On non-quadratic 4-dimensional Artin–Schelter regular algebras and 3-folds

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Abstract

In this short note we prove that non-quadratic 4-dimensional Artin–Schelter regular algebras do not give rise to smooth projective 3-folds. This is in contrast with the 3-dimensional case, where the quadric surface can be described using a cubic Artin–Schelter regular algebra, and therefore they give rise to noncommutative quadrics.

1 Introduction

In [1] Artin and Schelter introduced a class of noncommutative graded algebras whose homological and ring-theoretical properties mimick those of the usual polynomial algebra. They have become one of the most important objects in noncommutative algebraic geometry: given the correct Hilbert series they provide a noncommutative version of $\mathbb{P}^n$. They are defined as follows.

Definition 1. Let A be a connected graded k-algebra. We say it is *Artin–Schelter regular of dimension d* if

1. $\operatorname{gldim} A = d$
2. $\operatorname{GKdim} < +\infty$
3. A is Gorenstein, i.e.

$$\operatorname{Ext}^i_A(k, A) \cong \begin{cases} 0 & i \neq d \\ k(\ell) & i = d, \end{cases} \tag{1}$$

   where ℓ is the *Gorenstein parameter* of A.

Throughout this note we will assume moreover that all graded algebras are generated in degree 1, and that they are noetherian. It then follows for $d \leq 4$ that they are domains [2, theorem 3.9].

To a sufficiently nice graded algebra one associates its *noncommutative projective scheme* [4], which is the abelian category $\operatorname{qgr} A$, defined as the Serre quotient of the category of graded modules modulo the finite-dimensional graded modules. If A is a homogeneous coordinate ring for a projective variety X, then $\operatorname{coh} X \simeq \operatorname{qgr} A$.

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Artin–Schelter regular algebras have been classified in low dimensions. In dimension 2 there are only two types of algebras [1], the skew polynomial algebras $k\langle x,y\rangle/(xy-qyx)$ for some $q \in k^\times$ and the Jordan plane $k\langle x,y\rangle/(xy-yx-x^2)$, all of which have the same Hilbert series as $k[x,y]$. Moreover they are all twisted homogeneous coordinate rings of $\mathbb{P}^1$, and hence $\operatorname{qgr} A \simeq \operatorname{coh}\mathbb{P}^1$.

In dimension 3 there exists a complete classification of Artin–Schelter regular algebras [3]. An important feature is that there are now *two* possible Hilbert series. There now exist many algebras for which there is no smooth projective surface S such that $\operatorname{coh} S \simeq \operatorname{qgr} A$, so they truly give rise to noncommutative surfaces. But we do have the following equivalences

$$\begin{aligned} \operatorname{coh}\mathbb{P}^2 &\simeq \operatorname{qgr} k[x,y,z], \\ \operatorname{coh}\mathbb{P}^1\times\mathbb{P}^1 &\simeq \operatorname{qgr} k\langle x,y\rangle/(x^2y-yx^2, xy^2-y^2x), \end{aligned} \tag{2}$$

where $k[x,y,z]$ (resp. $k[x,y]/(x^2y-yx^2,xy^2-y^2x)$) is a quadratic (resp. cubic) Artin–Schelter regular algebra. The other algebras in the classification are therefore named *noncommutative planes* and *noncommutative quadrics*, depending on their Hilbert series.

Observe that $k[x,y,z,w]/(xy-zw)$ is *not* an Artin–Schelter regular algebra, because it has infinite global dimension.

In dimension 4 the complete classification is still far from finished. But given that $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$ arise using 3-dimensional Artin–Schelter regular algebras, one can ask whether all classes of 4-dimensional Artin–Schelter algebras give rise to (noncommutative) deformations of certain smooth projective 3-folds. This would give a geometric interpretation of these algebras like in the 3-dimensional case.

For the quadratic case there is the obvious equivalence $\operatorname{qgr} k[x,y,z,w] \simeq \operatorname{coh}\mathbb{P}^3$. The following theorem shows that there no such interpretation is possible for the non-quadratic cases, and this is the main result of this note.

Theorem 2. Let A be a non-quadratic 4-dimensional Artin–Schelter regular algebra. Then there does not exist a smooth projective 3-fold X such that $\operatorname{qgr} A \simeq \operatorname{coh} X$.

In particular, it does not make sense to call the noncommutative projective scheme associated to a non-quadratic 4-dimensional Artin–Schelter regular algebra a *noncommutative "something"*, where "something" stands for a smooth projective 3-fold.

The proof is given in section 2. We compute both the Hochschild homology of any 3-fold using the Hochschild–Kostant–Rosenberg decomposition, and the Hochschild homology of $\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A)$ using noncommutative motives, and observe that the former is always even-dimensional whilst the latter has dimension 5 or 7 in the non-quadratic case.

