Advanced topics in algebra: Hochschild (co)homology, and the Hochschild–Kostant–Rosenberg decomposition

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Introduction

What you are reading now are the lecture notes for a course on Hochschild (co)homology, taught at the University of Bonn, in the Sommersemester of 2017–2018. They are currently being written, and regularly updated. The table of contents is provisional.

The goal of the course is to give an introduction to Hochschild (co)homology, focussing on

1. its applications in *deformation theory* of algebras (and schemes)
2. and the role of the *Hochschild–Kostant–Rosenberg* decomposition in all this.

There are by many several texts on various aspects of Hochschild (co)homology. In particular the following books dedicate some chapters on Hochschild (co)homology:

1. chapter IX in Cartain–Eilenberg's *Homological algebra* [4],
2. the first chapters of Loday's *Cyclic homology* [11],
3. chapter 9 of Weibel's *An introduction to homological algebra* [14],
4. chapter 2 by Tsygan in Cuntz–Skandalis–Tsygan's *Cyclic homology in noncommutative geometry* [5],
5. chapter II by Schedler in the Bellamy–Rogalski–Schedler–Stafford–Wemyss' *Noncommutative algebraic geometry* [1].

There are also the following unpublished lecture notes:

1. Ginzburg's *Lectures on noncommutative geometry* [6]
2. Kaledin's Tokyo lectures [8] and Seoul lectures [9].

There is also Witherspoon's textbook-in-progress called *An introduction to Hochschild cohomology* [15], which is dedicated entirely to Hochschild cohomology and some its applications. So far this is the only textbook dedicated entirely to Hochschild (co)homology, and it is a good reference for things not covered in these notes.

Compared to the existing texts these notes aim to focus more on Hochschild (co)homology in algebraic geometry, using derived categories of smooth projective varieties. This point of view has been developed in several papers [2, 3, 10] and applied in many more dealing with semiorthogonal decompositions. But there is no comprehensive treatment, let alone starting from the basics of Hochschild (co)homology for algebras. These notes aim to fill this gap, where we start focussing on smooth projective varieties starting in the second half of chapter 2.

Now that we know what is supposed to go in this text, let us mention that the following will not be discussed: the relationship with algebraic K-theory via Chern characters, support varieties, deformation theory of abelian and dg categories, applications to Hopf algebras, topological versions of Hochschild (co)homology and related constructions, …

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Contents

Chapter 1

Algebras

Conventions Throughout these notes we will let k be a field. It is possible to develop much of the theory in the case for algebras which are flat over a commutative base ring without much extra effort, but we will not do so explicitly. The interested reader is invited to do so. There are also versions which are valid in a more general setting, but will refrain from discussing these.

At some points we will take k of characteristic zero, or algebraically closed. This will be mentioned explicitly.

If A is a k-algebra we will denote the *enveloping algebra* $A \otimes A^{\mathrm{opp}}$ of A by A^{e}, so that A-bimodules are the same as left A^{e}-modules.

1.1 Definition and first properties

1.1.1 Hochschild (co)chain complexes

We start with a seemingly ad hoc definition.

Definition 1. Let A be a k-algebra. The *bar complex* $\mathrm{C}_\bullet^{\mathrm{bar}}(A)$ of A is the cochain complex

$$(1.1)\quad \ldots \xrightarrow{\mathrm{d}_2} A \otimes_k A \otimes_k A \xrightarrow{\mathrm{d}_1} A \otimes_k A \to 0,$$

of A-bimodules, where we have $\mathrm{C}_n^{\mathrm{bar}}(A) := A^{\otimes n+2}$, hence $A \otimes_k A$ lives in degree 0, and the differentials $\mathrm{d}_n \colon \mathrm{C}_n^{\mathrm{bar}}(A) \to \mathrm{C}_{n-1}^{\mathrm{bar}}(A)$ are given by

$$(1.2)\quad \mathrm{d}_n(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}.$$

The A-bimodule structure (or equivalently left A^{e}-module structure) on $\mathrm{C}_n^{\mathrm{bar}}(A)$ is given by

$$(1.3)\quad (a \otimes b) \cdot (a_0 \otimes \ldots \otimes a_{n+1}) = aa_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}b.$$

We will also consider the morphism $\mathrm{d}_0 \colon A \otimes_k A \to A$, which by the formula for d_n is nothing but the multiplication morphism $\mu \colon A \otimes_k A \to A$.

Remark 2. The terminology "bar complex" originates from the fact that an element $a_0 \otimes \ldots \otimes a_{n+1}$ is sometimes denoted $a_0[a_1|\ldots|a_n]a_{n+1}$.

Before we start studying the bar complex (for instance, at this point we haven't proven it is a complex), we introduce the following morphisms:

$$s_n\colon A^{\otimes n+2} \to A^{\otimes n+3} : a_0 \otimes \ldots \otimes a_{n+1} \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_{n+1}. \tag{1.4}$$

Given that this is the first proof we will give details. We will see many similar proofs throughout the beginning of the notes, we will leave some of them as exercises.

