Advanced topics in algebra (V5A5)

Hochschild (co)homology, and the Hochschild–Kostant–Rosenberg decomposition

$$\mathrm{Def}_A(R) \cong \mathrm{MC}\left(\mathfrak{m}\widehat{\otimes}_k \mathrm{C}^\bullet(A)[1]\right)$$

$$\mathrm{HH}^i(X) \cong \bigoplus_{p+q=i} \mathrm{H}^p\left(X, \bigwedge^q \mathrm{T}_X\right)$$

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June 16, 2018

Introduction

What you are reading now are the lecture notes for a course on Hochschild (co)homology, taught at the University of Bonn, in the Sommersemester of 2017–2018. They are currently being written, and regularly updated. The table of contents is provisional.

The goal of the course is to give an introduction to Hochschild (co)homology, focussing on

1. its applications in *deformation theory* of algebras (and schemes)
2. and the role of the *Hochschild–Kostant–Rosenberg* decomposition in all this.

There are by many several texts on various aspects of Hochschild (co)homology. In particular the following books dedicate some chapters on Hochschild (co)homology:

1. chapter IX in Cartain–Eilenberg's *Homological algebra* [5],
2. the first chapters of Loday's *Cyclic homology* [21],
3. chapter 9 of Weibel's *An introduction to homological algebra* [29],
4. chapter 2 by Tsygan in Cuntz–Skandalis–Tsygan's *Cyclic homology in noncommutative geometry* [6],
5. chapter II by Schedler in the Bellamy–Rogalski–Schedler–Stafford–Wemyss' *Noncommutative algebraic geometry* [2].

There are also the following unpublished lecture notes:

1. Ginzburg's *Lectures on noncommutative geometry* [11]
2. Kaledin's Tokyo lectures [16] and Seoul lectures [17].

There is also Witherspoon's textbook-in-progress called *An introduction to Hochschild cohomology* [30], which is dedicated entirely to Hochschild cohomology and some its applications. So far this is the only textbook dedicated entirely to Hochschild (co)homology, and it is a good reference for things not covered in these notes.

Compared to the existing texts these notes aim to focus more on Hochschild (co)homology in algebraic geometry, using derived categories of smooth projective varieties. This point of view has been developed in several papers [3, 4, 20] and applied in many more dealing with semiorthogonal decompositions. But there is no comprehensive treatment, let alone starting from the basics of Hochschild (co)homology for algebras. These notes aim to fill this gap, where we start focussing on smooth projective varieties starting in the second half of chapter 2.

Now that we know what is supposed to go in this text, let us mention that the following will not be discussed: the relationship with algebraic K-theory via Chern characters, support varieties, deformation theory of abelian and dg categories, applications to Hopf algebras, topological versions of Hochschild (co)homology and related constructions, …

Contents

Chapter 1

Algebras

Conventions Throughout these notes we will let k be a field. It is possible to develop much of the theory in the case for algebras which are flat over a commutative base ring without much extra effort, but we will not do so explicitly. The interested reader is invited to do so. There are also versions which are valid in a more general setting, but will refrain from discussing these.

At some points we will take k of characteristic zero, or algebraically closed. This will be mentioned explicitly.

If A is a k-algebra we will denote the *enveloping algebra* $A \otimes A^{\mathrm{op}}$ of A by A^{e}, so that A-bimodules are the same as left A^{e}-modules.

1.1 Definition and first properties

1.1.1 Hochschild (co)chain complexes

We start with a seemingly ad hoc definition.

Definition 1. Let A be a k-algebra. The *bar complex* $\mathrm{C}^{\mathrm{bar}}_{\bullet}(A)$ of A is the cochain complex

$$(1.1)\quad \ldots \xrightarrow{\mathrm{d}_2} A \otimes_k A \otimes_k A \xrightarrow{\mathrm{d}_1} A \otimes_k A \to 0,$$

of A-bimodules, where we have $\mathrm{C}^{\mathrm{bar}}_n(A) := A^{\otimes n+2}$, hence $A \otimes_k A$ lives in degree 0, and the differentials $\mathrm{d}_n\colon \mathrm{C}^{\mathrm{bar}}_n(A) \to \mathrm{C}^{\mathrm{bar}}_{n-1}(A)$ are given by

$$(1.2)\quad \mathrm{d}_n(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}.$$

The A-bimodule structure (or equivalently left A^{e}-module structure) on $\mathrm{C}^{\mathrm{bar}}_n(A)$ is given by

$$(1.3)\quad (a \otimes b) \cdot (a_0 \otimes \ldots \otimes a_{n+1}) = a a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} b.$$

We will also consider the morphism $\mathrm{d}_0\colon A \otimes_k A \to A$, which by the formula for d_n is nothing but the multiplication morphism $\mu\colon A \otimes_k A \to A$.

Remark 2. The terminology "bar complex" originates from the fact that an element $a_0 \otimes \ldots \otimes a_{n+1}$ is sometimes denoted $a_0[a_1|\ldots|a_n]a_{n+1}$.

Before we start studying the bar complex (for instance, at this point we haven't proven it is a complex), we introduce the following morphisms:

$$(1.4)\quad s_n\colon A^{\otimes n+2} \to A^{\otimes n+3} : a_0 \otimes \ldots \otimes a_{n+1} \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_{n+1}.$$

Given that this is the first proof we will give details. We will see many similar proofs throughout the beginning of the notes, we will leave some of them as exercises.

Lemma 3. We have that

$$(1.5)\quad \mathrm{d}_{n+1} \circ s_n + s_{n-1} \circ \mathrm{d}_n = \mathrm{id}_{A^{\otimes n+2}}.$$

Proof. One computes that

$$(1.6)\quad \begin{aligned} &s_{n-1} \circ \mathrm{d}_n(a_0 \otimes \ldots \otimes a_n) \\ &\quad= \sum_{i=0}^{n} (-1)^i 1 \otimes a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}, \\ &\mathrm{d}_{n+1} \circ s_n(a_0 \otimes \ldots \otimes a_n) \\ &\quad= a_0 \otimes \ldots \otimes a_{n+1} + \sum_{i=1}^{n+1} (-1)^i 1 \otimes a_0 \otimes \ldots \otimes a_{i-1} a_i \otimes \ldots \otimes a_{n+1}, \end{aligned}$$

so everything but the identity cancels after reindexing. □

We can check that the d_i's indeed turn $\mathrm{C}^{\mathrm{bar}}_\bullet(A)$ into a chain complex.

Lemma 4. We have that $\mathrm{d}_{n-1} \circ \mathrm{d}_n = 0$.

Proof. Let us consider $n = 1$ first. Then $\mathrm{d}_0 \circ \mathrm{d}_1(a_0 \otimes a_1 \otimes a_2) = (a_0 a_1) a_2 - a_0(a_1 a_2)$, which is zero as A is associative.

For $n \geq 2$ we use induction, using (1.5). We have

$$(1.7)\quad \mathrm{d}_n \circ \mathrm{d}_{n+1} \circ s_n = \mathrm{d}_n - \mathrm{d}_n \circ s_{n-1} \circ \mathrm{d}_n = s_{n-2} \circ \mathrm{d}_{n-1} \circ \mathrm{d}_n = 0,$$

but as the image of s_n generates $A^{\otimes n+3}$ as a left A-module we get that $\mathrm{d}_n \circ \mathrm{d}_{n+1} = 0$. □

The bar complex didn't include A, but if we use the morphism $\mathrm{d}_0\colon A \otimes_k A \to A$ as defined above we get the following proposition.

Proposition 5. The bar complex of A is a free resolution of A as an A-bimodule, where the augmentation $\mathrm{d}_0\colon A \otimes_k A \to A$ is given by the multiplication.

Proof. By lemma 3 we see that the s_i's provides a contracting homotopy, hence the bar complex is exact, as a complex of A-bimodules.

We also check that the cokernel of d_1 is indeed the multiplication $A \otimes_k A \to A$. For this it suffices to observe that

$$(1.8)\quad \mathrm{d}_1(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2,$$

and that there exists a map $\operatorname{coker} \mathrm{d}_1 \to A$ mapping the class of $a_0 \otimes a_1$ to $a_0 a_1$. By the definition of d_1 it sends elements of $\operatorname{im} \mathrm{d}_1$ to zero, so it is well-defined. Its inverse is given by the morphism which sends a to $1 \otimes a$.

That $C_n^{\mathrm{bar}}(A)$ is free as an A-bimodule follows from the isomorphisms of A-bimodules

$$(1.9)\quad A^{\otimes n+2} \cong A^{\mathrm{e}} \otimes A^{\otimes n} \cong \bigoplus_{i\in I} A^{\mathrm{e}} \cdot 1 \otimes 1 \otimes a_i$$

where $\{a_i \mid i \in I\}$ is a vector space basis of $A^{\otimes n}$, and the first isomorphism is

$$(1.10)\quad a_0 \otimes \ldots \otimes a_{n+1} \mapsto (a_0 \otimes a_{n+1}) \otimes a_1 \otimes \ldots \otimes a_n.$$

□

Definition 6. Let A be a k-algebra, and M an A-bimodule. The *Hochschild chain complex* $\mathrm{C}_\bullet(A, M)$ is $M \otimes_{A^{\mathrm{e}}} \mathrm{C}_\bullet^{\mathrm{bar}}(A)$, considered as a complex of k-modules, with differential $\mathrm{id}_M \otimes \mathrm{d}_n$.

Its homology is the *Hochschild homology of A with values in M*, and will be denoted $\mathrm{HH}_\bullet(A, M)$. If $M = A$, we'll write $\mathrm{HH}_n(A)$.

Dual to this we could instead of the tensor product use the Hom-functor, and obtain the dual notion of Hochschild cohomology.

Definition 7. Let A be a k-algebra, and M an A-bimodule. The *Hochschild cochain complex* $\mathrm{C}^\bullet(A, M)$ is $\mathrm{Hom}_{A^{\mathrm{e}}}(\mathrm{C}_\bullet^{\mathrm{bar}}(A), M)$, considered as a complex of k-modules, with differential $\mathrm{Hom}(\mathrm{d}_n, \mathrm{id}_M)$.

Its cohomology is the *Hochschild cohomology of A with values in M*, and will be denoted $\mathrm{HH}^\bullet(A, M)$. If $M = A$, we'll write $\mathrm{HH}^n(A)$.

Remark 8. Observe that one can recover the bar complex from the Hochschild complex:

$$(1.11)\quad \mathrm{C}_\bullet^{\mathrm{bar}}(A) = \mathrm{C}_\bullet(A, A^{\mathrm{e}}).$$

Reinterpreting the Hochschild cochain complex The Hochschild (co)chain complexes were obtained by considering a specific free resolution of A as an A-bimodule, and constructing a (co)chain complex of vector spaces out of it. We can rephrase this complex of vector spaces a bit, where instead of $\mathrm{Hom}_{A^{\mathrm{e}}}(-,-)$ and $- \otimes_{A^{\mathrm{e}}} -$, we use $\mathrm{Hom}_k(-,-)$ and $- \otimes_k -$. This will be very useful for computations later on.

The proofs of the following two propositions follow from the fact that A^{e} only involves the first and last tensor factor of a bimodule in the bar complex. The following proposition is then a consequence of the adjunction

$$(1.12)\quad - \otimes_k A^{\mathrm{e}} \colon\ \mathrm{Mod}\, k \rightleftarrows \mathrm{Mod}\, A^{\mathrm{e}} : \mathrm{res}.$$

The explicit formula for the Hochschild differentials in (1.16) and (1.20) will be important for us in section 1.1.3.

Proposition 9. There exists an isomorphism of k-modules

$$(1.13)\quad \varphi\colon\ \mathrm{C}^n(A, M) \xrightarrow{\cong} \mathrm{Hom}_k(A^{\otimes n}, M),$$

given by

$$(1.14)\quad g \mapsto \Big[a_1 \otimes \ldots \otimes a_n \mapsto g(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)\Big],$$

whose inverse is given by

$$(1.15)\quad f \mapsto \Big[a_0 \otimes \ldots \otimes a_{n+1} \mapsto a_0 f(a_1 \otimes \ldots \otimes a_n) a_{n+1}\Big].$$

The differential in $\mathrm{Hom}_k(A^\bullet, M)$ is then given by

$$
\begin{aligned}
&\mathrm{d}_{\mathrm{Hoch}} f(a_1 \otimes \ldots \otimes a_{n+1}) \\
&\quad = a_1 f(a_2 \otimes \ldots \otimes a_{n+1}) \\
&\qquad + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\
&\qquad + (-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n) a_{n+1}
\end{aligned}
\tag{1.16}
$$

for $f \in \mathrm{Hom}_k(A^{\otimes n}, M)$.

