we have the formula

$$\mathbf{R}f_* \circ \mathbf{R}g_* \cong \mathbf{R}(f \circ g)_*. \tag{4}$$

The next functor is the derived tensor product, as the tensor product is in general only right exact. To prove that it is well-defined on the bounded derived category one has furthermore to assume that X is smooth. If this is the case, one can take a finite resolution using locally free sheaves [Huy06, proposition 3.26]. Using this bounded resolution one can then compute the derived tensor product as an underived tensor product with a complex (because tensoring with a locally free sheaf is exact).

Definition 6. Let X be a smooth projective scheme over a field k. Then the *derived tensor product* is the bifunctor

$$- \otimes^{\mathbf{L}} -\colon \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_X) \times \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_X) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_X). \tag{5}$$

The last functor that we will need to define a Fourier-Mukai transform is the higher inverse image $\mathbf{L}f^*$. Recall that the inverse image f^* is the composition of f^{-1} (which is exact) and the tensor product with the structure sheaf (which is only right exact). Hence it makes sense to consider the higher inverse images.

Definition 7. Let $f : X \to Y$ be a morphism of projective schemes over a field k. Then the *higher inverse image* $\mathbf{L}f^*$ is the total derived functor

$$\mathbf{L}f^*\colon \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_Y) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_X). \tag{6}$$

Remark 8. Again we have the (now contravariant) isomorphism

$$\mathbf{L}f^* \circ \mathbf{L}g^* \cong \mathbf{L}(g \circ f)^*. \tag{7}$$

Remark 9. Another important isomorphism is the *derived projection formula* [Huy06, (3.11)], which generalises the projection formula in the non-derived case which required a locally free assumption. We will need this later on.

$$\mathbf{R}f_*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} \mathcal{F}^\bullet \cong \mathbf{R}f_*(\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathbf{L}f^*(\mathcal{F}^\bullet)). \tag{8}$$

There are three other functors on the level of derived categories, but we will not need them in this exposition. They are nevertheless required to develop the full theory of Fourier-Mukai transforms.

Remark 10. From now on we will denote

$$\mathbf{D}^{\mathrm{b}}(X) := \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}_X). \tag{9}$$

Remark 11. From now on we will always assume that X and Y (and sometimes even Z) are smooth projective varieties over a field k. Whenever we need conditions on the field k this will be made explicit.

1.3 Relationship between $\mathbf{D}^{\mathrm{b}}(X)$ and the geometry of X

Having introduced the derived category of coherent sheaves of a variety X the question becomes: what can we recover from $\mathbf{D}^{\mathrm{b}}(X)$ about the geometry of X? First of all, we get information on the dimension.

Proposition 12 ([Huy06, proposition 4.1])**.** Let X and Y be as before. If X and Y are derived equivalent then $\dim X = \dim Y$.

Admittedly, if derived equivalences couldn't recognise dimension it would not be a strong tool. Besides recognising dimension (for which there might be easier solutions), derived equivalences are also very much related to canonical bundles.

Proposition 13 ([Huy06, proposition 4.1])**.** Let X and Y be as before. If X and Y are derived equivalent then ω_X and ω_Y have the same order.

But it does not stop here. Recall that the *anticanonical bundle* of X is ω_X^{-1}, the inverse of the canonical sheaf. We then see that the true power of derived equivalences depends on ω_X and ω_X^{-1}, a result by Bondal-Orlov.

Proposition 14 ([Huy06, proposition 4.11])**.** Let X and Y be as before. Assume that either ω_X or ω_X^{-1} is ample. If X and Y are derived equivalent then X is isomorphic to Y.

Hence, derived equivalence is only an interesting notion in the case where the (anti)canonical bundle is not ample. And if X is a K3 surface we have $\omega_X = \mathcal{O}_X$, so we might (and will) get interesting results here, as derived equivalence is not the same as isomorphism. But first we discuss the easier case of curves.

Example 15. By Riemann-Roch we know that the canonical bundle of a curve of genus $g \geq 2$ is ample, while the anticanonical bundle of $\mathbb{P}^1$ is ample. Hence if the derived categories of coherent sheaves on two curves of genus $g \neq 1$ are derived equivalent then the curves themselves are isomorphic. The only remaining case is that of an elliptic curve, which has a trivial canonical bundle.

Elliptic curves are an example of abelian varieties, and these were the objects Mukai was originally interested [Muk81]. Luckily, in this one-dimensional case, where everything is determined by the Hodge decomposition of weight 1, one can prove that they too are isomorphic if and only if they are derived equivalent.

So for all genera, we have that derived equivalence is the same as isomorphism. This shows that derived equivalence is a fine-grained measure. The results for higher-dimensional varieties on the other hand will show that it is not *too* fine-grained, and allows for interesting results.

2 Fourier-Mukai transforms

We have introduced derived categories as a computational tool. Fourier-Mukai transforms are another type of computational tool, which allow for very explicit manipulations. Due to time and space constraints it is unfortunately impossible to explain how one uses Fourier-Mukai transforms in the actual study of K3 surfaces (for which we refer to [Huy06]) but it is very worthwile to introduce them.

2.1 Definition

One could describe a Fourier-Mukai transform as a "derived correspondence". In case the term correspondence doesn't ring a bell, recall its definition[1].

Definition 16. Let X and Y be as before. A *correspondence* is a cycle in $X \times_k Y$.

There is an interesting "calculus of cycles" possible, using pushforward, pullbacks and intersections [Ful98, §10]. And every morphism $f : X \to Y$ yields a correspondence by considering its graph Γ_f. Hence one can interpret cycles as "generalised morphisms". This is one of the main ideas in the construction of motives [LNMC], the idea being that there are not enough morphisms between projective varieties.