Lemma 3. We have that

$$\mathrm{d}_{n+1} \circ s_n + s_{n-1} \circ \mathrm{d}_n = \mathrm{id}_{A^{\otimes n+2}}. \tag{1.5}$$

Proof. One computes that

$$\begin{aligned} &s_{n-1} \circ \mathrm{d}_n(a_0 \otimes \ldots \otimes a_n) \\ &\qquad = \sum_{i=0}^{n} (-1)^i 1 \otimes a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}, \\ &\mathrm{d}_{n+1} \circ s_n(a_0 \otimes \ldots \otimes a_n) \\ &\qquad = a_0 \otimes \ldots \otimes a_{n+1} + \sum_{i=1}^{n+1} (-1)^i 1 \otimes a_0 \otimes \ldots \otimes a_{i-1} a_i \otimes \ldots \otimes a_{n+1}, \end{aligned} \tag{1.6}$$

so everything but the identity cancels after reindexing. □

We can check that the d_i's indeed turn $\mathrm{C}^{\mathrm{bar}}_\bullet(A)$ into a chain complex.

Lemma 4. We have that $\mathrm{d}_{n-1} \circ \mathrm{d}_n = 0$.

Proof. Let us consider $n = 1$ first. Then $\mathrm{d}_0 \circ \mathrm{d}_1(a_0 \otimes a_1 \otimes a_2) = (a_0a_1)a_2 - a_0(a_1a_2)$, which is zero as A is associative.

For $n \geq 2$ we use induction, using (1.5). We have

$$\mathrm{d}_n \circ \mathrm{d}_{n+1} \circ s_n = \mathrm{d}_n - \mathrm{d}_n \circ s_{n-1} \circ \mathrm{d}_n = s_{n-2} \circ \mathrm{d}_{n-1} \circ \mathrm{d}_n = 0, \tag{1.7}$$

but as the image of s_n generates $A^{\otimes n+3}$ as a left A-module we get that $\mathrm{d}_n \circ \mathrm{d}_{n+1} = 0$. □

The bar complex didn't include A, but if we use the morphism $\mathrm{d}_0\colon A \otimes_k A \to A$ as defined above we get the following proposition.

Proposition 5. The bar complex of A is a free resolution of A as an A-bimodule, where the augmentation $\mathrm{d}_0\colon A \otimes_k A \to A$ is given by the multiplication.

Proof. By lemma 3 we see that the s_i's provides a contracting homotopy, hence the bar complex is exact, as a complex of A-bimodules.

We also check that the cokernel of d_1 is indeed the multiplication $A \otimes_k A \to A$. For this it suffices to observe that

$$\mathrm{d}_1(a_0 \otimes a_1 \otimes a_2) = a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2, \tag{1.8}$$

and that there exists a map $\operatorname{coker} \mathrm{d}_1 \to A$ mapping the class of $a_0 \otimes a_1$ to a_0a_1. By the definition of d_1 it sends elements of $\operatorname{im} \mathrm{d}_1$ to zero, so it is well-defined. Its inverse is given by the morphism which sends a to $1 \otimes a$.

That $\mathrm{C}_n^{\mathrm{bar}}(A)$ is free as an A-bimodule follows from the isomorphisms of A-bimodules

$$A^{\otimes n+2} \cong A^{\mathrm{e}} \otimes A^{\otimes n} \cong \bigoplus_{i \in I} A^{\mathrm{e}} \cdot 1 \otimes 1 \otimes a_i \tag{1.9}$$

where $\{a_i \mid i \in I\}$ is a vector space basis of $A^{\otimes n}$, and the first isomorphism is

$$a_0 \otimes \ldots \otimes a_{n+1} \mapsto (a_0 \otimes a_{n+1}) \otimes a_1 \otimes \ldots \otimes a_n. \tag{1.10}$$

□

Definition 6. Let A be a k-algebra, and M an A-bimodule. The *Hochschild chain complex* $\mathrm{C}_\bullet(A, M)$ is $M \otimes_{A^{\mathrm{e}}} \mathrm{C}_\bullet^{\mathrm{bar}}(A)$, considered as a complex of k-modules, with differential $\mathrm{id}_M \otimes \mathrm{d}_n$.

Its homology is the *Hochschild homology of A with values in M*, and will be denoted $\mathrm{HH}_\bullet(A, M)$. If $M = A$, we'll write $\mathrm{HH}_n(A)$.

Dual to this we could instead of the tensor product use the Hom-functor, and obtain the dual notion of Hochschild cohomology.

Definition 7. Let A be a k-algebra, and M an A-bimodule. The *Hochschild cochain complex* $\mathrm{C}^\bullet(A, M)$ is $\mathrm{Hom}_{A^{\mathrm{e}}}(\mathrm{C}_\bullet^{\mathrm{bar}}(A), M)$, considered as a complex of k-modules, with differential $\mathrm{Hom}(\mathrm{d}_n, \mathrm{id}_M)$.

Its cohomology is the *Hochschild cohomology of A with values in M*, and will be denoted $\mathrm{HH}^\bullet(A, M)$. If $M = A$, we'll write $\mathrm{HH}^n(A)$.