Proposition 10. There exists an isomorphism of k-modules

$$\psi\colon \mathrm{C}_\bullet(A, M) \xrightarrow{\cong} M \otimes_k A^\bullet \tag{1.17}$$

given by

$$\psi(m \otimes_{A^{\mathrm{e}}} a_0 \otimes \ldots \otimes a_{n+1}) = a_{n+1} m a_0 \otimes a_1 \otimes \ldots \otimes a_n, \tag{1.18}$$

whose inverse is given by

$$m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes_{A^{\mathrm{e}}} 1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1. \tag{1.19}$$

The differential $\mathrm{d}_{\mathrm{Hoch}}\colon M \otimes_k A^{\otimes n} \to M \otimes_k A^{\otimes n-1}$ is then given by

$$
\begin{aligned}
&\mathrm{d}_{\mathrm{Hoch}}(m \otimes a_1 \otimes \ldots \otimes a_n) \\
&\quad = m a_1 \otimes \ldots \otimes a_n \\
&\qquad + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\
&\qquad + (-1)^n a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1}
\end{aligned}
\tag{1.20}
$$

for $m \otimes a_1 \otimes \ldots \otimes a_n \in M \otimes_k A^{\otimes n}$.

Functoriality of Hochschild (co)homology Given an algebra morphism $f\colon A \to B$, or a bimodule morphism $g\colon M \to N$, we would like to understand how this interacts with taking Hochschild (co)homology. First of all: *Hochschild homology is covariantly functorial in both arguments.*

Proposition 11. Let $f\colon A \to B$ be an algebra morphism, and M a B-bimodule (which has an induced A-bimodule structure, denoted $f^*(M)$). Then

$$f_*\colon \mathrm{C}_\bullet(A, f^*(M)) \to \mathrm{C}_\bullet(B, M) : m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes f(a_1) \otimes \ldots \otimes f(a_n) \tag{1.21}$$

gives a functor $\mathrm{HH}_\bullet(-, M)$.

Let $g\colon M \to N$ be an A-bimodule morphism. Then

$$g_*\colon \mathrm{C}_\bullet(A, M) \to \mathrm{C}_\bullet(A, N) : m \otimes a_1 \otimes \ldots \otimes a_n \mapsto g(m) \otimes a_1 \otimes \ldots \otimes a_n \tag{1.22}$$

gives a functor $\mathrm{HH}_\bullet(A, -)$.

In particular, taking $M = A$ we can use the covariant functoriality in both arguments for Hochschild homology to get the following.

Corollary 12. Hochschild homology $\mathrm{HH}_\bullet(-)$ is a covariant functor from the category of associative k-algebras to the category of k-modules.

For Hochschild cohomology the situation is different: *Hochschild cohomology is contravariantly functorial in the first argument, and covariantly functorial in the second.*

Proposition 13. Let $f\colon A \to B$ be an algebra morphism, and M a B-bimodule (which has an induced A-bimodule structure). Then

(1.23) $f^*\colon \mathrm{C}^n(B,M) \to \mathrm{C}^n(A,M) : \varphi \mapsto \varphi \circ f^{\otimes n}$

gives a (contravariant) functor $\mathrm{HH}^\bullet(-,M)$.

Let $g\colon M \to N$ be an A-bimodule morphism. Then

(1.24) $g_*\colon \mathrm{C}^n(A,M) \to \mathrm{C}^n(A,N) : \varphi \mapsto g \circ \varphi$

gives a functor $\mathrm{HH}^\bullet(A,-)$.

Remark 14. So $\mathrm{HH}^\bullet(-)$ is *not* a functor (at least when we consider arbitrary morphisms between k-algebras), despite its appearance. We will come back to this in remark 20, and we will partially remedy this deficiency in section 2.3.

At this point it is also important that in some sources it is written that $\mathrm{HH}^\bullet(-)$ *is* a functor, see e.g. [22, §1.5.4]. But this is not the same functor, despite the similarity in notation! Indeed, in those situations one takes $M = A^\vee = \mathrm{Hom}_k(A,k)$ as the second argument. This makes the construction functorial (as the covariant functor in the second argument becomes contravariant), but one does not obtain the interpretation of Hochschild cohomology which will be used in this text. The construction in op. cit. has applications in studying cyclic cohomology and generalisations of the Chern character, which we will not go into here.

In section 2.3 we will greatly extend this functoriality for Hochschild homology, and discuss what can be done in the case of Hochschild cohomology. Remark that in the next section's corollary 16 we will obtain that Hochschild cohomology is a functor for Morita equivalences.

1.1.2 Hochschild (co)homology as Ext and Tor

In these notes we have *defined* Hochschild (co)homology as the (co)homology of an explicit (co)chain complex, which might seem ad hoc at first. But the bar complex of A being a free resolution of A as a bimodule over itself allows us to interpret Hochschild (co)homology in terms of more familiar constructions as explained in section 1.1.3.

Morover, the definition via the bar complex gives us an explicit description which will prove to be very useful in section 1.2 when we are discussing the extra structure on the Hochschild (co)chain complexes, which can conveniently be described by extra structure before taking cohomology. But it is of course an interesting question to find good intrinsic descriptions of the extra structure, and we will give further comments on this.

Theorem 15. There exist isomorphisms

(1.25) $\mathrm{HH}^i(A,M) \cong \mathrm{Ext}^i_{A^e}(A,M)$

and

(1.26) $\mathrm{HH}_i(A,M) \cong \mathrm{Tor}_i^{A^e}(A,M)$.

Proof. By proposition 5 the bar complex is a free resolution of A as an A-bimodule. In particular it can serve as a flat (resp. projective) resolution when computing the derived functors of $A \otimes_{A^e} -$ (resp. $\mathrm{Hom}_{A^e}(A, -)$). □

In particular, we have that

$$\begin{aligned} \mathrm{HH}^0(A, M) &\cong \mathrm{Hom}_{A^e}(A, M), \\ \mathrm{HH}_0(A, M) &\cong M \otimes_{A^e} A. \end{aligned} \tag{1.27}$$

But these descriptions are not necessarily very illuminating at this point. In section 1.1.3 we will give more concrete interpretations.

An important observation using theorem 15 is that the Hochschild cohomology of the A-bimodule M only depends on the category of A-bimodules. In this generality it is due to Rickard [25].

Corollary 16. Hochschild (co)homology is Morita invariant.

Proof. Assume that A and B are Morita equivalent through the bimodules ${}_AP_B$ and ${}_BQ_A$. The equivalences of categories are given by $P \otimes_A -$ and $Q \otimes_B -$, and these functors preserve projective resolutions. We obtain isomorphisms

$$\begin{aligned} \mathrm{Ext}^n_A(P \otimes_B -, -) &\cong \mathrm{Ext}^n_B(-, Q \otimes_A -) \\ \mathrm{Ext}^n_A(-, P \otimes_B -) &\cong \mathrm{Ext}^n_B(Q \otimes_A -, -) \\ \mathrm{Tor}^A_n(P \otimes_B -, -) &\cong \mathrm{Tor}^B_n(-, Q \otimes_A -) \\ \mathrm{Tor}^A_n(-, P \otimes_B -) &\cong \mathrm{Tor}^B_n(Q \otimes_A -, -) \end{aligned} \tag{1.28}$$

where we are only using the left module structure, and we have similar expressions when using the right module structure.

Using theorem 15 and these isomorphisms we get for every A-bimodule M that

$$\begin{aligned} \mathrm{HH}^n(A, M) &\cong \mathrm{Ext}^n_{A\otimes A^{\mathrm{op}}}(A, M) \\ &\cong \mathrm{Ext}^n_{A\otimes A^{\mathrm{op}}}(P \otimes_B Q, M) \\ &\cong \mathrm{Ext}^n_{B\otimes A^{\mathrm{op}}}(Q, Q \otimes_A M) \\ &\cong \mathrm{Ext}^n_{B\otimes B^{\mathrm{op}}}(B, Q \otimes_A M \otimes_A P \\ &\cong \mathrm{HH}^n(B, Q \otimes_A M \otimes_A P) \end{aligned} \tag{1.29}$$

and likewise for Hochschild homology. □

In section 2.3 we will greatly extend this Morita invariance to derived Morita invariance.

1.1.3 Interpretation in low degrees

We will now give an interpretation for Hochschild (co)homology in low degrees, where we can explicitly manipulate the bar complex, or rather its reinterpretation as in propositions 9 and 10. For this we observe that the differential of the Hochschild chain complex in low degrees is given by

$$\begin{array}{ccccc} M \otimes_k A \otimes_k A & \xrightarrow{\mathrm{d}} & M \otimes_k A & \xrightarrow{\mathrm{d}} & M \\ m \otimes a \otimes b & \longmapsto & ma \otimes b - m \otimes ab + bm \otimes a & & \\ & & m \otimes a & \longmapsto & ma - am, \end{array} \tag{1.30}$$

whilst for the Hochschild cochain complex $\mathrm{C}^\bullet(A, M)$

$$\begin{array}{ccccc} M & \xrightarrow{\mathrm{d}} & \mathrm{Hom}_k(A, M) & \xrightarrow{\mathrm{d}} & \mathrm{Hom}_k(A \otimes_k A, M) \\ m & \longmapsto & \mathrm{d}(m)\colon a \mapsto am - ma & & \\ & & f & \longmapsto & \mathrm{d}(f)\colon a \otimes b \mapsto af(b) - f(ab) + f(a)b \end{array} \tag{1.31}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_k(A \otimes_k A, M) & \xrightarrow{\mathrm{d}} & \mathrm{Hom}_k(A \otimes_k A \otimes_k A, M) \\ g & \longmapsto & \mathrm{d}(g)\colon a \otimes b \otimes c \mapsto ag(b \otimes c) - g(ab \otimes c) + g(a \otimes bc) - g(a \otimes b)c. \end{array} \tag{1.32}$$

Using these explicit descriptions in low degrees we can obtain the following.

Zeroth Hochschild homology

Proposition 17. We have that

$$\mathrm{HH}_0(A, M) \cong M/\langle am - ma \mid a \in A, m \in M\rangle \tag{1.33}$$

is the *module of coinvariants*. In particular, we have

$$\mathrm{HH}_0(A) \cong A/[A, A] = A_{\mathrm{ab}}. \tag{1.34}$$

Proof. This is immediate from the description of the morphism in (1.30). □

Remark 18. The vector space $[A, A]$ is usually not an ideal in A, so there is no obvious algebra structure on $\mathrm{HH}_0(A)$.

There is no one-size-fits-all description for Hochschild homology in higher degrees. But if A is commutative then a description in terms of differential forms is possible. We will come back to this in section 1.3.

Zeroth Hochschild cohomology

Proposition 19. We have that

$$\mathrm{HH}^0(A, M) \cong \{m \in M \mid \forall a \in A\colon am = ma\} \tag{1.35}$$

is the *submodule of invariants*. In particular, we have

$$\mathrm{HH}^0(A) \cong \mathrm{Z}(A). \tag{1.36}$$

Proof. This is immediate from the description of the morphism in (1.31). □

Remark 20. We can now give a new explanation of the non-functoriality of Hochschild cohomology using the interpretation of $\mathrm{HH}^0(A)$ as the center: taking the center of an algebra isn't a functor.

First Hochschild cohomology

Definition 21. A morphism $f\colon A \to M$ is a k-*derivation* if

$$f(ab) = af(b) + f(a)b. \tag{1.37}$$

We will denote the k-module of derivations by $\mathrm{Der}(A, M)$.

If $f = \mathrm{ad}_m$ for $m \in M$, where

$$\mathrm{ad}_m(a) = [a, m] = am - ma \tag{1.38}$$

then f is an *inner derivation.* We will denote the k-module of inner derivations by $\mathrm{InnDer}(A, M)$.

When $A = M$, we will use the notation $\mathrm{OutDer}(A)$ and $\mathrm{InnDer}(A)$. When A is commutative we will discuss derivations in more detail in section 1.3. For now, observe that in the commutative case there are no inner derivations.

Proposition 22. We have that

$$\mathrm{HH}^1(A, M) \cong \mathrm{OutDer}(A, M) := \mathrm{Der}(A, M)/\,\mathrm{InnDer}(A, M) \tag{1.39}$$

are the *outer derivations.* In particular we have that

$$\mathrm{HH}^1(A) \cong \mathrm{OutDer}(A). \tag{1.40}$$

Proof. The description of the morphism in (1.31) tells us that Hochschild 1-cocycles are derivations, whilst Hochschild 1-coboundaries are inner derivations. □

At this point the first Hochschild cohomology $\mathrm{HH}^1(A)$ is just a vector space. But we can equip it with a Lie bracket. This is just a small piece of the extra structure that we will see in section 1.2.

Lemma 23. Let $D_1, D_2\colon A \to A$ be derivations. Then $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ is also a derivation. Moreover, if $D_2 = \mathrm{ad}_a$ is an inner derivation, for some $a \in A$, then $[D_1, \mathrm{ad}_a] = \mathrm{ad}_{D_1(a)}$.

Proof. From

$$\begin{aligned}
[D_1, D_2](ab) &= D_1(D_2(ab)) - D_2(D_1(ab)) \\
&= D_1\left(aD_2(b) + D_2(a)b\right) - D_2(aD_1(b) + D_1(a)b)) \\
&= aD_1(D_2(b)) + D_1(a)D_2(b) + D_2(a)D_1(b) + D_1(D_2(a))b \\
&\qquad - aD_2(D_1(b)) - D_2(a)D_1(b) - D_1(a)D_2(b) - D_1(D_2(a))b \\
&= aD_1(D_2(b)) - aD_2(D_1(b)) + D_1(D_2(a))b - D_2(D_1(a))b \\
&= a[D_1, D_2](b) + [D_1, D_2](a)b
\end{aligned} \tag{1.41}$$

we get that $[D_1, D_2]$ is indeed a derivation.