The idea of a Fourier-Mukai transform is analogous to the idea of a correspondence as a generalised morhpism. Instead of looking at $X \times_k Y$ one could consider the bounded derived category of coherent sheaves on the product, i.e. $\mathbf{D}^{\mathrm{b}}(X \times_k Y)$. If we denote

$$\begin{tikzcd} X \times_k Y \arrow[r, "q"] \arrow[d, "p"] & X \\ Y & \end{tikzcd} \tag{10}$$

then it is possible to introduce the following definition.

Definition 17. Let X and Y be as before. Let $\mathcal{P}^\bullet \in \mathrm{Obj}(\mathbf{D}^{\mathrm{b}}(X \times_k Y))$. The *Fourier-Mukai transform* $\Phi_{\mathcal{P}^\bullet}$ associated to the *Fourier-Mukai kernel* $\mathcal{P}^\bullet$ is the functor

$$\Phi_{\mathcal{P}^\bullet} : \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y) : \mathcal{E}^\bullet \mapsto \mathbf{R}p_*\left(q^*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} \mathcal{P}^\bullet\right). \tag{11}$$

Because the projection $q \colon X \times_k Y \to X$ is flat we have that $\mathbf{L}q^* = q^*$, hence we can use the non-derived version in the definition.

The necessity of using derived categories is obvious from the use of $\mathbf{R}p_*$: in general the pushforward of a coherent sheaf is no longer coherent. If we use derived categories on the other hand we see that $\mathbf{R}p_*(\mathbf{D}^{\mathrm{b}}(X)) \subseteq \mathbf{D}^{\mathrm{b}}(Y)$ [Huy06, theorem 3.23]. Hence the tensor product should also be used in its derived form. Remark that, as it is the composition of three exact functors, a Fourier-Mukai transform is again exact.

Remark 18. This is completely analogous to the definition of the morphism α_* in intersection theory [Ful98, definition 16.1.2], which for a cycle α on $X \times_k Y$ is

$$\alpha_* \colon \mathrm{CH}^\bullet(X) \to \mathrm{CH}^\bullet(Y) : \beta \mapsto p_*\left(q^*(\beta) \cdot \alpha\right) \tag{12}$$

[1] Some variations exist, this one is based on [Ful98, §1.3]. If one is more interested in motives the condition that the cycle is finite and surjective over V is used [LNMC, definition 1.1]

Fourier transform	Fourier-Mukai transform
V	X
$V^\vee$	Y
$\mathrm{L}^2(V)$	$\mathbf{D}^\mathrm{b}(X)$
$\mathrm{L}^2(V^\vee)$	$\mathbf{D}^\mathrm{b}(Y)$
f	$\mathcal{E}^\bullet$
(reinterpretation)	q^*
real multiplication	$- \otimes^\mathbf{L} -$
$\mathrm{e}^{2\pi\mathrm{i}\langle -,-\rangle}$	$\mathcal{P}^\bullet$
$\int_V - \,\mathrm{d}v$	$\mathbf{R}p_*$

Table 1: Comparison between the Fourier transform and the Fourier-Mukai transform

We can also remark that the map $\alpha^*\colon \mathrm{CH}^\bullet(Y) \to \mathrm{CH}^\bullet(X)$ defined by exchanging p and q has an analogous Fourier-Mukai transform, which is defined by

$$(13)\quad \Phi_{\mathcal{P}^\bullet}^{Y\to X} : \mathbf{D}^\mathrm{b}(Y) \to \mathbf{D}^\mathrm{b}(X) : \mathcal{E}^\bullet \mapsto \mathbf{R}q_*\left(p^*(\mathcal{E}^\bullet) \otimes^\mathbf{L} \mathcal{P}^\bullet\right)$$

where the superscript optionally indicates the direction.

Remark 19. That this functor is named after Shigeru Mukai is not remarkable, as he is the one who introduced this notion to algebraic geometry [Muk81]. That there is also a reference to the theory of Fourier transforms might seem odd in the context of algebraic geometry, but already in this first paper Mukai notices the analogy between these functors and Fourier transforms.

Recall that the Fourier transform of an integrable function $f : V \to \mathbb{R}$ where V is a finite-dimensional real vectorspace is given by

$$(14)\quad \hat{f}(\alpha) := \int_V f(v)\mathrm{e}^{2\pi\mathrm{i}\langle v,\alpha\rangle}\,\mathrm{d}v.$$

and it gives an isometry between the function spaces[2] $\mathrm{L}^2(V)$ and $\mathrm{L}^2(V^\vee)$. In this case the complex $\mathcal{E}^\bullet$ in (11) corresponds to the function f, which acts on V. The Fourier-Mukai kernel corresponds to the kernel of the Fourier transform $\mathrm{e}^{2\pi\mathrm{i}\langle -,-\rangle}$. The derived tensor product corresponds to the multiplication of functions, and integration corresponds to the derived pushforward. The pullback does not have an immediate counterpart, because the multiplication of real numbers is already defined, hence no reinterpretation is necessary. The comparison is summarised in table 1.

Remark 20. Now a word on terminology. The Fourier transform from remark 19 gives an isometry between two function spaces. More in general we can replace the kernel function to get an *integral transform*, which does not necessarily induces an isometry. The same can be done for Fourier-Mukai transforms: if it is an equivalence of categories then it is *a* Fourier-Mukai transform (hence if one uses this terminology, all Fourier-Mukai transforms are equivalences) and we call X and Y *Fourier-Mukai partners*.