Remark 8. Observe that one can recover the bar complex from the Hochschild complex:

$$\mathrm{C}_\bullet^{\mathrm{bar}}(A) = \mathrm{C}_\bullet(A, A^{\mathrm{e}}). \tag{1.11}$$

Reinterpreting the Hochschild cochain complex The Hochschild (co)chain complexes were obtained by considering a specific free resolution of A as an A-bimodule, and constructing a (co)chain complex of vector spaces out of it. We can rephrase this complex of vector spaces a bit, where instead of $\mathrm{Hom}_{A^{\mathrm{e}}}(-,-)$ and $- \otimes_{A^{\mathrm{e}}} -$, we use $\mathrm{Hom}_k(-,-)$ and $- \otimes_k -$. This will be very useful for computations later on.

The proofs of the following two propositions follow from the fact that A^{e} only involves the first and last tensor factor of a bimodule in the bar complex. The explicit formula for the Hochschild differentials in (1.15) and (1.19) will be important for us in section 1.1.3.

Proposition 9. There exists an isomorphism of k-modules

$$\varphi\colon\ \mathrm{C}^n(A, M) \xrightarrow{\cong} \mathrm{Hom}_k(A^{\otimes n}, M), \tag{1.12}$$

given by

$$g \mapsto \left[a_1 \otimes \ldots \otimes a_n \mapsto g(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)\right], \tag{1.13}$$

whose inverse is given by

$$f \mapsto \left[a_0 \otimes \ldots \otimes a_{n+1} \mapsto a_0 f(a_1 \otimes \ldots \otimes a_n) a_{n+1}\right]. \tag{1.14}$$

The differential in $\mathrm{Hom}_k(A^\bullet, M)$ is then given by

$$\begin{aligned} &\mathrm{d}_{\mathrm{Hoch}} f(a_1 \otimes \ldots \otimes a_{n+1}) \\ &\quad = a_1 f(a_2 \otimes \ldots \otimes a_{n+1}) \\ &\qquad + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\ &\qquad + (-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n) a_{n+1} \end{aligned} \tag{1.15}$$

for $f \in \operatorname{Hom}_k(A^{\otimes n}, M)$.

Proposition 10. There exists an isomorphism of k-modules

$$(1.16)\quad \psi\colon\ \mathrm{C}_\bullet(A,M) \xrightarrow{\cong} M \otimes_k A^\bullet$$

given by

$$(1.17)\quad \psi(m \otimes_{A^e} a_0 \otimes \ldots \otimes a_{n+1}) = a_{n+1} m a_0 \otimes a_1 \otimes \ldots \otimes a_n,$$

whose inverse is given by

$$(1.18)\quad m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes m \otimes_{A^e} 1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1.$$

The differential $\mathrm{d}_{\mathrm{Hoch}}\colon M \otimes_k A^{\otimes n} \to M \otimes_k A^{\otimes n-1}$ is then given by

$$(1.19)\quad \begin{aligned} &\mathrm{d}_{\mathrm{Hoch}}(m \otimes a_1 \otimes \ldots \otimes a_n) \\ &\quad = m a_1 \otimes \ldots \otimes a_n \\ &\qquad + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ &\qquad + (-1)^n a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1} \end{aligned}$$

for $m \otimes a_1 \otimes \ldots \otimes a_n \in M \otimes_k A^{\otimes n}$.

Functoriality of Hochschild (co)homology Given an algebra morphism $f\colon A \to B$, or a bimodule morphism $g\colon M \to N$, we would like to understand how this interacts with taking Hochschild (co)homology. First of all: *Hochschild homology is covariantly functorial in both arguments.*

Proposition 11. Let $f\colon A \to B$ be an algebra morphism, and M a B-bimodule (which has an induced A-bimodule structure, denoted $f^*(M)$). Then

$$(1.20)\quad f_*\colon\ \mathrm{C}_\bullet(A, f^*(M)) \to \mathrm{C}_\bullet(B,M) : m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes f(a_1) \otimes \ldots \otimes f(a_n)$$

gives a functor $\mathrm{HH}_\bullet(-, M)$.

Let $g\colon M \to N$ be an A-bimodule morphism. Then

$$(1.21)\quad g_*\colon\ \mathrm{C}_\bullet(A,M) \to \mathrm{C}_\bullet(A,N) : m \otimes a_1 \otimes \ldots \otimes a_n \mapsto g(m) \otimes a_1 \otimes \ldots \otimes a_n$$

gives a functor $\mathrm{HH}_\bullet(A, -)$.

In particular, taking $M = A$ we can use the covariant functoriality in both arguments for Hochschild homology to get the following.

Corollary 12. Hochschild homology $\mathrm{HH}_\bullet(-)$ is a covariant functor from the category of associative k-algebras to the category of k-modules.

For Hochschild cohomology the situation is different: *Hochschild cohomology is contravariantly functorial in the first argument, and covariantly functorial in the second.*

Proposition 13. Let $f\colon A \to B$ be an algebra morphism, and M a B-bimodule (which has an induced A-bimodule structure). Then

$$(1.22)\quad f^*\colon\ \mathrm{C}^n(B,M) \to \mathrm{C}^n(A,M) : \varphi \mapsto \varphi \circ f^{\otimes n}$$

gives a (contravariant) functor $\mathrm{HH}^\bullet(-, M)$.

Let $g\colon M \to N$ be an A-bimodule morphism. Then

$$g_*\colon \mathrm{C}^n(A, M) \to \mathrm{C}^n(A, N) : \varphi \mapsto g \circ \varphi \tag{1.23}$$

gives a functor $\mathrm{HH}^\bullet(A, -)$.