Similarly we compute

$$\begin{aligned}
[D_1, \mathrm{ad}_a](b) &= D_1(\mathrm{ad}_a(b)) - \mathrm{ad}_a(D_1(b)) \\
&= D_1(ab - ba) - (aD_1(b) - D_1(b)a) \\
&= aD_1(b) + D_1(a)b - bD_1(a) - D_1(b)a - aD_1(b) + D_1(b)a \\
&= D_1(a)b - bD_1(a) \\
&= \mathrm{ad}_{D_1(a)}(b).
\end{aligned} \tag{1.42}$$

□

Corollary 24. $\mathrm{HH}^1(A)$ has the structure of a Lie algebra.

Proof. By lemma 23 we have that $\mathrm{Der}(A)$ is a Lie algebra (bilinearity and alternativity are trivial, the Jacobi identity is an easy computation), whilst $\mathrm{InnDer}(A) \subseteq \mathrm{Der}(A)$ is a Lie ideal. So $\mathrm{OutDer}(A)$ has the structure of a Lie algebra, and so does $\mathrm{HH}^1(A)$ via proposition 22. □

Second Hochschild cohomology The following discussion is the first aspect of why we care about Hochschild cohomology in the context of these lecture notes: deformation theory.

Definition 25. Let A be a k-algebra, and M an A-bimodule. A *square-zero extension* of A by M is a surjection $f\colon E \twoheadrightarrow A$ of k-algebras, such that

1. $(\ker f)^2 = 0$ (which implies that it has an A-bimodule structure),
2. $\ker f \cong M$ as A-bimodules[1].

To see that $\ker f$ indeed has an A-bimodule structure, let e be a lift of $a \in A$. We will define $a \cdot m = em$ and $m \cdot a = me$ for $m \in \ker f$. If e' is another lift, then $e - e' \in \ker f$, so $(e - e')m \in (\ker f)^2 = 0$ means $em = em'$ and $me = me'$.

So we have a sequence

$$0 \to M \to E \to A \to 0. \tag{1.43}$$

We will impose an equivalence relation on square-zero extensions.

Definition 26. We say that $f\colon E \to A$ and $f'\colon E' \to A$ are *equivalent* if there exists an algebra morphism $\varphi\colon E \to E'$ (necessarily an isomorphism), such that

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & A \arrow[r, "f"] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & E' \arrow[r, "f'"] & A \arrow[r] & 0
\end{tikzcd} \tag{1.44}$$

commutes.

Under our standing assumption on k being a field the sequence (1.43) is split as a sequence of vector spaces. If we choose a splitting $s\colon A \to E$ we get an isomorphism $E \cong A \oplus M$ of vector spaces. Using this decomposition the multiplication law on E can be written as

$$(a, m) \cdot (b, n) = (ab, an + mb + g(a, b)) \tag{1.45}$$

[1] We will introduce an equivalence relation to deal with the choice of isomorphism here.

for $g\colon A\otimes_k A \to M$. This morphism is called the *factor set*. The factor set is determined by the splitting s, which is not necessarily an algebra morphism, by $g(a,b) = s(ab) - s(a)s(b)$. One can check that the unit of E corresponds to $(1, -g(1,1))$ in this description.

If we consider the multiplication $(a,0)\cdot(b,0)\cdot(c,0)$ inside E, then the associativity of E is equivalent to

$$ag(b\otimes c) + g(a\otimes bc) - g(ab)c - g(ab\otimes c) = 0, \tag{1.46}$$

which corresponds to g being a Hochschild 2-cocycle, by (1.32).

But there was a choice of splitting $s\colon A\to E$ involved in the definition of g. If $s'\colon A\to E$ is another splitting, then we obtain a different factor set g'. Comparing them gives

$$\begin{aligned} g'(a,b) - g(a,b) &= (s'(a)s'(b) - s'(ab)) - (s(a)s(b) - s(ab)) \\ &= s'(a)\left(s'(b) - s(b)\right) - (s'(ab) - s(ab)) + (s'(a) - s(a))\, s(b). \end{aligned} \tag{1.47}$$

But this is precisely the Hochschild differential applied to $s - s'$, which is a morphism $A\to M$ by construction, using the M-bimodule structure on M as discussed above. So the choice of a factor set gives a well-defined cohomology class.

If $g = 0$, then we call E the *trivial extension*.

Theorem 27. There exists a bijection

$$\mathrm{HH}^2(A,M) \cong \mathrm{AlgExt}(A,M) \tag{1.48}$$

such that $0\in\mathrm{HH}^2(A,M)$ corresponds to the equivalence class of the trivial extension.

We will mostly be interested in the case where $M = A$. In this case we will call E an *square-zero deformation* or *first order deformation*. This is a particular case of an infinitesimal deformation, as will be discussed in section 1.5. When $M = A$, we are describing algebra structures on $A\oplus At$ such that $t^2 = 0$, so we can equivalently describe square-zero deformations of A as a $k[t]/(t^2)$-algebra E, such that $E\otimes_{k[t]/(t^2)} k \cong A$. The notion of equivalence becomes that of a $k[t]/(t^2)$-module automorphism which reduces to the identity when t is set to 0.

So far we haven't seen any examples of Hochschild cohomology, let alone an example where $\mathrm{HH}^2(A)\neq 0$. The following example gives an ad hoc description of a (non-trivial) infinitesimal deformation of the polynomial ring in 2 variables.

Example 28. Let $A = k[x,y]$. Then we can equip $k[x,y]\oplus tk[x,y]$ with a multiplication for which $y\cdot x = yx + t$, i.e. using the factor set $g(y,x) = 1$. This is an infinitesimal deformation of $k[x,y]$ in the direction of the Weyl plane. We will come back to this.

If $\mathrm{HH}^2(A) = 0$, then A does not have any square-zero deformations, and vice versa. Such algebras are called (infinitesimally, or absolutely) *rigid*[2].

Third Hochschild cohomology One can show that $\mathrm{HH}^3(A,M)$ classifies crossed bimodules, see [22, exercise E.1.5.1]. We will not discuss this here.

But we should at this point mention that the combination of $\mathrm{HH}^1(A)$, $\mathrm{HH}^2(A)$ and $\mathrm{HH}^3(A)$ will play an important role in the deformation theory of algebras, as discussed in section 1.5. The third Hochschild cohomology group will take on the role of obstruction space.

[2]Hochschild called such algebras *segregated* in his original paper.

1.1.4 Examples

Explicit computations with the bar complex are often difficult, and only work in very elementary cases. We will collect a few of these examples, but we will also discuss some examples in which there exists a much smaller resolution that we can use, instead of the bar complex.

From now on we will focus on the case where $M = A$, occasionally we will mention what happens in the general case.

Example 29 (The base field)**.** This case is completely trivial, but we observe that if $A = k$, then $A^{\mathrm{e}} \cong k$. So

$$(1.49)\quad \mathrm{HH}^i(k) \cong \begin{cases} k & i = 0 \\ 0 & i \geq 1 \end{cases}$$

and

$$(1.50)\quad \mathrm{HH}_i(k) \cong \begin{cases} k & i = 0 \\ 0 & i \geq 1. \end{cases}$$

This triviality will be useful when discussing exceptional collections in **??**.

Example 30 (The polynomial ring $k[t]$)**.** Instead of the bar complex we can use a very concrete resolution of $k[t]$ as a bimodule over itself. Observe that $k[t]^{\mathrm{e}} \cong k[x,y]$, and $k[t]$ as a $k[x,y]$-module has a free resolution

$$(1.51)\quad 0 \to k[x,y] \overset{\cdot(x-y)}{\to} k[x,y] \to k[t] \to 0.$$

From this we immediately see that

$$(1.52)\quad \mathrm{HH}_i(k[t]) \cong \begin{cases} k[t] & i = 0,1 \\ 0 & i \geq 2. \end{cases}$$

and

$$(1.53)\quad \mathrm{HH}^i(k[t]) \cong \begin{cases} k[t] & i = 0,1 \\ 0 & i \geq 2. \end{cases}$$

This agreement between Hochschild homology and cohomology is no coincidence: $k[t]$ is a so called 1-Calabi–Yau algebra, so Poincaré–Van den Bergh duality applies, as in appendix B.2.

Example 31 (Finite-dimensional algebras)**.** If A is a finite-dimensional k-algebra, then it is possible to construct a small projective resolution of A as an A-bimodule. For details one is referred to [12, §1.5][3]

Applying this to $A = kQ$, where Q is a connected acyclic quiver, the resolution takes on the form

$$(1.54)\quad 0 \to \bigoplus_{\alpha \in Q_1} A^{\mathrm{e}} e_{s(\alpha)} \otimes e_{t(\alpha)} \to \bigoplus_{v \in Q_0} A^{\mathrm{e}} e_v \otimes e_v \to A \to 0.$$

From the length of this resolution it is immediate that path algebras do not have deformations. Imposing relations on the quiver yields more complicated finite-dimensional algebras, and the explicit description of the resolution can be implemented in computer algebra, notably QPA[4].

[3] I should probably give a self-contained discussion.

[4] `https://www.gap-system.org/Packages/qpa.html`

More generally when A has finite global dimension we have the description

$$\mathrm{HH}_i(A) \cong \begin{cases} k^r & i = 0 \\ 0 & i \geq 1 \end{cases} \tag{1.55}$$

where r is the number of isomorphism classes of simple modules.

Example 32 (Truncated polynomial algebras $k[t]/(t^n)$)**.** Again we want to use a small resolution of $A = k[t]/(t^n)$ as a bimodule over itself. We will use a 2-periodic resolution for this, which immediately tells us that the Hochschild (co)homology is itself 2-periodic, i.e.

$$\begin{aligned} \mathrm{HH}^i(A, M) &\cong \mathrm{HH}^{i+2}(A, M) \\ \mathrm{HH}_i(A, M) &\cong \mathrm{HH}_{i+2}(A, M) \end{aligned} \tag{1.56}$$

for any A-bimodule M, and $i \geq 1$. This would of course be impossible to read off from the definition using bar resolution.

This 2-periodic resolution is defined as follows: let $u = t \otimes 1 - 1 \otimes t$ and $v = \sum_{i=0}^{n-1} t^{n-1-i} \otimes t^i$. Then we will use

$$\dots \xrightarrow{v\cdot} A^{\mathrm{e}} \xrightarrow{u\cdot} A^{\mathrm{e}} \xrightarrow{v\cdot} A^{\mathrm{e}} \xrightarrow{u\cdot} A^{\mathrm{e}} \xrightarrow{\mu} A \longrightarrow 0. \tag{1.57}$$

In exercise 33 a method of proving the exactness is suggested.

By applying $\mathrm{Hom}_{A^{\mathrm{e}}}(-, M)$ or $- \otimes_{A^{\mathrm{e}}} M$ to this sequence we get

$$0 \longrightarrow M \xrightarrow{0} M \xrightarrow{nt^{n-1}\cdot} M \xrightarrow{0} M \xrightarrow{nt^{n-1}\cdot} M \xrightarrow{0} \dots \tag{1.58}$$

We always have that

$$\mathrm{HH}^0(A, M) \cong \mathrm{HH}_0(A, M) \cong M, \tag{1.59}$$

which we could also deduce from propositions 17 and 19.

For $i \geq 1$ the description depends on $\operatorname{char} k$. If $\gcd(n, \operatorname{char} k) = 1$ we obtain for i even

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_i(A, M) \cong M/t^{n-1}M \tag{1.60}$$

and for i odd

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_i(A, M) \cong tM. \tag{1.61}$$

On the other hand, if $\gcd(n, \operatorname{char} k) \neq 1$, then the morphism which is multiplication by nt^{n-1} is the zero morphism, so the sequence splits, and we obtain

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_i(A, M) \cong M \tag{1.62}$$

for all $i \geq 1$.

1.1.5 Exercises

Exercise 33. Show that (1.57) is exact by showing that the maps s_i give a contracting homotopy, where for $i = -1$ we take $s_{-1}(1) = 1$, whilst for $m \geq 0$ we define

$$\begin{aligned} s_{2m}(1 \otimes t^j) &= -\sum_{l=1}^{j} t^{j-l} \otimes t^{l-1} \\ s_{2m+1}(1 \otimes x^j) &= \begin{cases} \delta_j^{n-1} \otimes 1 & j = n-1 \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{1.63}$$

Exercise 34. Let us denote $A = \mathrm{A}_1(k)$ the *first Weyl algebra*, defined as $k\langle x, y\rangle/(yx - xy - 1)$. It is the ring of differential operators on $\mathbb{A}_k^1 = \operatorname{Spec} k[x]$, where y corresponds to $\partial/\partial x$.