But not every choice of a Fourier-Mukai kernel $\mathcal{P}^\bullet$ will yield an equivalence (just take $\mathcal{P}^\bullet = 0$), in which case it is sometimes called an *integral transform*. In this text

[2] Mukai considered first te case of an abelian variety, which admits the notion of a dual. Hence it is more similar to the case of a classical Fourier transform as depicted here.

we follow the convention from [Huy06] and we will call every functor obtained by the choice of a Fourier-Mukai kernel a Fourier-Mukai transform, even if it is not an equivalence.

2.2 Some examples

We can now discuss a few choices of the Fourier-Mukai kernel $\mathcal{P}^\bullet$ and the associated Fourier-Mukai transform.

Example 21 ([Huy06, examples 5.4(i)]). Let $X = Y$. Let $\mathcal{P}^\bullet$ be $\mathcal{O}_\Delta$, the structure sheaf of the diagonal. In that case we see that, if we write $i\colon X \xrightarrow{\sim} \Delta \subset X \times_k X$,

$$
\begin{aligned}
&\Phi_{\mathcal{O}_\Delta}(\mathcal{E}^\bullet) \\
&\qquad = \mathbf{R}p_*\left(q^*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} \mathcal{O}_\Delta\right) && \text{definition } \Phi_{\mathcal{O}_\Delta} \\
&\qquad = \mathbf{R}p_*\left(q^*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} i_*(\mathcal{O}_X)\right) && \text{definition } \mathcal{O}_\Delta \\
&\qquad \cong \mathbf{R}p_*\left(i_*\left(i^*\left(q^*(\mathcal{E}^\bullet)\right) \otimes^{\mathbf{L}} \mathcal{O}_X\right)\right) && \text{projection formula (8)} \\
&\qquad \cong \mathbf{R}(p \circ i)_* \circ (q \circ i)^*(\mathcal{E}^\bullet) && \text{simplifying} \\
&\qquad \cong \mathcal{E}^\bullet && p \circ i = q \circ i = \mathrm{id}
\end{aligned}
\tag{15}
$$

hence the Fourier-Mukai transform is nothing but $\mathrm{id}\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X)$.

Example 22 ([Huy06, examples 5.4(ii)]). Let $f: X \to Y$ be a morphism. In that case we can consider $\mathcal{P}^\bullet := \mathcal{O}_{\Gamma_f}$. We obtain the isomorphisms

$$
\begin{aligned}
\Phi^{X\to Y}_{\mathcal{O}_{\Gamma_f}} &\cong \mathbf{R}f_* \\
\Phi^{Y\to X}_{\mathcal{O}_{\Gamma_f}} &\cong \mathbf{L}f^*.
\end{aligned}
\tag{16}
$$

Let us prove the first case, the second one being similar (p and q are exchanged, so the compositions with $\mathrm{id} \times f$ make the other functor become trivial). We have

$$
\begin{aligned}
&\Phi_{\mathcal{O}_{\Gamma_f}}(\mathcal{E}^\bullet) \\
&\qquad = \mathbf{R}p_*\left(q^*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} \mathcal{O}_{\Gamma_f}\right) && \text{definition } \Phi_{\mathcal{O}_{\Gamma_f}} \\
&\qquad = \mathbf{R}p_*\left(q^*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} \mathbf{R}(\mathrm{id} \times f)_*(\mathcal{O}_X)\right) && \text{definition } \mathcal{O}_{\Gamma_f} \\
&\qquad \cong \mathbf{R}p_*\left(\mathbf{R}(\mathrm{id} \times f)_*\left(\mathbf{L}(\mathrm{id} \times f)^* \circ q^*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} \mathcal{O}_X\right)\right) && (8) \\
&\qquad \cong \mathbf{R}(p \circ (\mathrm{id} \times f))_* \circ \mathbf{L}(q \circ (\mathrm{id} \times f))^*(\mathcal{E}^\bullet) && \text{rewriting} \\
&\qquad \cong \mathbf{R}f_* \circ \mathrm{id}^*(\mathcal{E}^\bullet) && \text{simplifying} \\
&\qquad \cong \mathbf{R}f_*(\mathcal{E}^\bullet).
\end{aligned}
\tag{17}
$$

This is again an instance of the mantra that Fourier-Mukai transforms are derived correspondences, because thechoice of the cycle Γ_f yields analogous statements for the Chow rings [Ful98, proposition 16.1.1].

Example 23 ([Huy06, examples 5.4(iii)]). Let $X = Y$. Let $\mathcal{L}$ be a line bundle on X.

Let $\mathcal{P}^\bullet := i_*(\mathcal{L})$, with i as in example 21. Then we see that

$$
\begin{aligned}
&\Phi_{\mathcal{O}_{i_*(\mathcal{L})}}(\mathcal{E}^\bullet) \\
&\qquad = \mathbf{R}p_*\left(q^*(\mathcal{E}^\bullet)\otimes^{\mathbf{L}} i_*(\mathcal{L})\right) && \text{definition } \Phi_{\mathcal{O}_{i_*(\mathcal{L})}} \\
&\qquad \cong \mathbf{R}p_*\left(i_*\left(\mathbf{L}i^*\circ q^*(\mathcal{E}^\bullet)\otimes^{\mathbf{L}}\mathcal{L}\right)\right) && (8) \\
&\qquad \cong \mathbf{R}(p\circ i)_*\left((q\circ i)^*(\mathcal{E}^\bullet)\otimes^{\mathbf{L}}\mathcal{L}\right) && \text{simplifying} \\
&\qquad \cong \mathcal{E}^\bullet\otimes^{\mathbf{L}}\mathcal{L}.
\end{aligned}
\tag{18}
$$

Hence we have found an *auto-equivalence* of the category $\mathbf{D}^{\mathrm{b}}(X)$, as the Fourier-Mukai kernel $i_*(\mathcal{L}^{-1})$ will yield an inverse functor.