Remark 14. So $\mathrm{HH}^\bullet(-)$ is *not* a functor (at least when we consider arbitrary morphisms between k-algebras), despite its appearance. We will come back to this in remark 20, and we will partially remedy this deficiency in section 2.3.

At this point it is also important that in some sources it is written that $\mathrm{HH}^\bullet(-)$ *is* a functor, see e.g. [12, §1.5.4]. But this is not the same functor, despite the similarity in notation! Indeed, in those situations one takes $M = A^\vee = \mathrm{Hom}_k(A, k)$ as the second argument. This makes the construction functorial (as the covariant functor in the second argument becomes contravariant), but one does not obtain the interpretation of Hochschild cohomology which will be used in this text. The construction in op. cit. has applications in studying cyclic cohomology and generalisations of the Chern character, which we will not go into here.

In section 2.3 we will greatly extend this functoriality for Hochschild homology, and discuss what can be done in the case of Hochschild cohomology. Remark that in the next section's corollary 16 we will obtain that Hochschild cohomology is a functor for Morita equivalences.

1.1.2 Hochschild (co)homology as Ext and Tor

In these notes we have *defined* Hochschild (co)homology as the (co)homology of an explicit (co)chain complex, which might seem ad hoc at first. But the bar complex of A being a free resolution of A as a bimodule over itself allows us to interpret Hochschild (co)homology in terms of more familiar constructions as explained in section 1.1.3.

Morover, the definition via the bar complex gives us an explicit description which will prove to be very useful in section 1.2 when we are discussing the extra structure on the Hochschild (co)chain complexes, which can conveniently be described by extra structure before taking cohomology. But it is of course an interesting question to find good intrinsic descriptions of the extra structure, and we will give further comments on this.

Theorem 15. There exist isomorphisms

$$\mathrm{HH}^i(A, M) \cong \mathrm{Ext}^i_{A^e}(A, M) \tag{1.24}$$

and

$$\mathrm{HH}_i(A, M) \cong \mathrm{Tor}_i^{A^e}(A, M). \tag{1.25}$$

Proof. By proposition 5 the bar complex is a free resolution of A as an A-bimodule. In particular it can serve as a flat (resp. projective) resolution when computing the derived functors of $A \otimes_{A^e} -$ (resp. $\mathrm{Hom}_{A^e}(A, -)$). □

In particular, we have that

$$\begin{aligned} \mathrm{HH}^0(A, M) &\cong \mathrm{Hom}_{A^e}(A, M), \\ \mathrm{HH}_0(A, M) &\cong M \otimes_{A^e} A. \end{aligned} \tag{1.26}$$

But these descriptions are not necessarily very illuminating at this point. In section 1.1.3 we will give more concrete interpretations.

An important observation using theorem 15 is that the Hochschild cohomology of the A-bimodule M only depends on the category of A-bimodules. In this generality it is due to Rickard [13].

Corollary 16. Hochschild (co)homology is Morita invariant.

Proof. Assume that A and B are Morita equivalent through the bimodules ${}_AP_B$ and ${}_BQ_A$. The equivalences of categories are given by $P \otimes_A -$ and $Q \otimes_B -$, and these functors preserve projective resolutions. We obtain isomorphisms

$$\begin{aligned}
&\mathrm{Ext}^n_A(P \otimes_B -, -) \cong \mathrm{Ext}^n_B(-, Q \otimes_A -)\\
&\mathrm{Ext}^n_A(-, P \otimes_B -) \cong \mathrm{Ext}^n_B(Q \otimes_A -, -)\\
&\mathrm{Tor}^A_n(P \otimes_B -, -) \cong \mathrm{Tor}^B_n(-, Q \otimes_A -)\\
&\mathrm{Tor}^A_n(-, P \otimes_B -) \cong \mathrm{Tor}^B_n(Q \otimes_A -, -)
\end{aligned} \tag{1.27}$$

where we are only using the left module structure, and we have similar expressions when using the right module structure.

Using theorem 15 and these isomorphisms we get for every A-bimodule M that

$$\begin{aligned}
\mathrm{HH}^n(A, M) &\cong \mathrm{Ext}^n_{A\otimes A^{\mathrm{opp}}}(A, M)\\
&\cong \mathrm{Ext}^n_{A\otimes A^{\mathrm{opp}}}(P \otimes_B Q, M)\\
&\cong \mathrm{Ext}^n_{B\otimes A^{\mathrm{opp}}}(Q, Q \otimes_A M)\\
&\cong \mathrm{Ext}^n_{B\otimes B^{\mathrm{opp}}}(B, Q \otimes_A M \otimes_A P\\
&\cong \mathrm{HH}^n(B, Q \otimes_A M \otimes_A P)
\end{aligned} \tag{1.28}$$

and likewise for Hochschild homology. ◻

In section 2.3 we will greatly extend this Morita invariance to derived Morita invariance.