Let V be a 2-dimensional vector space, and choose a basis $\{v, w\}$. Show that

$$0 \longrightarrow A^{\mathrm{e}} \otimes \textstyle\bigwedge^2 V \xrightarrow{f} A^{\mathrm{e}} \otimes V \xrightarrow{g} A^{\mathrm{e}} \longrightarrow 0 \tag{1.64}$$

where

$$f(1 \otimes 1 \otimes v \wedge w) = (1 \otimes x - x \otimes 1) \otimes w - (1 \otimes y - y \otimes 1) \otimes v \tag{1.65}$$

and

$$\begin{aligned} g(1 \otimes 1 \otimes v) &= 1 \otimes x - x \otimes 1 \\ g(1 \otimes 1 \otimes u) &= 1 \otimes y - y \otimes 1 \end{aligned} \tag{1.66}$$

gives a free resolution of A. Using this, show that

$$\begin{aligned} \mathrm{HH}^i(A) &= \begin{cases} k & i = 0 \\ 0 & i \neq 0 \end{cases}, \\ \mathrm{HH}_i(A) &= \begin{cases} k & i = 2 \\ 0 & i \neq 2 \end{cases}. \end{aligned} \tag{1.67}$$

This apparent duality between Hochschild homology and cohomology is not a coincidence in this case, see appendix B.2.

Exercise 35. We have seen that $\mathrm{HH}_\bullet(-)$ is a (covariant) functor. Show that

1. it sends products to direct sums, i.e.

$$\mathrm{HH}_\bullet(A \times B) \cong \mathrm{HH}_\bullet(A) \oplus \mathrm{HH}_\bullet(B), \tag{1.68}$$

2. it preserves sequential limits, i.e. if $A_i \to A_{i+1}$ for $i \in \mathbb{N}$ is a sequence of algebra morphisms, then

$$\mathrm{HH}_\bullet(\varinjlim A_i) \cong \varinjlim \mathrm{HH}_\bullet(A_i). \tag{1.69}$$

Now fixing A, show that $\mathrm{HH}_\bullet(A, -)$ sends a short exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{1.70}$$

of A-bimodules to a long exact sequence

$$\ldots \to \mathrm{HH}_n(A, M') \to \mathrm{HH}_n(A, M) \to \mathrm{HH}_n(A, M'') \to \ldots. \tag{1.71}$$

Exercise 36. Prove propositions 11 and 13.

Exercise 37. Let A be an associative k-algebra.

1. Show that

$$\mathrm{HH}_i(A, A^{\mathrm{e}}) \cong \begin{cases} A & i = 0 \\ 0 & i \geq 1. \end{cases} \tag{1.72}$$

2. Explain why the analogous statement is not true for $\mathrm{HH}^i(A, A^{\mathrm{e}})$.

1.2 Extra structure on Hochschild (co)homology

Hochschild homology and cohomology have a rich structure: they are more than just k-modules, which is how we defined them in the previous section. We will discuss the following structure in these notes.

1. Hochschild cohomology has both the structure of an associative algebra and a Lie algebra;
2. Hochschild homology is both a module and a representation over Hochschild cohomology;
3. if A is commutative, then Hochschild homology itself has an algebra structure.

We will take $A = M$ throughout here.

Observe that this is not an exhaustive list of the extra structure. We will not discuss the action of $\mathrm{HH}^\bullet(A)$ on $\mathrm{Ext}^\bullet_A(M, N)$ (see [30, §1.6]), the cut coproduct on Hochschild homology, generalisations of the structures discussed here when the A-bimodule has an algebra structure of its own, similar structures on the variations on cyclic homology, …

1.2.1 Hochschild cohomology is a Gerstenhaber algebra

The first aspect that we deal with is the algebraic structure on Hochschild cohomology (and Hochschild cochains): it is both

- a graded-commutative algebra,
- a graded Lie (super-)algebra,

and these structures are compatible: we will call such a structure a Gerstenhaber algebra, see definition 54.

For Hochschild cochains the situation is somewhat more complicated, as some properties are only true *up to homotopy*. For now we will not go into many details regarding this, this might change later on in the notes.

Observe that we have already seen a small part of the algebra structure in proposition 19, and of the Lie algebra structure in corollary 24. We will now extend these structures to the entire Hochschild cohomology of A, and discuss their compatibility.

Originally the Lie bracket on Hochschild cochains was introduced by Gerstenhaber in [9] to prove that the multiplication on Hochschild cohomology is graded-commutative. But this Lie bracket is also very important for deformation theory, we will come back to this in section 1.5.

Associative algebra structure: cup product We will start with introducing the associative multiplication, both on $\mathrm{C}^\bullet(A)$, and by compatibility with the differential, on $\mathrm{HH}^\bullet(A)$. The graded-commutativity will have to wait for now.

Definition 38. Let $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$ be Hochschild cochains. The *cup product* of f and g is the element $f \cup g$ defined by

(1.73) $f \cup g(a_1 \otimes \ldots \otimes a_{m+n}) = f(a_1 \otimes \ldots \otimes a_m)g(a_{m+1} \otimes \ldots \otimes a_{m+n}).$

Lemma 39. The cup product makes $\mathrm{C}^\bullet(A)$ into a differential graded algebra, i.e. the cup product is associative and satisfies the *graded Leibniz rule*

(1.74) $\mathrm{d}_{m+n+1}(f \cup g) = \mathrm{d}_{m+1}(f) \cup g + (-1)^m f \cup \mathrm{d}_{n+1}(g).$

where $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$.

Proof. Associativity is immediate, as $(f \cup g) \cup h$ and $f \cup (g \cup h)$ involve multiplication inside A, which is associative.

The compatibility with the differential is the following computation, which follows immediately from the definitions. For the left-hand side we have

$$
\begin{aligned}
&\mathrm{d}_{m+n+1}(f \cup g)(a_1 \otimes \dots a_{m+n+1}) \\
&\quad = a_1 (f \cup g)(a_2 \otimes \dots \otimes a_{m+n+1}) \\
&\qquad + \sum_{i=1}^{m+n} (-1)^i (f \cup g)(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{m+n+1}) \\
&\qquad + (-1)^{m+n+1} (f \cup g)(a_1 \otimes \dots \otimes a_{m+n}) a_{m+n+1}
\end{aligned}
\tag{1.75}
$$

whilst for the right-hand side we have

$$
\begin{aligned}
&(\mathrm{d}_{m+1}(f) \cup g)(a_1 \otimes \dots \otimes a_{m+n+1}) \\
&\quad = a_1 f(a_2 \otimes \dots \otimes a_{m+1}) g(a_{m+2} \otimes \dots \otimes a_{m+n+1}) \\
&\qquad + \sum_{i=1}^{m} f(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{m+1}) g(a_{m+2} \otimes \dots \otimes a_{m+n+1}) \\
&\qquad + (-1)^{m+1} f(a_1 \otimes \dots \otimes a_m) a_{m+1} g(a_{m+2} \otimes \dots \otimes a_{m+n+1})
\end{aligned}
\tag{1.76}
$$

and

$$
\begin{aligned}
&(-1)^m (f \cup \mathrm{d}_{n+1}(g))(a_1 \otimes \dots \otimes a_{m+n+1}) \\
&\quad = (-1)^m f(a_1 \otimes \dots \otimes a_m) a_{m+1} g(a_{m+2} \otimes \dots \otimes a_{m+n+1}) \\
&\qquad + \sum_{i=1}^{n} (-1)^{m+i} f(a_1 \otimes \dots \otimes a_m) g(a_{m+1} \otimes \dots \otimes a_{m+i} a_{m+i+1} \otimes \dots \otimes a_{m+n+1}) \\
&\qquad + (-1)^{m+n+1} f(a_1 \otimes \dots \otimes a_m) g(a_{m+1} \otimes \dots \otimes a_{m+n}) a_{m+n+1}.
\end{aligned}
\tag{1.77}
$$

It suffices to identify the last and first terms of (1.76) and (1.77), and reindex the summation in (1.77) to run from $m + 1$ to $n + m$ to get the equality. ◻

By taking cohomology of the Hochschild cochain complex we get the following corollary.

Corollary 40. The Hochschild cohomology $\mathrm{HH}^\bullet(A)$ is a graded associative algebra.

This is only the first aspect of the algebraic structure of $\mathrm{C}^\bullet(A)$. Before we define the Lie bracket, we should mention that the cup product on the level of cohomology is actually commutative! This is one of the main results of [9].

Proposition 41. The Hochschild cohomology $\mathrm{HH}^\bullet(A)$ is a graded-commutative algebra, i.e. for $f \in \mathrm{HH}^m(A)$ and $g \in \mathrm{HH}^n(A)$ we have that

$$f \cup g = (-1)^{mn} g \cup f. \tag{1.78}$$

The proof of this result will require the Gerstenhaber bracket which will be defined shortly. We will show that the difference between $f \cup g$ and $g \cup f$ for two Hochschild cochains has a precise description as the differential of the circle product $f \circ g$, so that it vanishes in cohomology.

Observe that in proposition 19 we saw that $\mathrm{HH}^0(A) \cong \mathrm{Z}(A)$, so we at least already knew that the degree zero part was a commutative subalgebra. It turns out that in a precise sense Hochschild cohomology can be seen as a *derived center*.

Remark 42. Using theorem 15 we have another graded-commutative algebra structure on Hochschild cohomology, given by the Yoneda product on Ext-groups. One can show that the cup product and Yoneda product are actually identified under the isomorphism (1.25). We refer to [30] for details.

Lie algebra structure: Gerstenhaber bracket Next up is a Lie bracket on Hochschild cochains, which like the product is compatible with the Hochschild differential, hence descends to a Lie bracket on Hochschild cohomology.

Definition 43. Let $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$ be Hochschild cochains. Let us denote[5] the element $f \circ_i g$ of $\mathrm{C}^{m+n-1}(A)$, for $i = 1, \ldots, m$, defined by

$$(1.79)\quad f \circ_i g(a_1 \otimes \ldots \otimes a_{m+n-1}) = f\left(a_1 \otimes \ldots \otimes a_{i-1} \otimes g(a_i \otimes \ldots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \ldots \otimes a_{m+n-1}\right).$$

The *circle product* of the Hochschild cochains f and g is the element $f \circ g \in \mathrm{C}^{m+n-1}(A)$ defined by

$$(1.80)\quad f \circ g \coloneqq \sum_{i=1}^{m} (-1)^{(i-1)(n+1)} f \circ_i g.$$

This circle product equips $\mathrm{C}^\bullet(A)$ with the structure of a so called *pre-Lie algebra*. In particular, it is not associative. We will not be interested in this structure on its own, as we are only interested in the structure induced by the following definition.

Definition 44. Let $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$ be Hochschild cochains. Then their *Gerstenhaber bracket* is the element $[f, g] \in \mathrm{C}^{m+n-1}(A)$ defined by

$$(1.81)\quad [f, g] \coloneqq f \circ g - (-1)^{(m-1)(n-1)} g \circ f.$$

The way Gerstenhaber proves essential properties of his bracket depends greatly on a detailed analysis of $- \circ_i -$ and $- \circ -$, and for details[6] one is referred to [9]. We will only summarise the intermediate steps in what follows.

Example 45. The Gerstenhaber product will play an important role when studying the deformation theory of algebras. If $m = n = 2$, and $f \in \mathrm{C}^2(A)$, then (at least when the characteristic is not 2)

$$(1.82)\quad [f, f] = 2\left(f(f(a \otimes b) \otimes c) - f(a \otimes f(b \otimes c))\right).$$

Remark 46. Observe that the definition of the Gerstenhaber bracket did *not* use the algebra structure on A. But one can observe that the multiplication morphism $\mu\colon A \otimes_k A \to A$ is the coboundary of the identity, and

$$(1.83)\quad \mathrm{d}(f) = [f, -\mu]$$

makes the link between the algebra structure on A, the Hochschild differential and the Gerstenhaber bracket.

Even more is true: the cup product can also be expressed in terms of the $- \circ_i -$, as

$$(1.84)\quad f \cup g = (\mu \circ_0 f) \circ_{m-1} g$$

where $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$.

[5] The ambiguity with composition of functions is intentional: indeed, for $m = n = 1$ the circle product really *is* the composition of Hochschild 1-cochains.

[6] Gerstenhaber himself writes on page 86 of [10] that the Poisson identity relating the Gerstenhaber bracket to the cup product follows from "a (nasty) computation". I am not going to argue with this judgment.

The skew symmetry and Jacobi identity are discussed in [9, theorem 1]. These follow rather straightforwardly from the pre-Lie structure. Establishing that $\mathrm{C}^\bullet(A)$ has a pre-Lie structure is done by using that of a *pre-Lie system*, which takes all the $- \circ_i -$ into account. It is shown in [9, theorem 2] how such a pre-Lie system induces a pre-Lie algebra structure.

Proposition 47. Let $f \in \mathrm{C}^m(A, A)$, $g \in \mathrm{C}^n(A)$ and $h \in \mathrm{C}^p(A)$ be Hochschild cochains. Then

skew symmetry $[f, g] = -(-1)^{(m-1)(n-1)}[g, f]$

Jacobi identity $(-1)^{(m-1)(p-1)}[f, [g, h]] + (-1)^{(p-1)(n-1)}[h, [f, g]] + (-1)^{(n-1)(m-1)}[g, [h, f] = 0$

Remark 48. The skew symmetry means that we are considering graded Lie superalgebras, we will not consider graded Lie algebras that in the strict sense of the word.