Example 24 ([Huy06, examples 5.4(iv)])**.** Another example of an auto-equivalence (besides the obvious example 21) can be obtained by taking $\mathcal{P}^\bullet := \mathcal{O}_\Delta[1]$. In that case we see that by moving the shift on $\mathcal{O}_\Delta[1]$ around in (15) the resulting functor is nothing but the shift on $\mathbf{D}^{\mathrm{b}}(X)$, i.e.

$$\Phi_{\mathcal{O}_\Delta[1]}(\mathcal{E}^\bullet)\cong\mathcal{E}^\bullet[1]. \tag{19}$$

One could ask the question

> What does the group of auto-equivalences of the derived category look like?

This is a difficult question, which in the case of abelian varieties has a complete answer. The case of K3 surfaces is yet not solved. See §3.5 for some discussion on this.

2.3 Properties of Fourier-Mukai transforms

As a Fourier-Mukai transform has a definition which is suited to explicit calculations one can expect that it satisfies certain properties. First of all, a Fourier-Mukai transform is exact (which is obviously not true for an arbitrary functor).

Proposition 25 ([Huy06, remark 5.3])**.** Fourier-Mukai transforms are exact.

Fourier-Mukai transforms moreover generalise the adjointness properties of the derived functors that one encounters in Grothendieck duality.

Proposition 26 ([Huy06, proposition 5.9])**.** Let $\Phi_{\mathcal{P}^\bullet}:\mathbf{D}^{\mathrm{b}}(X)\to\mathbf{D}^{\mathrm{b}}(Y)$ be a Fourier-Mukai transform. There exist a left and right adjoint to this functor, which are both Fourier-Mukai transforms, whose Fourier-Mukai kernels are given by

$$\mathcal{P}^\bullet_{\mathrm{L}} := \mathcal{P}^{\bullet,\vee}\otimes^{\mathbf{L}} p^*(\omega_Y[\dim Y]) \tag{20}$$

for the left adjoint and

$$\mathcal{P}^\bullet_{\mathrm{R}} := \mathcal{P}^{\bullet,\vee}\otimes^{\mathbf{L}} q^*(\omega_X[\dim X]) \tag{21}$$

in case of the right adjoint.

Remark 27. The dual of a complex is given by

$$\mathcal{P}^{\bullet,\vee} := \mathbf{R}\mathrm{Hom}(\mathcal{E}^\bullet,\mathcal{O}_X). \tag{22}$$

For the definition and properties of $\mathbf{R}$Hom see [Huy06, §3].

A last property is an explicit formula of the composition of Fourier-Mukai transforms.

Proposition 28 ([Huy06, proposition 5.10])**.** Let X, Y and Z be as before. Consider two Fourier-Mukai transforms $\Phi_{\mathcal{P}^\bullet}\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ and $\Phi_{\mathcal{Q}^\bullet}\colon \mathbf{D}^{\mathrm{b}}(Y) \to \mathbf{D}^{\mathrm{b}}(Z)$. The composition is again a Fourier-Mukai transform $\Phi_{\mathcal{R}^\bullet}$, whose Fourier-Mukai kernel is given by

$$\mathcal{R}^\bullet := \mathbf{R}\pi_{X,Z,*}\left(\pi_{X,Y}^*(\mathcal{P}^\bullet) \otimes^{\mathbf{L}} \pi_{Y,Z}^*(\mathcal{Q}^\bullet)\right) \tag{23}$$

where

$$\begin{tikzcd} & X \times_k Y \times_k Z \arrow[dl, "\pi_{X,Y}"'] \arrow[d, "\pi_{X,Z}"] \arrow[dr, "\pi_{Y,Z}"] & \\ X \times_k Y & X \times_k Z & Y \times_k Z. \end{tikzcd} \tag{24}$$

2.4 Bondal-Orlov theorem

The following theorem explains why Fourier-Mukai transforms are interesting functors: given some conditions on an arbitrary functor we can represent it as a Fourier-Mukai transform [Huy06, theorem 5.14].

Theorem 29 (Bondal-Orlov)**.** Let X and Y be as before. Let $F\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a fully faithful and exact functor. If there exists both a left and right adjoint to F, then there exists a complex $\mathcal{P}^\bullet \in \mathrm{Obj}(\mathbf{D}^{\mathrm{b}}(X \times_k Y))$ which is unique up to isomorphism such that $F \cong \Phi_{\mathcal{P}^\bullet}$.

Most often the theorem is used in the form of the following corollary.

Corollary 30. If, in the situation of theorem 29, F is an equivalence, then F is representable as a Fourier-Mukai transform.

The results discussed in §1.3 can now be reproved using this representability argument. And recall the discussed in remark 20 about the choice of terminology.

A first application of this result is a classical theorem by Gabriel, which gives a way of recognising isomorphic varieties through their (non-derived!) categories of coherent sheaves. There exists a proof without using Fourier-Mukai transforms, but it is a nice application of the Bondal-Orlov representability theorem.

Proposition 31 ([Huy06, corollary 5.24])**.** Let X and Y be as before. Then X and Y are isomorphic if and only if there exists an equivalence $\mathrm{Coh}(X) \cong \mathrm{Coh}(Y)$. Moreover, the equivalence is necessarily given by tensoring with a line bundle.