1.1.3 Interpretation in low degrees

We will now give an interpretation for Hochschild (co)homology in low degrees, where we can explicitly manipulate the bar complex, or rather its reinterpretation as in propositions 9 and 10. For this we observe that the differential of the Hochschild chain complex in low degrees is given by

$$\begin{array}{ccccc}
M \otimes_k A \otimes_k A & \xrightarrow{\mathrm{d}} & M \otimes_k A & \xrightarrow{\mathrm{d}} & M \\
m \otimes a \otimes b & \longmapsto & ma \otimes b - m \otimes ab + bm \otimes a & & \\
& & m \otimes a & \longmapsto & ma - am,
\end{array} \tag{1.29}$$

whilst for the Hochschild cochain complex $\mathrm{C}^\bullet(A, M)$

$$
\begin{array}{rcl}
M & \xrightarrow{\mathrm{d}} \mathrm{Hom}_k(A, M) & \xrightarrow{\quad\mathrm{d}\quad} \mathrm{Hom}_k(A \otimes_k A, M) \\
m & \longmapsto \mathrm{d}(m)\colon a \mapsto am - ma & \\
& f & \longmapsto \mathrm{d}(f)\colon a \otimes b \mapsto af(b) - f(ab) + f(a)b
\end{array}
\tag{1.30}
$$

and

$$
\begin{array}{rcl}
\mathrm{Hom}_k(A \otimes_k A, M) & \xrightarrow{\quad\mathrm{d}\quad} & \mathrm{Hom}_k(A \otimes_k A \otimes_k A, M) \\
g & \longmapsto & \mathrm{d}(g)\colon a \otimes b \otimes c \mapsto ag(b \otimes c) - g(ab \otimes c) + g(a \otimes bc) - g(a \otimes b)c.
\end{array}
\tag{1.31}
$$

Using these explicit descriptions in low degrees we can obtain the following.

Zeroth Hochschild homology

Proposition 17. We have that

$$\mathrm{HH}_0(A, M) \cong M/\langle am - ma \mid a \in A, m \in M\rangle \tag{1.32}$$

is the *module of coinvariants*. In particular, we have

$$\mathrm{HH}_0(A) \cong A/[A, A] = A_{\mathrm{ab}}. \tag{1.33}$$

Proof. This is immediate from the description of the morphism in (1.29). □

Remark 18. The vector space $[A, A]$ is usually not an ideal in A, so there is no obvious algebra structure on $\mathrm{HH}_0(A)$.

There is no one-size-fits-all description for Hochschild homology in higher degrees. But if A is commutative then a description in terms of differential forms is possible. We will come back to this in section 1.3.

Zeroth Hochschild cohomology

Proposition 19. We have that

$$\mathrm{HH}^0(A, M) \cong \{m \in M \mid \forall a \in A\colon am = ma\} \tag{1.34}$$

is the *submodule of invariants*. In particular, we have

$$\mathrm{HH}^0(A) \cong \mathrm{Z}(A). \tag{1.35}$$

Proof. This is immediate from the description of the morphism in (1.30). □

Remark 20. We can now give a new explanation of the non-functoriality of Hochschild cohomology using the interpretation of $\mathrm{HH}^0(A)$ as the center: taking the center of an algebra isn't a functor.

First Hochschild cohomology

Definition 21. A morphism $f\colon A \to M$ is a k-*derivation* if

$$f(ab) = af(b) + f(a)b. \tag{1.36}$$

We will denote the k-module of derivations by $\operatorname{Der}(A, M)$.

If $f = \operatorname{ad}_m$ for $m \in M$, where

$$\operatorname{ad}_m(a) = [a, m] = am - ma \tag{1.37}$$

then f is an *inner derivation*. We will denote the k-module of inner derivations by $\operatorname{InnDer}(A, M)$.

When $A = M$, we will use the notation $\operatorname{OutDer}(A)$ and $\operatorname{InnDer}(A)$. When A is commutative we will discuss derivations in more detail in section 1.3. For now, observe that in the commutative case there are no inner derivations.

Proposition 22. We have that

$$\operatorname{HH}^1(A, M) \cong \operatorname{OutDer}(A, M) \coloneqq \operatorname{Der}(A, M)/\operatorname{InnDer}(A, M) \tag{1.38}$$

are the *outer derivations*. In particular we have that

$$\operatorname{HH}^1(A) \cong \operatorname{OutDer}(A). \tag{1.39}$$

Proof. The description of the morphism in (1.30) tells us that Hochschild 1-cocycles are derivations, whilst Hochschild 1-coboundaries are inner derivations. □

At this point the first Hochschild cohomology $\operatorname{HH}^1(A)$ is just a vector space. But we can equip it with a Lie bracket. This is just a small piece of the extra structure that we will see in section 1.2.

Lemma 23. Let $D_1, D_2\colon A \to A$ be derivations. Then $[D_1, D_2] \coloneqq D_1 \circ D_2 - D_2 \circ D_1$ is also a derivation. Moreover, if $D_2 = \operatorname{ad}_a$ is an inner derivation, for some $a \in A$, then $[D_1, \operatorname{ad}_a] = \operatorname{ad}_{D_1(a)}$.