Proof. The skew symmetry follows easily by replacing $[-, -]$ with its definition as the commutator of the circle product, and observing that the four terms appear with opposite signs.

For the proof of the Jacobi identity, one is referred to [9], as explained above. □

The next step is the compatibility with the Hochschild differential. In other words

Proposition 49. Let $f \in \mathrm{C}^m(A, A)$, $g \in \mathrm{C}^n(A)$ and $h \in \mathrm{C}^p(A)$ be Hochschild cochains. Then

$$\mathrm{d}([f, g]) = (-1)^{n-1}[\mathrm{d}(f), g] + [f, \mathrm{d}(g)]. \tag{1.85}$$

Proof. This follows from (1.83) and the Jacobi identity from proposition 47, applying (1.83) to $[f, g]$. □

From this we get the following corollary, which will be important for the deformation theory of algebras, see section 1.5. Recall that the axioms for a differential graded Lie algebra are precisely given by the results of proposition 47, except that there is a shift in the degree appearing.

Corollary 50. $\mathrm{C}^{\bullet+1}(A, A)$ is a differential graded Lie algebra.

Recall that in corollary 24 we saw that $\mathrm{HH}^1(A)$ has the structure of a Lie algebra. The following result tells us that it is a Lie subalgebra in degree 0 of a graded Lie algebra. It is clear from the definition of the Gerstenhaber bracket for elements in $\mathrm{C}^1(A)$ and the definition of the Lie algebra structure on $\mathrm{HH}^1(A)$ that they agree.

Proposition 51. $\mathrm{HH}^{\bullet+1}(A)$ is a graded Lie algebra.

Let us consider this graded Lie algebra structure in a special case.

Example 52. The Lie algebra $\mathrm{HH}^1(A)$ consisting of outer derivations acts on the Hochschild cohomology space $\mathrm{HH}^0(A)$, which we have shown to be the center $\mathrm{Z}(A)$ of A. If D is a derivation, and $z \in \mathrm{Z}(A)$ a central element, then

$$[D, z] = D \circ z - z \circ D = D \circ z = D(z) \tag{1.86}$$

commutes with every element $a \in A$, as one checks easily.

Commutativity of the cup product We can now prove the commutativity of the cup product on the level of cohomology. The main ingredient is given in proposition 53, which is a computation depending on the notion of a pre-Lie algebra that can be found in [9, theorem 3]. We will not reproduce it here[7].

[7] It is an interesting exercise to compute things in low degree, to get a feel for the formulas and the role of the Hochschild differential.

Proposition 53. Let $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$ be Hochschild cochains. Then

$$f \cup g - (-1)^{mn} g \cup f = \mathrm{d}(g) \circ f + (-1)^m \,\mathrm{d}(g \circ f) + (-1)^{m-1} g \circ \mathrm{d}(f) \tag{1.87}$$

But this leads us immediately to the proof of the graded-commutativity of $\mathrm{HH}^\bullet(A)$.

Proof of proposition 41. In the notation of proposition 53, if f and g are Hochschild cocycles, then (1.87) becomes

$$f \cup g - (-1)^{mn} g \cup f = \mathrm{d}_{n+m+1}(f \circ g). \tag{1.88}$$

So the difference between the commutator of two cocycles is a coboundary, and it vanishes when taking cohomology. ◻

Gerstenhaber algebra structure The cup product and Gerstenhaber bracket on Hochschild cohomology define the structure of a super-commutative algebra and a graded Lie superalgebra. They are moreover compatible in the following sense. We assign a name to this structure, because as it turns out, this is *not* the only natural example of such a structure. We will discuss polyvector fields, and their connection to Hochschild cohomology, in section 1.3.

Definition 54. A graded vector space $A^\bullet$ is a *Gerstenhaber algebra* if

1. $A^\bullet$ has an (associative) super-commutative multiplication of degree 0;
2. $A^\bullet$ has a super-Lie bracket of degree -1;
3. these two structures are related via the *Poisson identity*

   $$[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|} b[a, c]. \tag{1.89}$$

Written out in full detail, we have that

$$\begin{aligned} |ab| &= |a| + |b| \\ ab &= (-1)^{|a||b|} ba \end{aligned} \tag{1.90}$$

for the multiplication, and

$$\begin{aligned} |[a, b]| &= |a| + |b| - 1 \\ [a, b] &= -(-1)^{(|a|-1)(|b|-1)} [b, a] \end{aligned} \tag{1.91}$$

for the Lie bracket.

The Poisson identity then tells us that $a \mapsto [a, -] \colon A^p \to A^{p-1}$ is a derivation of degree $p - 1$.

Proposition 55. Let A be an associative k-algebra. Then $\mathrm{HH}^\bullet(A)$ is a Gerstenhaber algebra.

Proof. In proposition 41 and proposition 51 we have discussed the algebra and Lie algebra structure. The missing ingredient is the compatibility between these to structures through the Poisson identity. The proof of this goes along the same lines as the commutativity of the Gerstenhaber product: one shows that on the level of Hochschild cochains the obstruction to the Poisson identity is a certain coboundary given in [9, theorem 5]. This is a quite technical computation, and we will not reproduce it here. ◻

Remark 56. The cup product and Gerstenhaber bracket on the level of Hochschild cochain complexes do *not* satisfy the Poisson identity, nor is the dg algebra structure graded-commutative, so they do not give an immediate dg translation of a Gerstenhaber algebra structure. But there are homotopical versions of this structure, such as that of a B_∞-and G_∞-algebra, which fixes this incompatibility by introducing higher homotopies.

At this point we should mention that these (and other) homotopical structures form part of the program on the Deligne conjecture[8]). We will not go further into this for the time being, but this operadic picture is an important modern incarnation of the extra structure that we have discussed up to now.

In proposition 83 we will see another example of a Gerstenhaber algebra. These two examples are very closely related, and their story forms one of the main topics of these notes.

1.2.2 Hochschild homology is a Gerstenhaber module for Hochschild cohomology

For arbitrary algebras A there is no internal structure[9] on $\mathrm{HH}_\bullet(A)$ or $\mathrm{HH}_\bullet(A, M)$. But there are interesting *actions* of $\mathrm{HH}^\bullet(A)$ on $\mathrm{HH}_\bullet(A)$, such that $\mathrm{HH}_\bullet(A)$ is

- a module under the graded-commutative multiplication,
- a representation for the Gerstenhaber bracket

which are compatible in a certain way. The combination of these structures will be called a Gerstenhaber module, and they constitute an important part of the so-called Gerstenhaber (pre)calculus on the pair $(\mathrm{C}^\bullet(A), \mathrm{C}_\bullet(A))$. As we will not discuss this again until the very end[10] we will content ourselves with giving the definitions.

Observe that there are no good written proofs of the compatibility of these operations with the Hochschild differentials. Feel free to take this up as a challenge.

The cap product First up, the action by multiplication, i.e. the module structure.

Definition 57. Let M be an A-bimodule. Let $f \in \mathrm{C}^n(A)$ and $m \otimes a_1 \otimes \ldots \otimes a_p \in \mathrm{C}_p(A, M)$. Then their *cap product* is

$$f \cap (m \otimes a_1 \otimes \ldots \otimes a_p) = \begin{cases} (-1)^n m f(a_1 \otimes \ldots \otimes a_n) \otimes a_{n+1} \otimes \ldots a_p & p \geq n \\ 0 & p < n \end{cases} \tag{1.92}$$

which is an element of $\mathrm{C}_{p-n}(A, M)$.

One can then prove the following result.

Proposition 58. $\mathrm{C}_i(A, M)$ is a differential graded module over $\mathrm{C}^\bullet(A)$.

From this we get the following.

Corollary 59. $\mathrm{HH}_\bullet(A, M)$ is a graded module for the graded-commutative algebra $\mathrm{HH}^\bullet(A)$.

Remark 60. In particular we have that $\mathrm{HH}_i(A, M)$ is a module over $\mathrm{HH}^0(A) \cong \mathrm{Z}(A)$.

[8] Stated in 1993 in a letter to Gerstenhaber–May–Stasheff, now a theorem with proofs due to Tamarkin, McClure–Smith, Kontsevich–Soibelman, …

[9] If A is commutative we discuss the shuffle product in section 1.2.3.

[10] At least for now. The interested reader is invited to prove the following properties him- or herself.

The Lie derivative The next step is the action by the Lie bracket.

Definition 61. Let $f \in \mathrm{C}^{n+1}(A)$ and $a_0 \otimes a_1 \otimes \ldots \otimes a_p \in \mathrm{C}_p(A, M)$. Then the *Lie derivative* of $a_0 \otimes a_1 \otimes \ldots \otimes a_p$ with respect to f is

$$
\begin{aligned}
\mathrm{L}_f(a_0 \otimes a_1 \otimes \ldots \otimes a_p) &= \sum_{i=0}^{p-n} (-1)^{ni} a_0 \otimes \ldots \otimes a_{i-1} \otimes f(a_i \otimes \ldots \otimes a_{i+n}) \otimes a_{i+n+1} \ldots \otimes a_p \\
&+ \sum_{j=p-n}^{p-1} (-1)^{p(j+1)} f(a_{j+1} \otimes \ldots \otimes a_p \otimes a_0 \otimes \ldots \otimes a_{n-p+j}) \otimes a_{p-n+j+1} \otimes \ldots \otimes a_j
\end{aligned}
\tag{1.93}
$$

One can then prove the following result.

Proposition 62. $\mathrm{C}_i(A)$ is a differential graded Lie representation over $\mathrm{C}^{\bullet+1}(A)$.

From this we get the following.

Corollary 63. $\mathrm{HH}_\bullet(A)$ is a representation of the graded Lie algebra $\mathrm{HH}^{\bullet+1}(A)$.

We can combine these into the notion of a Gerstenhaber module, and discuss the notion of a Gerstenhaber (pre)calculus. We will not do this for now.

1.2.3 The shuffle product on Hochschild homology

In general $\mathrm{HH}_\bullet(A)$ is only a graded $\mathrm{HH}^\bullet(A)$-module. But if A is commutative we can equip it with its own product. The algebra structure on $\mathrm{HH}_\bullet(A)$ for A commutative is actually induced using a pairing

$$\mathrm{C}_\bullet(A, M) \otimes_k \mathrm{C}_\bullet(B, N) \to \mathrm{C}_\bullet(A \otimes_k B, M \otimes_k N) \tag{1.94}$$

which is defined for arbitrary algebras A and B, and bimodules M and N (unlike in the rest of this section we will use M and N to make the formulas a bit more transparent, but we will have $M = A$ and $N = B$ in applications). This will be the shuffle product from the title of this section.

Definition 64. A *(p, q)-shuffle* is an element σ of Sym_{p+q} such that $\sigma(i) < \sigma(j)$ whenever

1. $1 \leq i < j \leq p$,
2. or $p + 1 \leq i < j \leq p + q$.

The subset of (p, q)-shuffles inside the symmetric group is denoted $\mathrm{Sh}_{p,q}$.

We can define an action of Sym_n on $\mathrm{C}_n(A, M)$, by setting

$$\sigma \cdot (m \otimes a_1 \otimes \ldots \otimes a_n) \coloneqq m \otimes a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)} \tag{1.95}$$

for $\sigma \in \mathrm{Sym}_n$ and $m \otimes a_1 \otimes \ldots a_n \in \mathrm{C}_n(A, M)$.

Definition 65. The *(p, q)-shuffle product* for A and B is the morphism

$$\mathrm{sh}_{p,q}(-,-) = - \times -\colon \mathrm{C}_p(A, M) \otimes_k \mathrm{C}_q(B, N) \to \mathrm{C}_{p+q}(A \otimes_k B, M \otimes_k N) \tag{1.96}$$

which sends $(m \otimes a_1 \otimes \ldots a_p) \otimes (n \otimes b_1 \otimes \ldots \otimes b_q)$ to

$$\sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)\sigma \cdot \Big((m \otimes n) \otimes (a_1 \otimes 1) \otimes \ldots \otimes (a_p \otimes 1) \otimes (1 \otimes b_1) \otimes \ldots \otimes (1 \otimes b_q)\Big) \tag{1.97}$$

The next lemma shows that the Hochschild homology differential is a graded derivation for the shuffle product. For a proof, see [22, proposition 4.2.2].