For more applications of Fourier-Mukai transforms we refer to [Huy06], in which

1. geometric criteria for fully faithfulness and equivalences are given [Huy06, §7];
2. the applications of derived equivalences to abelian varieties is discussed [Huy06, §9];
3. the use of Fourier-Mukai transforms in the classification of arbitrary surfaces is explained [Huy06, §12].

In the next section we will discuss the results of [Huy06, §10], where the theory is applied to K3 surfaces.

3 The derived Torelli theorem

In this section we will provide the necessary tools to state the derived Torelli theorem. The classical version of the global Torelli theorem is a way of recognising isomorphisms by using a cohomological criterion. The notion of derived equivalence is a (moderate) generalisation of isomorphism, and in the case of K3 surfaces it is possible to again give a purely cohomological criterion to decide whether two K3 surfaces are derived equivalent.

3.1 Preliminaries: the K-theoretic Fourier-Mukai transform

From now on we will assume that $k = \mathbb{C}$, as we will consider the relationship between what is happening in the derived category (where we consider sheaves for the Zariski topology) and singular cohomology (with the analytic topology). For this to make sense we need to work over the complex numbers, so using Serre's GAGA we can go back and forth between the algebraic and analytic context [GAGA]. In remark 41 the case of characteristic p is discussed. Technically speaking the assumption on the characteristic is not necessary for this paragraph, but it will be for the next. So we might as well assume it yet now.

We want to descend from derived categories to cohomology. To do so we pass through K-theory. Only a rough sketch of the construction is provided here, [Huy06, §5.2] contains the full details.

Recall that the Grothendieck group $\mathrm{K}(X)$ of a variety X is the free abelian group generated by the locally free sheaves, where we mod out the relationship

(25) $[\mathcal{E}_0] + [\mathcal{E}_2] = [\mathcal{E}_1]$

if the $\mathcal{E}_i$ fit in a short exact sequence

(26) $0 \to \mathcal{E}_0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to 0.$

As we are working on a smooth projective variety, we know that every coherent sheaf has a finite resolution of locally free sheaves [Huy06, proposition 3.26]. Hence to every coherent sheaf a well-defined element in the Grothendieck group can be associated. The tensor product moreover induces a multiplicative structure, hence we obtain a ring structure on $\mathrm{K}(X)$.

We first define the map

(27) $[-]\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathrm{K}(X) : \mathcal{F}^\bullet \mapsto [\mathcal{F}^\bullet] = \sum_{i\in\mathbb{Z}} (-1)^i [\mathcal{F}^i] = \sum_{i\in\mathbb{Z}} [\mathcal{H}^i(\mathcal{F}^\bullet)].$

Because tensoring with a locally free sheaf is exact we obtain a morphism that is compatible with both the additive and multiplicative structure.

We want to define a K-theoretic version of the Fourier-Mukai transform. As the construction of $\mathrm{K}(X)$ is contravariant, the pullback of locally free sheaves without further ado induces a map

(28) $f^*\colon \mathrm{K}(Y) \to \mathrm{K}(X).$

For the covariant functoriality of a morphism $f : X \to Y$, we have to assume it is a projective (or rather, proper) morphism, which allows us to define (using the surjectivity of $[-]$)

$$(29)\quad f_!\colon \mathrm{K}(X)\to\mathrm{K}(Y) : [\mathcal{F}^\bullet]\mapsto\sum_{i\in\mathbb{Z}}(-1)^i[\mathbf{R}f_*(\mathcal{F}^\bullet)].$$

We can then define, in complete analogy with definition 17 and remark 18

Definition 32. Let X and Y be as before, where $k=\mathbb{C}$. Let $e\in\mathrm{K}(X\times_k Y)$. The *K-theoretic Fourier-Mukai transform* Φ_e^{K} is defined by

$$(30)\quad \Phi_e^{\mathrm{K}}\colon \mathrm{K}(X)\to\mathrm{K}(Y) : f\mapsto p_!(q^*(f)\otimes e).$$

These maps are moreover compatible with $\mathbf{R}f_*$ and $\mathbf{L}f^*$, in the sense that

$$(31)\quad \begin{tikzcd} \mathbf{D}^{\mathrm{b}}(Y) \arrow[r, "\mathbf{L}f^*"] \arrow[d, "{[-]}"'] & \mathbf{D}^{\mathrm{b}}(X) \arrow[d, "{[-]}"] \\ \mathrm{K}(Y) \arrow[r, "f^*"'] & \mathrm{K}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{D}^{\mathrm{b}}(X) \arrow[r, "\mathbf{R}f_*"] \arrow[d, "{[-]}"'] & \mathbf{D}^{\mathrm{b}}(Y) \arrow[d, "{[-]}"] \\ \mathrm{K}(X) \arrow[r, "f_!"'] & \mathrm{K}(Y) \end{tikzcd}$$

commute. Hence we have that, using the assignment $\mathcal{P}^\bullet\mapsto[\mathcal{P}^\bullet]$ of the Fourier-Mukai kernel $\mathcal{P}^\bullet$ the diagram

$$(32)\quad \begin{tikzcd} \mathbf{D}^{\mathrm{b}}(X) \arrow[r, "\Phi_{\mathcal{P}^\bullet}"] \arrow[d, "{[-]}"'] & \mathbf{D}^{\mathrm{b}}(Y) \arrow[d, "{[-]}"] \\ \mathrm{K}(X) \arrow[r, "\Phi^{\mathrm{K}}_{[\mathcal{P}^\bullet]}"'] & \mathrm{K}(Y) \end{tikzcd}$$

commutes.