Proof. From

$$\begin{aligned}
[D_1, D_2](ab) &= D_1(D_2(ab)) - D_2(D_1(ab)) \\
&= D_1\left(aD_2(b) + D_2(a)b\right) - D_2(aD_1(b) + D_1(a)b)) \\
&= aD_1(D_2(b)) + D_1(a)D_2(b) + D_2(a)D_1(b) + D_1(D_2(a))b \\
&\qquad - aD_2(D_1(b)) - D_2(a)D_1(b) - D_1(a)D_2(b) - D_1(D_2(a))b \\
&= aD_1(D_2(b)) - aD_2(D_1(b)) + D_1(D_2(a))b - D_2(D_1(a))b \\
&= a[D_1, D_2](b) + [D_1, D_2](a)b
\end{aligned} \tag{1.40}$$

we get that $[D_1, D_2]$ is indeed a derivation.

Similarly we compute

$$\begin{aligned}
[D_1, \operatorname{ad}_a](b) &= D_1(\operatorname{ad}_a(b)) - \operatorname{ad}_a(D_1(b)) \\
&= D_1(ab - ba) - (aD_1(b) - D_1(b)a) \\
&= aD_1(b) + D_1(a)b - bD_1(a) - D_1(b)a - aD_1(b) + D_1(b)a \\
&= D_1(a)b - bD_1(a) \\
&= \operatorname{ad}_{D_1(a)}(b).
\end{aligned} \tag{1.41}$$

□

Corollary 24. $\mathrm{HH}^1(A)$ has the structure of a Lie algebra.

Proof. By lemma 23 we have that $\mathrm{Der}(A)$ is a Lie algebra (bilinearity and alternativity are trivial, the Jacobi identity is an easy computation), whilst $\mathrm{InnDer}(A) \subseteq \mathrm{Der}(A)$ is a Lie ideal. So $\mathrm{OutDer}(A)$ has the structure of a Lie algebra, and so does $\mathrm{HH}^1(A)$ via proposition 22. □

Second Hochschild cohomology The following discussion is the first aspect of why we care about Hochschild cohomology in the context of these lecture notes: deformation theory.

Definition 25. Let A be a k-algebra, and M an A-bimodule. A *square-zero extension* of A by M is a surjection $f\colon E \twoheadrightarrow A$ of k-algebras, such that

1. $(\ker f)^2 = 0$ (which implies that it has an A-bimodule structure),
2. $\ker f \cong M$ as A-bimodules.

To see that $\ker f$ indeed has an A-bimodule structure, let e be a lift of $a \in A$. We will define $a \cdot m = em$ and $m \cdot a = me$ for $m \in \ker f$. If e' is another lift, then $e - e' \in \ker f$, so $(e-e')m \in (\ker f)^2 = 0$ means $em = em'$ and $me = me'$.

So we have a sequence

$$0 \to M \to E \to A \to 0. \tag{1.42}$$

We will impose an equivalence relation on square-zero extensions.

Definition 26. We say that $f\colon E \to A$ and $f'\colon E' \to A$ are *equivalent* if there exists an algebra morphism $\varphi\colon E \to E'$ (necessarily an isomorphism), such that

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & A \arrow[r, "f"] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & E' \arrow[r, "f'"] & A \arrow[r] & 0 \end{tikzcd} \tag{1.43}$$

commutes.

Under our standing assumption on k being a field the sequence (1.42) is split as a sequence of vector spaces. If we choose a splitting $s\colon A \to E$ we get an isomorphism $E \cong A \oplus M$ of vector spaces. Using this decomposition the multiplication law on E can be written as

$$(a,m)\cdot(b,n) = (ab, an + mb + g(a,b)) \tag{1.44}$$

for $g\colon A \otimes_k A \to M$. This morphism is called the *factor set*. The factor set is determined by the splitting s, which is not necessarily an algebra morphism, by $g(a,b) = s(ab) - s(a)s(b)$. One can check that the unit of E corresponds to $(1, -g(1,1))$ in this description.

If we consider the multiplication $(a,0)\cdot(b,0)\cdot(c,0)$ inside E, then the associativity of E is equivalent to

$$ag(b\otimes c) + g(a \otimes bc) - g(ab)c - g(ab \otimes c) = 0, \tag{1.45}$$

which corresponds to g being a Hochschild 2-cocycle, by (1.31).

But there was a choice of splitting $s\colon A \to E$ involved in the definition of g. If $s'\colon A \to E$ is another splitting, then we obtain a different factor set g'. Comparing them gives

$$\begin{aligned} g'(a,b) - g(a,b) &= (s'(a)s'(b) - s'(ab)) - (s(a)s(b) - s(ab)) \\ &= s'(a)\left(s'(b) - s(b)\right) - (s'(ab) - s(ab)) + (s'(a) - s(a))\,s(b). \end{aligned} \tag{1.46}$$

But this is precisely the Hochschild differential applied to $s - s'$, which is a morphism $A \to M$ by construction, using the M-bimodule structure on M as discussed above. So the choice of a factor set gives a well-defined cohomology class.

If $g = 0$, then we call E the *trivial extension.*

Theorem 27. There exists a bijection

$$(1.47)\quad \mathrm{HH}^2(A, M) \cong \mathrm{AlgExt}(A, M)$$

such that $0 \in \mathrm{HH}^2(A, M)$ corresponds to the equivalence class of the trivial deformation.