Lemma 66. Let $m \otimes a_1 \otimes \ldots \otimes a_p \in \mathrm{C}_p(A, M)$ and $n \otimes b_1 \otimes \ldots \otimes b_q \in \mathrm{C}_q(B, N)$ be Hochschild chains. Then

$$\begin{aligned} &\mathrm{d}\left((m \otimes a_1 \otimes \ldots \otimes a_p) \times (n \otimes b_1 \otimes \ldots \otimes b_q)\right) \\ &\quad = \mathrm{d}(m \otimes a_1 \otimes \ldots \otimes a_p) \times (n \otimes b_1 \otimes \ldots \otimes b_q) + (-1)^p (m \otimes a_1 \otimes \ldots \otimes a_p) \times \mathrm{d}(n \otimes b_1 \otimes \ldots \otimes b_q). \end{aligned} \tag{1.98}$$

Proof. Let us write the ith summand of the differential as in (1.20) by d_i, indexed by $i = 0, \ldots, n$. Let us moreover write

$$(m \otimes a_1 \otimes \ldots \otimes a_p) \times (n \otimes b_1 \otimes \ldots \otimes b_q) = \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)(m \otimes n) \otimes c_1 \otimes \ldots \otimes c_{p+q} \tag{1.99}$$

where c_i is in the set $\{a_1 \otimes 1, \ldots, a_p \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_q\}$. Now consider

$$\mathrm{d}_i((m \otimes n) \otimes c_1 \otimes \ldots \otimes c_{p+q}, \tag{1.100}$$

for $i = 0, \ldots, n$. We now explain what happens with (1.100) on a case-by-case analysis.

- If $i = 0$, then $c_1 = a_1 \otimes 1$ (resp. $c_1 = 1 \otimes b_1$), and (1.100) appears in the first summand (resp. second summand) of the right-hand side of (1.98).
- The case $i = n$ is similar.
- If $i = 1, \ldots, n-1$ then we distinguish two cases:
  1. If c_i and c_{i+1} are elements of the form $a \otimes 1$ (resp. $1 \otimes b$) then they appear in the first (resp. second summand) of the right-hand side of (1.98).
  2. Otherwise we can permute them, as they will still arise from the application of a different (p, q)-shuffle, in which case we can cancel them, as they appear with opposite signs in the shuffle product.

□

Using the shuffle product we can construct the Künneth formula for Hochschild homology: we will combine the (p, q)-shuffles in the following way

$$\mathrm{sh}_n := \sum_{p+q=n} \mathrm{sh}_{p,q} \colon (\mathrm{C}_\bullet(A) \otimes_k \mathrm{C}_\bullet(B))_n = \bigoplus_{p+q=n} \mathrm{C}_p(A) \otimes_k \mathrm{C}_q(B) \to \mathrm{C}_n(A \otimes_k B). \tag{1.101}$$

Proposition 67. The morphism $\mathrm{sh}_\bullet$ is a morphism of chain complexes.

Proof. By lemma 66 we can express $\mathrm{d} \circ \mathrm{sh}_{p,q}(-,-)$ in terms of $\mathrm{sh}_{p-1,q}(\mathrm{d}(-),-)$ and $\mathrm{sh}_{p,q-1}(-,\mathrm{d}(-))$, which with the appropriate signs gives the differential in the tensor product of chain complexes. □

But $\mathrm{sh}_\bullet$ is not just an morphism of chain complexes: it is actually a quasi-isomorphism. The proof of this result can be found [29, §9.4].

Theorem 68 (Künneth formula)**.** The shuffle product $\mathrm{sh}_\bullet$ induces an isomorphism

$$\mathrm{HH}_\bullet(A) \otimes_k \mathrm{HH}_\bullet(B) \cong \mathrm{HH}_\bullet(A \otimes_k B). \tag{1.102}$$

Remark 69. Observe that a similar statement is not true for Hochschild cohomology, at least not without conditions on A and B. In exercise 73 a suggestion for a counterexample is given. In [29, §9.4] the condition that at least one of them is finite-dimensional is used. It is not clear to me whether this can be generalised.

If we now impose commutativity, then the multiplication gives us a morphism of algebras

$$\mu\colon A \otimes_k A \to A. \tag{1.103}$$

Using functoriality of the Hochschild chain complex, we obtain a morphism

$$\mathrm{C}_\bullet(A \otimes_k A) \to \mathrm{C}_\bullet(A). \tag{1.104}$$

One can then prove that this equips the Hochschild chain complex with the structure of a commutaitve differential graded algebra [29, proposition 9.4.2], and therefore we have the following.

Proposition 70. $\mathrm{HH}_\bullet(A)$ is a graded-commutative algebra.

1.2.4 Exercises

Exercise 71. Let $\mathfrak{g}$ be a Lie algebra. Equip $\bigwedge^\bullet \mathfrak{g}$ with the exterior product as multiplication, and the unique extension of the Lie bracket on $\bigwedge^1 \mathfrak{g}$ to all of $\bigwedge^\bullet \mathfrak{g}$. Show that this is a Gerstenhaber algebra.

Exercise 72. Use the definition of the circle product to check remark 46.

Exercise 73. Let K, L be fields of infinite transcendence degree over k. Then

$$\mathrm{HH}^\bullet(K \otimes_k L) \not\cong \mathrm{HH}^\bullet(K) \otimes_k \mathrm{HH}^\bullet(L). \tag{1.105}$$

Exercise 74. Explain how (1.84) gives an alternative proof that $\mathrm{d}^2 = 0$, using only the graded Lie algebra structure of $\mathrm{C}^\bullet(A)$.

1.3 The Hochschild–Kostant–Rosenberg isomorphism

The goal of this section is to discuss the Hochschild–Kostant–Rosenberg isomorphism, which identifies the Hochschild (co)homology of a regular *commutative* k-algebra A with its polyvector fields and differential forms. It is given as [14, theorem 5.2], where the interpretation from theorem 15 is used to make the link with Hochschild (co)homology.

To understand where the isomorphism comes from, recall that we have identifications

$$(1.106)\quad \begin{cases} \mathrm{HH}^0(A) \cong A & \text{proposition 19} \\ \mathrm{HH}^1(A) \cong \mathrm{Der}(A) & \text{proposition 22} \end{cases}$$

and

$$(1.107)\quad \begin{cases} \mathrm{HH}_0(A) \cong A & \text{proposition 17} \\ \mathrm{HH}_1(A) \cong \Omega^1_A & \text{proposition 79} \end{cases}$$

where the identification for $\mathrm{HH}_1(A)$ stricto sensu is not yet known[11].

Then the Hochschild–Kostant–Rosenberg isomorphism (see theorem 98) tells us that we can get *all* of the Hochschild (co)homology by taking exterior powers of what we have in degree 1, and that this is an isomorphism of graded commutative algebras: by propositions 41 and 70 we have that $\mathrm{HH}^\bullet(A)$ and $\mathrm{HH}_\bullet(A)$ are graded commutative, and the exterior product is graded commutative by construction.

1.3.1 Polyvector fields and differential forms

Let us introduce the module Ω^1_A which already made an appearance in (1.107) without being defined.

Definition 75. The *module of Kähler differentials* Ω^1_A is the A-module which is generated by the symbols $\mathrm{d}a$ for $a \in A$, subject to the relations[12]

$$(1.108)\quad \mathrm{d}(\lambda a + \mu b) = \lambda \mathrm{d}a + \mu \mathrm{d}b$$

for all $\lambda, \mu \in k$ and $a, b \in A$, and

$$(1.109)\quad \mathrm{d}ab = a\mathrm{d}b + b\mathrm{d}a$$

for all $a, b \in A$.

The module of Kähler differentials appears in many ways in this context. First of all, it satisfies a well-known universal property: it co-represents the functor of derivations, via the *universal derivation*

$$(1.110)\quad \mathrm{d}\colon A \to \Omega^1_A : a \mapsto \mathrm{d}\,a.$$

Proposition 76. We have an isomorphism

$$(1.111)\quad \mathrm{Hom}_A(\Omega_A, M) \cong \mathrm{Der}(A, M)$$

sending $\alpha\colon \Omega_A \to M$ to $\mathrm{d}\circ\alpha : A \to M$, giving an isomorphism of functors $\mathrm{Hom}_A(\Omega_A, -) \cong \mathrm{Der}(A, -)$.

[11] The dilemma is whether to give a preliminary discussion of Ω^1_A in section 1.1.3, or postpone it until we have time to discuss it in detail. We have opted for the latter.

[12] To be precise, and consistent with the notation in the literature, we should denote this by $\Omega^1_{A/k}$, making the dependence on the base field explicit. But then we should also do this for $\mathrm{Der}(A)$, and likewise for $\mathrm{HH}^\bullet$ and $\mathrm{HH}_\bullet$, which we won't.

Recall from proposition 22 that $\mathrm{Der}(A, M) \cong \mathrm{HH}^1(A, M)$. So we have that $\mathrm{HH}^1(A) \cong \mathrm{Der}(A) \cong \mathrm{Hom}_A(\Omega_A, A)$. This is the first ingredient in the proof of the Hochschild–Kostant–Rosenberg isomorphism for Hochschild cohomology. In geometric notation, when $X = \operatorname{Spec} A$, we have $\mathrm{HH}^1(A) \cong \mathrm{T}_X$.

There is a second description of the Kähler differentials which is useful to us. For a proof of this standard fact one is referred to [26, tag 00RW].

Proposition 77. Let $I := \ker(\mu\colon A \otimes_k A \to A)$. Then the morphism

(1.112) $\Omega^1_A \to I/I^2 : a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$

is an isomorphism of A-modules.

Remark 78. If A is noncommutative, then one denotes $\Omega^1_{\mathrm{nc}}(A) := I$ the bimodule of noncommutative differential forms on A. In that case (1.111) takes on the form

(1.113) $\mathrm{Der}(A, M) \cong \mathrm{Hom}_{A^e}(\Omega^1_{\mathrm{nc}}(A), M)$.

For more information, one is referred to [11, §10] or [30, §3.2].

Finally we can relate Ω^1_A to Hochschild homology, just like we have already done for Hochschild cohomology, which is the first step in understanding the Hochschild–Kostant–Rosenberg isomorphism for Hochschild homology.

Proposition 79. Let M be a symmetric A-bimodule. Then

(1.114) $\mathrm{HH}_1(A, M) \cong M \otimes_A \Omega^1_A$.

In particular we have

(1.115) $\mathrm{HH}_1(A) \cong \Omega^1_A$.

Proof. By assumption the morphism $M \otimes_k A \to M$ is the zero morphism, as this is the Hochschild differential as in (1.30) and M is symmetric, so $\mathrm{HH}_1(A, M)$ is the quotient of $M \otimes_k A$ by the subspace generated by $ma \otimes b - m \otimes ab + bm \otimes a$. So the morphism $\mathrm{HH}_1(A, M) \to M \otimes_A \Omega^1_A$ sending $m \otimes a$ to $m \otimes \mathrm{d}a$ is well-defined by (1.109).

In the other direction we consider the morphism $M \otimes_A \Omega^1_A \to \mathrm{C}_1(A, M)$ sending $m \otimes a\mathrm{d}b$ to $ma \otimes b$. This morphism lands in $\mathrm{Z}_1(A, M)$ by assumption, and one checks that the maps on cohomology are inverse. □

Remark 80. It is important that M is symmetric: exercise 37 gives an example where this fails.

1.3.2 Gerstenhaber algebra structure on polyvector fields

Using $\mathrm{Der}(A)$ we can construct a new Gerstenhaber algebra, which will be closely related to the Gerstenhaber algebra structure on Hochschild cohomology. We will do this by considering $\bigwedge^\bullet \mathrm{Der}(A)$, the polyvector fields (or multiderivations) on A. On $\bigwedge^\bullet \mathrm{Der}(A)$ we can consider the exterior product of polyvector fields, which equips it with the structure of a graded commutative algebra.

The space of derivations is the algebraic version of the vector fields on a manifold. As such, it is equipped with a Lie bracket. We can extend this Lie bracket to all of $\bigwedge^\bullet \mathrm{Der}(A)$ in the following way.

Definition 81. Let $\alpha_1 \wedge \ldots \wedge \alpha_m \in \bigwedge^m \operatorname{Der}(A)$ and $\beta_1 \wedge \ldots \wedge \beta_n \in \bigwedge^m \operatorname{Der}(A)$ be polyvector fields. Their *Schouten–Nijenhuis bracket*[13] is given by

(1.116)
$$[\alpha_1\wedge\ldots\wedge\alpha_m, \beta_1\wedge\ldots\wedge\beta_n] \coloneqq \sum_{i=1}^m\sum_{j=1}^n(-1)^{i+j+m-1}[\alpha_i,\beta_j]\wedge\alpha_1\wedge\ldots\wedge\widehat{\alpha_i}\wedge\ldots\wedge\alpha_m\wedge\beta_1\wedge\ldots\wedge\widehat{\beta_j}\wedge\ldots\wedge\beta_n.$$

This bracket is the unique extension to a graded Lie algebra structure when one imposes that $[D,z] = D(z)$ for $D \in \mathrm{HH}^1(A)$ and $z \in \mathrm{HH}^0(A) \cong A$ as in example 52 and $[D_1,D_2] = D_1\circ D_2 - D_2\circ D_1$ for $D_1,D_2 \in \mathrm{HH}^1(A)$ as in corollary 24. The following lemma is proved by staring at the signs.

Lemma 82. The Schouten–Nijenhuis bracket equips $\operatorname{Der}(A)$ with the structure of a graded Lie algebra.

Because the Schouten–Nijenhuis bracket was defined in terms of the generators of the algebra, we obtain the following.

Proposition 83. The exterior product and Schouten–Nijenhuis bracket equip $\bigwedge^\bullet \operatorname{Der}(A)$ with the structure of a Gerstenhaber algebra.