3.2 Preliminaries: the cohomological Fourier-Mukai transform

Having passed from $\mathbf{D}^{\mathrm{b}}(X)$ to $\mathrm{K}(X)$, we would further like to pass to $\mathrm{H}^\bullet(X,\mathbb{Q})$. Again there is a ring structure on rational cohomology.
To a morphism $f:X\to Y$ one has an associated pullback

$$(33)\quad f^*\colon \mathrm{H}^\bullet(Y,\mathbb{Q})\to\mathrm{H}^\bullet(X,\mathbb{Q}).$$

By the standing assumptions we have Poincaré duality, which allows us to define the pushforward

$$(34)\quad f_*\colon \mathrm{H}^\bullet(X,\mathbb{Q})\to\mathrm{H}^{\bullet+2\dim Y-2\dim X}(Y,\mathbb{Q})$$

as its dual. We are sufficiently used to the recurring definitions now to write

Definition 33. Let X and Y be as before, where $k=\mathbb{C}$. Let $\alpha\in\mathrm{H}^\bullet(X\times_k Y,\mathbb{Q})$. The *cohomological Fourier-Mukai transform* Φ_α^{H} is defined by

$$(35)\quad \Phi_\alpha^{\mathrm{H}}\colon \mathrm{H}^\bullet(X,\mathbb{Q})\to\mathrm{H}^\bullet(Y,\mathbb{Q}) : \beta\mapsto p_*(q^*(\beta)\cdot\alpha).$$

We still have to find a way to descend from the K-theoretic Fourier-Mukai transform to the cohomological one. This can be done using the Chern character. Recall (see for instance [Har77, §A.3 and A.4]) that it is defined as

$$(36)\quad \mathrm{ch}(\mathcal{L}) := \exp\left(\mathrm{c}_1(\mathcal{L})\right)=\sum_{i=0}^{+\infty}\frac{\mathrm{c}_1(\mathcal{L})^i}{i!}$$

for $\mathcal{L}$ a line bundle, where c_1 denotes the first Chern class, i.e. $c_1(\mathcal{L})$ is the image of $\mathcal{L}$ under the morphism

$$\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathcal{O}_X^\times) \to \mathrm{H}^2(X, \mathbb{Z}). \tag{37}$$

To define this on arbitrary locally free sheaves one has to do some work, involving fibre bundles over X and writing a locally free sheaf as successive extensions of line bundles.

To get a compatibility between the K-theoretic and cohomological Fourier-Mukai transforms one furthermore has to take the Todd class in account. This way it will be possible to get the desired commutativity. Recall that the Todd class of a line bundle $\mathcal{L}$ is defined as

$$\mathrm{td}(\mathcal{L}) = \frac{c_1(\mathcal{L})}{1 - \exp(-c_1(\mathcal{L}))} \tag{38}$$

(see e.g. [Har77, §A.4]). We then define $\mathrm{td}(X) := \mathrm{td}(\mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf. The compatibility between the Fourier-Mukai transforms results by a big theorem: Grothendieck-Riemann-Roch.

Theorem 34 ([Huy06, theorem 5.26])**.** Let X and Y be as before. Let $f : X \to Y$ be a projective morphism. Let $e \in \mathrm{K}(X)$. Then we have

$$\mathrm{ch}\left(f_!(e)\right) \cdot \mathrm{td}(Y) = f_*\left(\mathrm{ch}(e) \cdot \mathrm{td}(X)\right). \tag{39}$$

The final step in going down from K-theory (resp. derived categories) to cohomology is associating a cohomology class to each element of $\mathrm{K}(X)$ (resp. object of $\mathbf{D}^\mathrm{b}(X)$).

Definition 35. Let $e \in \mathrm{K}(X)$ (resp. $\mathcal{E}^\bullet \in \mathrm{Obj}(\mathbf{D}^\mathrm{b}(X))$). We define the *Mukai vector* of e (resp. $\mathcal{E}^\bullet$) by the cohomology class

$$\mathrm{v}(e) := \mathrm{ch}(e) \cdot \sqrt{\mathrm{td}(X)} \tag{40}$$

respectively

$$\mathrm{v}(\mathcal{E}^\bullet) := \mathrm{v}([\mathcal{E}^\bullet]) = \mathrm{ch}([\mathcal{E}^\bullet]) \cdot \sqrt{\mathrm{td}(X)}. \tag{41}$$

Hence the step from K-theory to cohomology is obtained by the morphism

$$\mathrm{v}\colon \mathrm{K}(X) \to \mathrm{H}^\bullet(X, \mathbb{Q}) \tag{42}$$

and we have the desired compatibility in the sense that

$$\begin{tikzcd} \mathrm{K}(X) \arrow[r, "\Phi_e^\mathrm{K}"] \arrow[d, "\mathrm{v}"'] & \mathrm{K}(Y) \arrow[d, "\mathrm{v}"] \\ \mathrm{H}^\bullet(X, \mathbb{Q}) \arrow[r, "\Phi_{\mathrm{v}(e)}^\mathrm{H}"'] & \mathrm{H}^\bullet(Y, \mathbb{Q}) \end{tikzcd} \tag{43}$$

commutes, or equivalently

$$\Phi_{\mathrm{v}(e)}^\mathrm{H}\left(\mathrm{ch}(f) \cdot \sqrt{\mathrm{td}(X)}\right) = \mathrm{ch}\left(\Phi_e^\mathrm{K}(f)\right) \cdot \sqrt{\mathrm{td}(Y)} \tag{44}$$

for every $e \in \mathrm{K}(X \times_k Y)$ and $f \in \mathrm{K}(X)$.