We will mostly be interested in the case where $M = A$. In this case we will call E an *square-zero deformation.* This is a particular case of an infinitesimal deformation, as will be discussed in section 1.5. When $M = A$, we are describing algebra structures on $A \oplus At$ such that $t^2 = 0$, so we can equivalently describe square-zero deformations of A as a $k[t]/(t^2)$-algebra E, such that $E \otimes_{k[t]/(t^2)} k \cong A$. The notion of equivalence becomes that of a $k[t]/(t^2)$-module automorphism which reduces to the identity when t is set to 0.

So far we haven't seen any examples of Hochschild cohomology, let alone an example where $\mathrm{HH}^2(A) \neq 0$. The following example gives an ad hoc description of a (non-trivial) infinitesimal deformation of the polynomial ring in 2 variables.

Example 28. Let $A = k[x, y]$. Then we can equip $k[x, y] \oplus tk[x, y]$ with a multiplication for which $y \cdot x = yx + t$, i.e. using the factor set $g(y, x) = 1$. This is an infinitesimal deformation of $k[x, y]$ in the direction of the Weyl plane. We will come back to this.

Third Hochschild cohomology One can show that $\mathrm{HH}^3(A, M)$ classifies crossed bimodules, see [12, exercise E.1.5.1]. We will not discuss this here.

But we should at this point mention that the combination of $\mathrm{HH}^1(A)$, $\mathrm{HH}^2(A)$ and $\mathrm{HH}^3(A)$ will play an important role in the deformation theory of algebras, as discussed in section 1.5. The third Hochschild cohomology group will take on the role of obstruction space.

1.1.4 Examples

Explicit computations with the bar complex are often difficult, and only work in very elementary cases. We will collect a few of these examples, but we will also discuss some examples in which there exists a much smaller resolution that we can use, instead of the bar complex.

From now on we will focus on the case where $M = A$, occasionally we will mention what happens in the general case.

Example 29 (The polynomial ring $k[t]$)**.** Instead of the bar complex we can use a very concrete resolution of $k[t]$ as a bimodule over itself. Observe that $k[t]^{\mathrm{e}} \cong k[x, y]$, and $k[t]$ as a $k[x, y]$-module has a free resolution

$$(1.48)\quad 0 \to k[x, y] \xrightarrow{\cdot(x-y)} k[x, y] \to k[t] \to 0.$$

From this we immediately see that

$$(1.49)\quad \mathrm{HH}_i(k[t]) \cong \begin{cases} k[t] & i = 0, 1 \\ 0 & i \geq 2. \end{cases}$$

and

$$\mathrm{HH}^i(k[t]) \cong \begin{cases} k[t] & i = 0, 1 \\ 0 & i \geq 2. \end{cases} \tag{1.50}$$

This agreement between Hochschild homology and cohomology is no coincidence: $k[t]$ is a so called 1-Calabi–Yau algebra, so Poincaré–Van den Bergh duality applies, as in appendix B.2.

Example 30 (Finite-dimensional algebras)**.** If A is a finite-dimensional k-algebra, then it is possible to construct a small projective resolution of A as an A-bimodule. For details one is referred to [7, §1.5][1]

Applying this to $A = kQ$, where Q is a connected acyclic quiver, the resolution takes on the form

$$0 \to \bigoplus_{\alpha\in Q_1} A^{\mathrm{e}} e_{s(\alpha)} \otimes e_{t(\alpha)} \to \bigoplus_{v\in Q_0} A^{\mathrm{e}} e_v \otimes e_v \to A \to 0. \tag{1.51}$$

From the length of this resolution it is immediate that path algebras do not have deformations. Imposing relations on the quiver yields more complicated finite-dimensional algebras, and the explicit description of the resolution can be implemented in computer algebra, notably QPA[2].

Example 31 (Truncated polynomial algebras $k[t]/(t^n)$)**.** Again we want to use a small resolution of $A = k[t]/(t^n)$ as a bimodule over itself. We will use a 2-periodic resolution for this, which immediately tells us that the Hochschild (co)homology is itself 2-periodic, i.e.

$$\begin{aligned} \mathrm{HH}^i(A, M) &\cong \mathrm{HH}^{i+2}(A, M) \\ \mathrm{HH}_i(A, M) &\cong \mathrm{HH}_{i+2}(A, M) \end{aligned} \tag{1.52}$$

for any A-bimodule M, and $i \geq 1$. This would of course be impossible to read off from the definition using bar resolution.

This 2-periodic resolution is defined as follows: let $u = t \otimes 1 - 1 \otimes t$ and $v = \sum_{i=0}^{n-1} t^{n-1-i} \otimes t^i$. Then we will use

$$\ldots \xrightarrow{v\cdot} A^{\mathrm{e}} \xrightarrow{u\cdot} A^{\mathrm{e}} \xrightarrow{v\cdot} A^{\mathrm{e}} \xrightarrow{u\cdot} A^{\mathrm{e}} \xrightarrow{\mu} A \longrightarrow 0. \tag{1.53}$$

In exercise 32 a method of proving the exactness is suggested.

By applying $\mathrm{Hom}_{A^{\mathrm{e}}}(-, M)$ or $- \otimes_{A^{\mathrm{e}}} M$ to this sequence we get

$$0 \longrightarrow M \xrightarrow{0} M \xrightarrow{nt^{n-1}\cdot} M \xrightarrow{0} M \xrightarrow{nt^{n-1}\cdot} M \xrightarrow{0} \ldots \tag{1.54}$$

We always have that

$$\mathrm{HH}^0(A, M) \cong \mathrm{HH}_0(A, M) \cong M, \tag{1.55}$$

which we could also deduce from propositions 17 and 19.