Remark 84. We have not yet precisely defined what the dg version of a Gerstenhaber algebra is (as it requires to understand operations up to homotopy), so it's not clear what exactly the extra structure induced by the cup product and Gerstenhaber bracket on $\mathrm{C}^\bullet(A)$ is. But observe that we can equip $\bigwedge^\bullet \operatorname{Der}(A)$ with the zero differential, in which case it will be a (strict) dg Gerstenhaber algebra. Then the Hochschild–Kostant–Rosenberg isomorphism for Hochschild cohomology can be upgraded to Kontsevich formality: a quasi-isomorphism of "dg Gerstenhaber algebras" between $\bigwedge^\bullet \operatorname{Der}(A)$ and $\mathrm{C}^\bullet(A)$, i.e. Hochschild cochains are quasi-isomorphic to their cohomology, and we know exactly what this cohomology is. We might discuss formality results later on in these notes.

1.3.3 Gerstenhaber module structure on differential forms

This section will be expanded at some point.

1.3.4 The Hochschild–Kostant–Rosenberg isomorphism: Hochschild homology

We now come to the first proof of the Hochschild–Kostant–Rosenberg isomorphism, for smooth commutative algebras. We will do this in a rather classical fashion for now, based on [22, §1.3, §3.4]. The proof for smooth projective varieties in section 3.3 will use more advanced machinery. It should be remarked that it *is* actually possible to globalise the current proof without using the machinery of derived categories and Atiyah classes in an essential way, and maybe this will be discussed too at some point.

The statement of the Hochschild–Kostant–Rosenberg isomorphism in this setting is the following.

Theorem 85. Let A be a smooth k-algebra and M a symmetric A-bimodule. Then the antisymmetrisation morphism (1.118) induces an isomorphism

(1.117) $\Omega^n_{A/k} \otimes_A M \to \mathrm{HH}_n(A,M).$

When $A = M$ this isomorphism is an isomorphism of graded k-algebras.

[13] Sometimes just Schouten bracket. I like to speculate that this is purely for pronunciation reasons.

The proof given in this section naturally splits in two pieces:

1. constructing the antisymmetrisation morphism $\epsilon_n \colon M \otimes_A \Omega^n_A \to \mathrm{HH}_n(A, M)$ of A-modules (no smoothness is required here);
2. showing that it is an isomorphism by checking it at every maximal ideal, using the description of Hochschild homology as Tor and an explicit free resolution (the Koszul resolution) in the local setting (smoothness is required here).

The construction of the morphism is done in proposition 89, and checking that it locally is an isomorphism is done after we prove proposition 95.

Note that, if we would only be interested in $\mathrm{HH}_n(A)$ and not $\mathrm{HH}_n(A, M)$, the construction of the morphism in step 1 can be done via a universal property, based on the graded-commutative algebra structure from proposition 70. We will take this approach in the case of Hochschild cohomology in section 1.3.5. Observe that this was the generality of the original paper of Hochschild–Kostant–Rosenberg, i.e. they only considered $\mathrm{HH}_\bullet(A)$ and $\mathrm{HH}^\bullet(A)$.

The antisymmetrisation morphism In proposition 79 we saw that the first Hochschild homology is isomorphic to the Kähler differentials, with the morphism $\Omega^1_A \otimes_A M \to \mathrm{HH}_1(A, M)$ being of the form $m \otimes a\mathrm{d}b \mapsto ma \otimes b$. We can extend these morphisms to differential n-forms and $\mathrm{HH}_n(A, M)$ in the following way. First we introduce the *antisymmetrisation map*

$$\epsilon_n \colon M \otimes_k \bigwedge^n A \to \mathrm{C}_n(A, M) : m \otimes a_1 \wedge \ldots \wedge a_n \mapsto \sum_{\sigma \in \mathrm{Sym}_n} \mathrm{sgn}(\sigma)\sigma \cdot m \otimes a_1 \otimes \ldots \otimes a_n \tag{1.118}$$

where the action of σ is defined analogously to (1.95). Remark that from this point on we will have to be careful about whether $\otimes$ or $\bigwedge$ is taken over k or A.

We want to turn this into a morphism $M \otimes_A \Omega^n_A \to \mathrm{HH}_n(A, M)$, so we need to show that

1. ϵ_n is compatible with the Hochschild differential;
2. it factors through $M \otimes_A \Omega^n_A$.

To do the first, we will use a technical trick, inspired by Chevalley–Eilenberg (co)homology for Lie algebras. If $\mathfrak{g}$ is a Lie algebra, and M a Lie module over it, then the *Chevalley–Eilenberg differential* is

$$\begin{aligned} &\mathrm{d}_{\mathrm{CE}} \colon M \otimes_k \bigwedge^n \mathfrak{g} \to M \otimes_k \bigwedge^{n-1} \mathfrak{g} \\ &m \otimes g_1 \wedge \ldots \wedge g_n \mapsto \sum_{i=1}^n (-1)^{i-1}[m, g_i] \otimes g_1 \wedge \ldots \wedge \widehat{g_i} \wedge \ldots g_n \\ &\qquad + \sum_{1 \le i < j \le n} (-1)^{i+j-1} m \otimes [g_i, g_j] \wedge \ldots \wedge \widehat{g_i} \wedge \ldots \wedge \widehat{g_j} \wedge \ldots \wedge g_n. \end{aligned} \tag{1.119}$$

The role of this differential in Chevalley–Eilenberg cohomology, which is the cohomology theory for Lie algebras parallel to Hochschild cohomology for associative algebras, will eventually be explained in appendix A.3.

We will apply it to $\mathfrak{g} = A$, where A is considered as a Lie algebra via the commutator bracket. In particular, when A is commutative this is an abelian Lie algebra. But the following proposition holds without commutativity assumption.

Proposition 86. The diagram

$$\begin{tikzcd} M \otimes_k \bigwedge^n A \arrow[r, "\epsilon_n"] \arrow[d, "\mathrm{d}_{\mathrm{CE}}"] & \mathrm{C}_n(A, M) \arrow[d, "\mathrm{d}"] \\ M \otimes_k \bigwedge^{n-1} A \arrow[r, "\epsilon_{n-1}"] & \mathrm{C}_{n-1}(A, M) \end{tikzcd} \tag{1.120}$$

commutes for all $n \geq 0$.

The proof goes via induction. We will need the following technical (but easy) lemma, where

$$\begin{aligned} \mathrm{ad}_n(a) \colon\ & \mathrm{C}_n(A, M) \to \mathrm{C}_n(A, M) \\ & m \otimes a_1 \otimes \ldots \otimes a_n \mapsto \sum_{i=0}^{n} m \otimes a_1 \otimes \ldots \otimes [a, a_i] \otimes \ldots \otimes a_n \end{aligned} \tag{1.121}$$

is an extension of the notion of inner derivation to $\mathrm{C}_n(A, M)$, and

$$\begin{aligned} h_n(a) \colon\ & \mathrm{C}_n(A, M) \to \mathrm{C}_{n+1}(A, M) \\ & m \otimes a_1 \otimes \ldots \otimes a_n \mapsto \sum_{i=0}^{n} (-1)^i m \otimes a_1 \otimes \ldots \otimes a_i \otimes a \otimes a_{i+1} \otimes \ldots \otimes a_n \end{aligned} \tag{1.122}$$

will provide a null-homotopy for our newly defined $\mathrm{ad}_n(a)$, and an inductive way to describe ϵ_n as in (1.125).

Lemma 87. We have that

$$-\mathrm{ad}_n(a) = \mathrm{d} \circ h_n(a) + h_{n-1}(a) \circ \mathrm{d}\,. \tag{1.123}$$

In particular $\mathrm{ad}_n(a) \colon \mathrm{HH}_n(A, M) \to \mathrm{HH}_n(A, M)$ is zero, as for $n = 0$ in proposition 22.

Proof. The term $\mathrm{d} \circ h_n(a)$ gives $[a, a_i]$ by considering the Hochschild differential for the summands containing $a \wedge a_i$ and $a_i \wedge a$. The term $h_{n-1}(a) \circ \mathrm{d}$ cancels all the other summands. □

We can now give the proof of proposition 86.

Proof of proposition 86. The statement for $n = 0$ is vacuous as the lower line is zero. For $n = 1$ we have that $\epsilon_0 = \mathrm{id}_M$ and $\epsilon_1 = \mathrm{id}_{M \otimes_k A}$. As

$$\mathrm{d}_{\mathrm{CE}}(m \otimes a) = [m, a] = ma - am = \mathrm{d}(m \otimes a) \tag{1.124}$$

the diagram commutes.

Let us assume that $\mathrm{d} \circ \epsilon_n = \epsilon_{n-1} \circ \mathrm{d}_{\mathrm{CE}}$. By construction we have that

$$\epsilon_{n+1}(m \otimes a_1 \wedge \ldots \wedge a_n \wedge a_{n+1}) = (-1)^n h_n(a_{n+1}) \epsilon_n(m \otimes a_1 \wedge \ldots \wedge a_n), \tag{1.125}$$

so

$$
\begin{aligned}
&\mathrm{d}\circ\epsilon_{n+1}(m\otimes a_1\wedge\ldots\wedge a_{n+1}) \\
&\quad= (-1)^n\,\mathrm{d}\circ h_n(a_{n+1})\circ\epsilon_n(m\otimes a_1\wedge\ldots\wedge a_n) && (1.125) \\
&\quad= (-1)^n(-\operatorname{ad}_n(a_{n+1}) - h_n(a_{n+1})\circ\mathrm{d})\circ\epsilon_n(m\otimes a_1\wedge\ldots\wedge a_n) && \text{lemma 87} \\
&\quad= (-1)^{n+1}\operatorname{ad}_n(a_{n+1})\circ\epsilon_n(m\otimes a_1\wedge\ldots\wedge a_n) \\
&\qquad\quad+ (-1)^{n-1}h_{n-1}(a_{n+1})\circ\epsilon_{n-1}\circ\mathrm{d}_{\mathrm{CE}}(m\otimes a_1\wedge\ldots\wedge a_n) \\
&\quad= (-1)^{n+1}\operatorname{ad}_n(a_{n+1})\circ\epsilon_n(m\otimes a_1\wedge\ldots\wedge a_n) \\
&\qquad\quad+ \epsilon_n\left(\mathrm{d}_{\mathrm{CE}}(m\otimes a_1\wedge\ldots\wedge a_n)\wedge a_{n+1}\right) && (1.125) \\
&\quad= \epsilon_n\circ\mathrm{d}_{\mathrm{CE}}(m\otimes a_1\wedge\ldots\wedge a_n)\wedge a_{n+1}).
\end{aligned}
\tag{1.126}
$$

∎

Corollary 88. If A is commutative and M symmetric, then $\operatorname{im}(\epsilon_n)\subseteq \mathrm{Z}_n(A)$. In particular, there exists a morphism

$$\epsilon_n\colon M\otimes_k \bigwedge^n A\to \mathrm{HH}_n(A,M). \tag{1.127}$$

Proof. The Chevalley–Eilenberg differential is identically zero in this case. ∎

Now we can check that the antisymmetrisation indeed defines a morphism of the desired form.

Proposition 89. Let A be commutative, and M a symmetric A-bimodule. Then the morphism (1.127) factors as

$$
\begin{tikzcd}
M\otimes_k \bigwedge^n A \arrow[r, "\epsilon_n"] \arrow[d] & \mathrm{HH}_n(A,M) \\
M\otimes_A \Omega_A^n \arrow[ur, "\epsilon_n"] &
\end{tikzcd}
\tag{1.128}
$$

where we will recycle the symbol ϵ_n for the morphism that we are interested in.

Proof. Recall that Ω_A^1 is generated by the symbols $\mathrm{d}a$, and hence Ω_A^n by the symbols $\mathrm{d}a_1\wedge\ldots\wedge\mathrm{d}a_n$.

We need to check that ϵ_n is compatible with the relations imposed on Ω_A^1 and that we can go from a tensor product over k to a tensor product over A. By the definition of ϵ_n we can assume that the product ab is the first position. We need to show that

$$\epsilon_n(m\otimes ab\wedge a_2\wedge\ldots\wedge a_n) - \epsilon_n(ma\otimes b\wedge a_2\wedge\ldots\wedge a_n) - \epsilon_n(mb\otimes a\wedge a_2\wedge\ldots\wedge a_n) \tag{1.129}$$

is actually zero in homology, as this expresses the relation $\mathrm{d}ab = a\mathrm{d}b + b\mathrm{d}a$, together with the change from $-\otimes_k-$ to $-\otimes_A-$.

If $n=0$ then there is nothing to check. If $n=1$ we have that (1.129) is $\mathrm{d}(m\otimes a\otimes b)$. More generally one can check that

$$(1.129) = -\,\mathrm{d}\left(\sum_{\sigma\in S}\operatorname{sgn}(\sigma)\sigma\cdot(m\otimes a\otimes b\otimes a_2\otimes\ldots\otimes a_n\right) \tag{1.130}$$

where $S=\{\sigma\in\mathrm{Sym}_{n+1}\mid\sigma(1)<\sigma(2)\}$. ∎

So for now we have only used commutativity of A. We will continue the proof of the Hochschild–Kostant–Rosenberg isomorphism after a short digression.