3.3 Preliminaries: the Hodge structure

The following proposition shows that, despite some drawbacks of going down to cohomology (in general the Chern character is not surjective), we have not lost too much information.

Proposition 36 ([Huy06, proposition 5.33]). Let X and Y be as before. Consider a Fourier-Mukai kernel $\mathcal{P}^\bullet \in \mathrm{Obj}(\mathbf{D}^{\mathrm{b}}(X \times_k Y))$. If this choice of kernel yields an equivalence, then the corresponding cohomological Fourier-Mukai transform

$$(45)\quad \Phi^{\mathrm{H}}_{\mathrm{v}([\mathcal{P}^\bullet])} \colon \mathrm{H}^\bullet(X,\mathbb{Q}) \to \mathrm{H}^\bullet(Y,\mathbb{Q})$$

is a bijection of $\mathbb{Q}$-vectorspaces.

Remark that in this result we have: *no* compatibility with the grading and *no* compatibility with the multiplicative structure! It is furthermore impossible to obtain this (except for trivial cases).

But there is a compatibility with the Hodge structure

$$(46)\quad \mathrm{H}^n(X,\mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X).$$

This compatibility will be related to the fact that Chern classes are of type (p,p), hence we can obtain a factorisation of the Mukai vector defined in definition 35

$$(47)\quad \begin{tikzcd} \mathrm{K}(X) \arrow[r, "\mathrm{v}"] \arrow[dr, "\mathrm{v}"'] & \bigoplus_{p\in\mathbb{Z}} \mathrm{H}^{p,p}(X) \cap \mathrm{H}^{2p}(X,\mathbb{Q}) \arrow[d] \\ & \mathrm{H}^\bullet(X,\mathbb{Q}). \end{tikzcd}$$

The compatibility with the Hodge decomposition is a bit counterintuitive, but should be interpreted in terms of the quadratic form on the cohomology [Huy06, proposition 5.44].

Proposition 37 ([Huy06, proposition 5.39]). Let X and Y be as before. Consider a Fourier-Mukai kernel $\mathcal{P}^\bullet \in \mathrm{Obj}(\mathbf{D}^{\mathrm{b}}(X))$ such that $\Phi_{\mathcal{P}^\bullet}$ is an equivalence. Then the corresponding cohomological Fourier-Mukai transform

$$(48)\quad \Phi^{\mathrm{H}}_{\mathrm{v}([\mathcal{P}^\bullet])} \colon \mathrm{H}^\bullet(X,\mathbb{Q}) \to \mathrm{H}^\bullet(Y,\mathbb{Q})$$

induces isomorphisms

$$(49)\quad \bigoplus_{p-q=i} \mathrm{H}^{p,q}(X) \cong \bigoplus_{p-q=i} \mathrm{H}^{p,q}(Y)$$

for $i = -\dim X, \ldots, \dim X$.

Remark 38. Hence if we take dimension we obtain that derived equivalence implies that

$$(50)\quad \sum_{p-q=i} \mathrm{h}^{p,q}(X) = \sum_{p-q=i} \mathrm{h}^{p,q}(Y).$$

There is a conjecture by Kontsevich that says that derived equivalence even implies

$$(51)\quad \mathrm{h}^{p,q}(X) = \mathrm{h}^{p,q}(Y).$$

It is about time that we restrict ourselves to the case of K3 surfaces, and leave the realm of glorious generality and arbitrary dimensions. The first thing we have to do is introduce a pairing on $\mathrm{H}^\bullet(X,\mathbb{Z})$. We now that for a K3 surface $\mathrm{H}^\bullet(X,\mathbb{Z})$ is concentrated in degrees 0, 2 and 4. Without further ado (the general situation requires another definition) we then define the *Mukai pairing* to be

$$\big\langle(\alpha_0,\alpha_2,\alpha_4),(\beta_0,\beta_2,\beta_4)\big\rangle \coloneqq \alpha_2\cdot\beta_2-\alpha_0\cdot\beta_4-\alpha_4\cdot\beta_0 \tag{52}$$

which is an element of $\mathrm{H}^4(X,\mathbb{Z})\cong\mathbb{Z}$. This extends the pairing on $\mathrm{H}^2(X,\mathbb{Z})$ by a certain sign, and we obtain that with this intersection pairing the cohomology $\mathrm{H}^\bullet(X,\mathbb{Z})$ is a unimodular even lattice, together with the isomorphism

$$\mathrm{H}^\bullet(X,\mathbb{Z})\cong 2(-\mathrm{E}_8)\oplus 4U. \tag{53}$$

One last thing before we can give results on K3 surfaces is a *Hodge structure of weight 2* on $\mathrm{H}^\bullet(X,\mathbb{Z})$. We extend the Hodge structure on $\mathrm{H}^2(X,\mathbb{Z})$ by setting the direct sum $\mathrm{H}^0(X,\mathbb{C})\oplus\mathrm{H}^4(X,\mathbb{C})$ to be of type $(1,1)$. We will denote this Hodge structure together with the Mukai pairing by $\tilde{\mathrm{H}}^\bullet(X,\mathbb{Z})$. This means that

$$\begin{aligned}\tilde{\mathrm{H}}^{1,1}(X)&=\mathrm{H}^0(X)\oplus\mathrm{H}^4(X)\oplus\mathrm{H}^{1,1}(X),\\ \tilde{\mathrm{H}}^{2,0}(X)&=\mathrm{H}^{2,0}(X).\end{aligned} \tag{54}$$

3.4 Derived Torelli

For completeness' and comparison's sake we recall the global Torelli theorem.