For $i \geq 1$ the description depends on $\operatorname{char} k$. If $\gcd(n, \operatorname{char} k) = 1$ we obtain for i even

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_i(A, M) \cong M/t^{n-1}M \tag{1.56}$$

and for i odd

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_i(A, M) \cong tM. \tag{1.57}$$

[1] I should probably give a self-contained discussion.

[2] https://www.gap-system.org/Packages/qpa.html

On the other hand, if $\gcd(n, \operatorname{char} k) \neq 1$, then the morphism which is multiplication by nt^{n-1} is the zero morphism, so the sequence splits, and we obtain

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_i(A, M) \cong M \tag{1.58}$$

for all $i \geq 1$.

1.1.5 Exercises

Exercise 32. Show that (1.53) is exact by showing that the maps s_i give a contracting homotopy, where for $i = -1$ we take $s_{-1}(1) = 1$, whilst for $m \geq 0$ we define

$$\begin{aligned} s_{2m}(1 \otimes t^j) &= -\sum_{l=1}^{j} t^{j-l} \otimes t^{l-1} \\ s_{2m+1}(1 \otimes x^j) &= \begin{cases} \delta_j^{n-1} \otimes 1 & j = n-1 \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{1.59}$$

Exercise 33. Let us denote $A = \mathrm{A}_1(k)$ the *first Weyl algebra*, defined as $k\langle x, y\rangle/(yx - xy - 1)$. It is the ring of differential operators on $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, where y corresponds to $\partial/\partial x$.

Let V be a 2-dimensional vector space, and choose a basis $\{v, w\}$. Show that

$$0 \longrightarrow A^{\mathrm{e}} \otimes \textstyle\bigwedge^2 V \xrightarrow{f} A^{\mathrm{e}} \otimes V \xrightarrow{g} A^{\mathrm{e}} \longrightarrow 0 \tag{1.60}$$

where

$$f(1 \otimes 1 \otimes v \wedge w) = (1 \otimes x - x \otimes 1) \otimes w - (1 \otimes y - y \otimes 1) \otimes v \tag{1.61}$$

and

$$\begin{aligned} g(1 \otimes 1 \otimes v) &= 1 \otimes x - x \otimes 1 \\ g(1 \otimes 1 \otimes u) &= 1 \otimes y - y \otimes 1 \end{aligned} \tag{1.62}$$

gives a free resolution of A. Using this, show that

$$\begin{aligned} \mathrm{HH}^i(A) &= \begin{cases} k & i = 0 \\ 0 & i \neq 0 \end{cases}, \\ \mathrm{HH}_i(A) &= \begin{cases} k & i = 2 \\ 0 & i \neq 2 \end{cases}. \end{aligned} \tag{1.63}$$

This apparent duality between Hochschild homology and cohomology is not a coincidence in this case, see appendix B.2.

Exercise 34. We have seen that $\mathrm{HH}_\bullet(-)$ is a (covariant) functor. Show that

1. it sends products to direct sums, i.e.

   $$\mathrm{HH}_\bullet(A \times B) \cong \mathrm{HH}_\bullet(A) \oplus \mathrm{HH}_\bullet(B), \tag{1.64}$$

2. it preserves sequential limits, i.e. if $A_i \to A_{i+1}$ for $i \in \mathbb{N}$ is a sequence of algebra morphisms, then

   $$\mathrm{HH}_\bullet(\varinjlim A_i) \cong \varinjlim \mathrm{HH}_\bullet(A_i). \tag{1.65}$$

Now fixing A, show that $\mathrm{HH}_\bullet(A, -)$ sends a short exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{1.66}$$

of A-bimodules to a long exact sequence

$$\ldots \to \mathrm{HH}_n(A, M') \to \mathrm{HH}_n(A, M) \to \mathrm{HH}_n(A, M'') \to \ldots. \tag{1.67}$$

Exercise 35. Prove propositions 11 and 13.

1.2 Extra structure on Hochschild (co)homology

1.3 The Hochschild–Kostant–Rosenberg isomorphism

1.4 Variations on Hochschild (co)homology

This will be skipped during the course, unless there is time and interest to revisit the noncommutative calculus of Hochschild (co)homology and cyclic homology at the end of the course.

1.5 Formal deformation theory of algebras

Chapter 2

Differential graded categories

2.1 Enhancements of triangulated categories

2.2 Hochschild cohomology for differential graded categories

2.3 Limited functoriality for Hochschild cohomology

2.4 Fourier–Mukai transforms

2.5 Hochschild (co)homology in algebraic geometry

2.6 Semi-orthogonal decompositions

Chapter 3

Schemes

3.1 Polyvector fields

3.2 Atiyah classes

3.3 The Hochschild–Kostant–Rosenberg decomposition

3.4 Riemann–Roch versus Hochschild homology

3.5 Căldăraru's conjecture

Appendix A

Preliminaries

Appendix B

Additional topics

B.1 Kontsevich's formality theorems

B.2 Calabi–Yau algebras and Poincaré–Van den Bergh duality

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