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2 Proof

Hochschild homology on the noncommutative side The following result is a generalisation of [1, theorem 1.5(i)], and shows that the "finite over center" condition in proposition 1.20 of op. cit. can be removed. It is taken from [6, proposition 1.4].

Lemma 3. Let A be a 4-dimensional Artin–Schelter regular algebra. Then the Hilbert series of A is one of the following:

1. $\mathrm{h}_A(t) = \dfrac{1}{(1-t)^4}$, so A has 4 generators;
2. $\mathrm{h}_A(t) = \dfrac{1}{(1-t)^3(1-t^2)}$, so A has 3 generators;
3. $\mathrm{h}_A(t) = \dfrac{1}{(1-t)^2(1-t^2)(1-t^3)}$, so A has 2 generators.

Example 4. The commutative polynomial ring $k[x,y,z,w]$ is 4-dimensional Artin–Schelter regular, and has Hilbert series of type 1. This is the quadratic case, which is excluded in theorem 2. Central extensions of 3-dimensional cubic Artin–Schelter regular algebras give rise to the Hilbert series of type 2.

From the Hilbert series we can read off the Gorenstein parameter of an Artin–Schelter regular algebra as explained in [8, remark 1.17]: we have that $\ell = \deg \mathrm{h}_A(t)^{-1}$. By Orlov's celebrated result we obtain a full and strong exceptional collection of length ℓ in $\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A)$ [7, corollary 18], and as in [8, theorem 1.16] this allows us to compute the additive invariants of this derived category. We obtain the following corollary when we specialise to Hochschild homology.

Corollary 5. Let A be a 4-dimensional Artin–Schelter regular algebra. Then

$$\dim_k \mathrm{HH}_\bullet(\mathbf{D}^{\mathrm{b}}(\operatorname{qgr} A)) = \begin{cases} 4 & \text{if } A \text{ has 4 generators,} \\ 5 & \text{if } A \text{ has 3 generators,} \\ 7 & \text{if } A \text{ has 2 generators.} \end{cases} \tag{3}$$

Proof. This follows from the discussion above and lemma 3. □

Hochschild homology on the commutative side On the other hand, for a smooth projective 3-fold we easily obtain the following restriction on the dimension of the Hochschild homology by the symmetry in the Hodge diamond. Recall that the Hodge numbers are defined as $\mathrm{h}^{p,q}(X) := \dim_k \mathrm{H}^q(X, \Omega_X^q)$ and that the Hochschild homology of X (which agrees with the Hochschild homology of $\mathbf{D}^{\mathrm{b}}(X)$) can be described using the Hochschild–Kostant–Rosenberg decomposition as

$$\mathrm{HH}_i(X) \cong \bigoplus_{p-q=i} \mathrm{H}^q(X, \Omega_X^p). \tag{4}$$

In particular, the dimension of $\mathrm{HH}_\bullet(X)$ is the sum of all the Hodge numbers.

Lemma 6. Let X be a smooth projective variety of odd dimension. Then $\mathrm{HH}_\bullet(X)$ (as an ungraded vector space) is even-dimensional.

Proof. By Serre duality we have that $\mathrm{h}^{p,q}(X) = \mathrm{h}^{n-p,n-q}(X)$, where $n = \dim X$. There is an even number of Hodge numbers, and there are no fix points under the Serre involution, so their sum is even. □

The proof of theorem 2 is now complete, as an equivalence $\operatorname{qgr} A \simeq \operatorname{coh} X$ would give an isomorphism of Hochschild homologies.

References

[1] Michael Artin and William Schelter. “Graded algebras of global dimension 3”. In: *Advances in Mathematics* 66.2 (1987), pp. 171–216.

[2] Michael Artin, John Tate, and Michel Van den Bergh. “Modules over regular algebras of dimension 3”. In: *Inventiones Mathematicae* 106.2 (1991), pp. 335–388.

[3] Michael Artin, John Tate, and Michel Van den Bergh. “Some algebras associated to automorphisms of elliptic curves”. In: *The Grothendieck Festschrift, Vol. I.* Vol. 86. Progr. Math. Birkhäuser, 1990, pp. 33–85.

[4] Michael Artin and James Zhang. “Noncommutative projective schemes”. In: *Advances in Mathematics* 109.2 (1994), pp. 228–287.

[5] Andrew Harder and Ludmil Katzarkov. *Perverse sheaves of categories and some applications.* arXiv: `1708.01181v1 [math.AG]`.

[6] Di Ming Lu, John Palmieri, Quan Shui Wu, and James Zhang. “Regular algebras of dimension 4 and their A_∞-Ext-algebras”. In: *Duke Mathematical Journal* 137.3 (2007), pp. 537–584.

[7] Dmitri Orlov. “Derived categories of coherent sheaves and triangulated categories of singularities”. In: *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II.* Vol. 270. Progr. Math. Birkhäuser, 2009, pp. 503–531.

[8] Goncalo Tabuada. *Invariants of noncommutative projective schemes.* arXiv: `1702.04712v2 [math.AG]`.