Theorem 39 (Global Torelli theorem)**.** Let X and Y be two K3 surfaces. Then X and Y are isomorphic if and only if there exists a Hodge isometry between $\mathrm{H}^2(X,\mathbb{Z})$ and $\mathrm{H}^2(Y,\mathbb{Z})$.

The derived version of this theorem, which is due to Mukai and Orlov, is the following. Again we obtain a statement in terms of the lattices in the cohomology groups.

Theorem 40 (Derived Torelli theorem)**.** Let X and Y be two K3 surfaces. Then X and Y are derived equivalent if and only if there exists a Hodge isometry between $\tilde{\mathrm{H}}^\bullet(X,\mathbb{Z})$ and $\tilde{\mathrm{H}}^\bullet(Y,\mathbb{Z})$.

Remark 41. One can of course ask himself: is derived equivalence the same as isomorphism? The Torelli theorems do not tell us whether having a Hodge isometry for H^2 is the same as having a Hodge isometry for $\tilde{\mathrm{H}}^\bullet$. Intuitively it is easier to have a Hodge isometry for $\tilde{\mathrm{H}}^\bullet$, because there are more dimensions to fiddle with, one does not necessarily have to preserve the H^2 part.

The answer is remarkable: there exist only *finitely many* non-isomorphic Fourier-Mukai partners for each K3 surface, and for every $N\geq 1$ there exists a K3 surface X such that it has at least N non-isomorphic Fourier-Mukai partners [Ogu02]. What might even be more remarkable is that the proof depends on a result on "almost primes" in analytic number theory.

3.5 Auto-equivalences of K3 surfaces

Recall from §1.3 that in case of an ample (anti)canonical bundle things become rather trivial. The same applies to the auto-equivalences of $\mathbf{D}^{\mathrm{b}}(X)$, which have an explicit description. In §2.2 we have seen examples of auto-equivalences. We obtain that these examples are all the possibilities we have if ω_X or ω_X^{-1} is ample (and derived equivalence corresponds to isomorphism).

Proposition 42 ([Huy06, proposition 4.17])**.** Let X be as before. Assume that it has an ample (anti)canonical bundle. Then we have

$$\text{(55)}\quad \mathrm{Aut}\left(\mathbf{D}^{\mathrm{b}}(X)\right) \cong \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)).$$

The case that originally interested Mukai was that of abelian varieties, where all the information can be obtained from the H^1. This makes the arguments required to study the derived category and its auto-equivalences tractable. For the proofs of this result, and the required constructions (e.g. the dual of an abelian variety) we refer to [Huy06, §9], to be able to give the final statement we repeat [Huy06, definition 9.46].

Definition 43. Let A be an abelian variety. We define the subgroup $\mathrm{U}(A)$ of $\mathrm{Aut}(A\times\widehat{A})$ by

$$\text{(56)}\quad \mathrm{U}(A) := \left\{ f \in \mathrm{Aut}(A \times \widehat{A}) \mid f^{-1} = \widetilde{f} \right\}$$

where $\widetilde{f}$ is obtained by first decomposing f as

$$\text{(57)}\quad f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} : A \times \widehat{A} \to A \times \widehat{A}$$

and then shuffling this to

$$\text{(58)}\quad \widetilde{f} := \begin{pmatrix} \widehat{f_4} & -\widehat{f_2} \\ -\widehat{f_3} & \widehat{f_1} \end{pmatrix} : A \times \widehat{A} \to A \times \widehat{A}.$$

This yields the following structural result

Proposition 44 ([Huy06, corollary 9.57])**.** Let A be an abelian variety. We have a short exact sequence

$$\text{(59)}\quad 1 \to \mathbb{Z} \oplus (A \times \widehat{A}) \to \mathrm{Aut}\left(\mathbf{D}^{\mathrm{b}}(A)\right) \to \mathrm{U}(A) \to 1.$$

To obtain this result one uses a “derived Torelli” for abelian varieties, which gives a cohomological criterion for derived equivalences [Huy06, corollary 9.50]. Another very strong result that can be obtained for abelian varieties is that every derived equivalence can be written as a Fourier-Mukai transform whose kernel is the shift of a single sheaf (and not a whole complex) [Huy06, proposition 9.53].

In the case of K3 surfaces we can try to get the same structural description, but there are certain places where the lack of an analogue of the strong results for abelian varieties poses a problem. More specifically, the main issue in obtaining a full description of $\mathrm{Aut}(\mathbf{D}^{\mathrm{b}}(X))$ for X a K3 surface is the lack of a good understanding of

$$\text{(60)}\quad \ker\left(\mathrm{Aut}\left(\mathbf{D}^{\mathrm{b}}(X)\right) \to \mathrm{Aut}(\tilde{\mathrm{H}}^{\bullet}(X,\mathbb{Z}))\right).$$

Remark 45. Further reading material could be [Huy05]. This overview article discusses several variants of the global Torelli theorem. The classical and derived ones are familiar now, but there exists statements of these in the context of *twisted* coherent sheaves. If one likes gerbes, Azumaya algebras, Brauer groups and the like this is the place to go for your favourite statements on K3 surfaces in a twisted context.

Another interesting article is the application of derived equivalences in characteristic p. The lattice-theoretic results relying on singular homology in the characteristic 0 case do not generalise immediately to the case of characteristic p, so it is interesting to see that things have such the nice lifting discussed in this article [LO11].

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