The projection morphism Before we continue with the Hochschild–Kostant–Rosenberg decomposition for smooth algebras we can prove something for arbitrary commutative algebras over fields of characteristic 0, by constructing a morphism in the opposite direction.

$$\pi_n\colon \mathrm{C}_n(A,M) \to M \otimes_A \Omega^n_A : m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_n. \tag{1.131}$$

This morphism is again compatible with the Hochschild differential.

Lemma 90. We have that $\pi_n \circ \mathrm{d} = 0$ for all $n \geq 0$.

Proof. Using the relation $\mathrm{d}ab = a\mathrm{d}b + b\mathrm{d}a$ after applying π_{n-1} to the expression (1.20) allows one to pair off terms with opposite signs. □

Corollary 91. There exists a morphism

$$\pi_n\colon \mathrm{HH}_n(A,M) \to M \otimes_A \Omega^n_A. \tag{1.132}$$

Proposition 92. The composition $\pi_n \circ \epsilon_n$ is multiplication by $n!$.

Proof. We have the equality

$$m \otimes \mathrm{d}a_{\sigma^{-1}(1)} \wedge \ldots \wedge \mathrm{d}a_{\sigma^{-1}(n)} = \mathrm{sgn}(\sigma) m \otimes \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_n \tag{1.133}$$

so this term appears $n!$ times. □

In characteristic zero we therefore obtain the following corollary.

Corollary 93. If $\mathrm{char}\, k = 0$, then $M \otimes_A \Omega^n_A$ is a direct summand of $\mathrm{HH}_n(A,M)$.

This leads to the *λ-decomposition* or *Hodge decomposition* of Hochschild homology, but we will not develop this further for now. The interested reader is referred to [22, §4.5]. Just be warned that what is called the Hochschild–Kostant–Rosenberg decomposition in section 3.3 is sometimes referred to as the Hodge decomposition, especially in earlier papers. We should stress that

1. in the affine setting the Hodge decomposition is only interesting in the presence of singularities, and in the smooth case it reduces to the Hochschild–Kostant–Rosenberg isomorphism;
2. in the smooth and projective setting the Hochschild–Kostant–Rosenberg decomposition was originally proved only for Hochschild cohomology, whence the name Hodge decomposition was used, but as the Hochschild–Kostant–Rosenberg decomposition for Hochschild homology is a transpose (see section 3.3) of the Hodge decomposition arising in Hodge theory, this leads to an unfortunate clash of terminology, which is avoided in these notes.

Computing Tor via the Koszul resolution We have seen in theorem 15 that Hochschild homology can be described using Tor, as the bar complex provided a free resolution of A as a bimodule. We will need another explicit free resolution in the computation of Tor for the proof of the Hochschild–Kostant–Rosenberg isomorphism, when A is a smooth local k-algebra. This will be provided by the Koszul complex, which is a standard object in algebra and algebraic geometry. For more information one is referred to [8, §17], we will only recall some notation and facts.

Definition 94. Let A be a commutative ring. Let $f\colon M = A^{\oplus n} \to A$ be a morphism of A-modules. Then the *Koszul complex* associated to f is

$$0 \to \bigwedge^n M \to \ldots \to \bigwedge^1 M \to A \to 0 \tag{1.134}$$

where

$$\mathrm{d}\colon \bigwedge^j M \to \bigwedge^{j-1} M : m_1 \wedge \ldots \wedge m_j \mapsto \sum_{i=1}^{j} (-1)^{i+1} f(m_i) m_1 \wedge \ldots \wedge \widehat{m}_{i-1} \wedge \ldots \wedge m_j. \tag{1.135}$$

One can check that this is indeed a complex, but more importantly, when the morphism f corresponds to a regular sequence for an ideal I, then it is actually a free resolution of A/I.

Recall that $f = (a_1, \ldots, a_n)$ is a regular sequence if a_{i+1} is not a zero-divisor in $A/(a_1, \ldots, a_i)$. In definition 96 we relate this to smoothness of a k-algebra.

We prove the following general result, which will be applied to the local rings we encounter after applying the local-to-global principle.

Proposition 95. Let B be a commutative local ring, and I an ideal of B generated by a regular sequence $\mathrm{g} = (g_1, \ldots, g_n)$. Then the isomorphism

$$\epsilon_1\colon I/I^2 \xrightarrow{\cong} \mathrm{Tor}_1^B(B/I, B/I) \tag{1.136}$$

induces an isomorphism

$$\epsilon_\bullet\colon \bigwedge\nolimits_{B/I}^{\bullet} I/I^2 \xrightarrow{\cong} \mathrm{Tor}_\bullet^B(B/I, B/I) \tag{1.137}$$

of graded-commutative algebras.

Proof. The Koszul complex provides a free resolution

$$0 \to \bigwedge\nolimits_B^n B^{\oplus n} \to \ldots \to \bigwedge\nolimits_B^2 B^{\oplus n} \to B^{\oplus n} \to B \to B/I \to 0 \tag{1.138}$$

of B/I as a B-module which we can use to compute Tor:

$$\begin{aligned} \mathrm{Tor}_\bullet^B(B/I, B/I) &\cong \mathrm{H}_\bullet\left((\bigwedge\nolimits_B^\bullet B^{\oplus n}) \otimes_B B/I, \mathrm{d}_{\mathrm{g}} \otimes_B \mathrm{id}_{B/I} \right) \\ &\cong \mathrm{H}_\bullet\left((\bigwedge\nolimits_B^\bullet B^{\oplus n}) \otimes_B B/I, 0 \right) \\ &\cong \bigwedge\nolimits_B^\bullet (B/I)^{\oplus n} \\ &\cong \bigwedge\nolimits_B^\bullet I/I^2 \end{aligned} \tag{1.139}$$

The second isomorphism follows from the observation that d_{g} has coefficients landing in $I \subseteq B$, as $\mathrm{d}_{\mathrm{g}}\colon \bigwedge^{k+1} B^{\oplus n} \to \bigwedge^k B^{\oplus n}$ has the form

$$\mathrm{d}_{\mathrm{g}}(v_0 \wedge \ldots \wedge v_k) = \sum_{i=0}^{k} (-1)^i \mathrm{g}(v_i) v_0 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k. \tag{1.140}$$

As I is generated by a regular sequence, we have that I/I^2 is a free B/I-module of rank n, generated by the classes of the elements in the sequence.

Finally, to check that (1.137) is an isomorphism of graded *algebras*, observe that the algebra structure on the right is described by an external product (much like the shuffle product), which can be computed via the exterior product of Koszul complexes. □

As throughout the entirety of these notes we will let k be a field. We have the following equivalent definitions for smoothness.

Definition 96. Let A be a flat k-algebra, locally of finite type. We say that A is *smooth* (over k) if one of the following equivalent conditions holds:

1. for all $\mathfrak{m}$ a maximal ideal of A the kernel of $\mu_{\mathfrak{m}}\colon (A\otimes_k A)_{\mu^{-1}(\mathfrak{m})}\to A_{\mathfrak{m}}$ is generated by a regular sequence;
2. the kernel of $\mu\colon A\otimes_k A\to A$ is a locally complete intersection;
3. for all $\mathfrak{p}$ a prime ideal of A we have that $\dim_{k(\mathfrak{p})}\Omega^1_{A/k}\otimes_A k(\mathfrak{p}) = \dim_{\mathfrak{p}}\operatorname{Spec} A$.

Let us remark that in characteristic 0 smoothness at a point $\mathfrak{p}\in\operatorname{Spec} A$ is equivalent to $\Omega^1_{A/k,\mathfrak{p}}$ being free of finite rank, and the ring $A_{\mathfrak{p}}$ being regular[14] (which is an absolute notion).

Having introduced smoothness, we can put it to good use in proving the main theorem of this section.

Proof of theorem 85. We have constructed a morphism

$$(1.141)\quad \epsilon_n\colon \Omega^n_A\otimes_A M\to \mathrm{HH}_n(A,M)$$

of A-modules. We can check whether it is an isomorphism by checking it after localising at every maximal ideal $\mathfrak{m}$ of A, i.e. $\epsilon_n\otimes_A A_{\mathfrak{m}}$ needs to be an isomorphism for every $\mathfrak{m}$. For the left-hand side we have the following compatibility with localisation

$$(1.142)\quad (\Omega^n_A\otimes_A M)\otimes_A A_{\mathfrak{m}}\cong \Omega^n_{A_{\mathfrak{m}}/k}\otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}}.$$

For the right-hand side we need an isomorphism

$$(1.143)\quad \mathrm{HH}_n(A,M)\otimes_A A_{\mathfrak{m}}\cong \mathrm{HH}_n(A_{\mathfrak{m}},M_{\mathfrak{m}}),$$

so in the construction of $\epsilon_\bullet$ we can assume that $(A,\mathfrak{m})$ is a local ring. To do this, let us denote $I := \ker(\mu\colon A\otimes_k A\to A)$. As $\operatorname{Spec} A\to\operatorname{Spec} A\otimes_k A$ is a closed morphism we have that $\mathfrak{n} := \mu^{-1}(\mathfrak{m})$ is a maximal ideal of $A\otimes_k A\cong A^{\mathrm{e}}$. There exists an isomorphism

$$(1.144)\quad \operatorname{Tor}^{A^{\mathrm{e}}}_n(A,M)\otimes_A A_{\mathfrak{m}}\cong \operatorname{Tor}^{(A^{\mathrm{e}})_{\mathfrak{n}}}_n(A_{\mathfrak{m}},M_{\mathfrak{m}})\cong \operatorname{Tor}^{A_{\mathfrak{m}}\otimes_k A_{\mathfrak{m}}}_n(A_{\mathfrak{m}},M_{\mathfrak{m}})$$

by flat base change for Tor.

By the definition of smoothness we have that $I_{\mathfrak{n}}$ is generated by a regular sequence of length $\dim A$. In the notation of proposition 95 we take $B := A\otimes_k A$, and I the ideal that cuts out A. □

So we get that the Hochschild homology $\mathrm{HH}_\bullet(A)$ for a smooth algebra is concentrated in finitely many degrees (where it consists of projective modules of finite rank). There is actually a converse to this, characterising smoothness in terms of the vanishing of Hochschild homology, see [1].

1.3.5 The Hochschild–Kostant–Rosenberg isomorphism: Hochschild cohomology

One could follow a similar approach to proving the Hochschild–Kostant–Rosenberg isomorphism for Hochschild cohomology. But in the special case of $\mathrm{HH}^\bullet(A)$ one can take a shortcut, avoiding checking explicitly that things are compatible with the differential, etc.

Indeed, as $\mathrm{HH}^\bullet(A)$ is a graded commutative algebra, the identification $\mathrm{HH}^1(A)\cong\operatorname{Der}(A)$ extends via the universal property of the exterior product to a morphism

$$(1.145)\quad \bigwedge^\bullet\operatorname{Der}(A)\to\mathrm{HH}^\bullet(A).$$

[14] In positive characteristic it only implies regularity.

To check that it locally is an isomorphism if A is a smooth k-algebra we will use the following result, which says that Ext commutes with localisation at a prime ideal. It is a special case of [29, proposition 3.3.10].

Lemma 97. Let A be a noetherian ring, and M, N be A-modules where M is moreover finitely generated. Let $\mathfrak{p}$ be a prime ideal of A, then

$$\mathrm{Ext}^n_A(M,N)_{\mathfrak{p}} \cong \mathrm{Ext}^n_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \tag{1.146}$$

for all $n \geq 0$.

Then one can recycle the argument for Hochschild homology verbatim to obtain the following result.

Theorem 98. Let A be a smooth k-algebra. Then there exist an isomorphism of graded-commutative algebras

$$\mathrm{HH}^\bullet(A) \cong \bigwedge^\bullet \mathrm{Der}(A). \tag{1.147}$$

Remark 99. Instead of Hochschild (co)homology we can also consider Hochschild (co)chains on one hand, and the exterior powers of derivations (resp. differential forms) as a complex with zero differential on the other. Then we have constructed a quasi-isomorphism between these complexes. But e.g. on the level of Hochschild cohomology it is not a quasi-isomorphism of differential graded algebras. Fixing this is part of the theory of Kontsevich's formality, which we might get back to in appendix B.1.

Remark 100. If we write $X = \operatorname{Spec} A$, then theorem 98 can be rewritten as

$$\mathrm{HH}^\bullet(A) \cong \Gamma(X, \bigwedge^\bullet \mathrm{T}_X) \tag{1.148}$$

and

$$\mathrm{HH}_\bullet(A) \cong \Gamma(X, \Omega^\bullet_{X/k}) \tag{1.149}$$

In section 3.3 we will generalise this result to the non-affine setting. In this situation the Hochschild–Kostant–Rosenberg isomorphism becomes a Hochschild–Kostant–Rosenberg decomposition, as in section 3.3: the higher cohomology of polyvector fields and differential forms starts playing a role.

1.3.6 Gerstenhaber calculus

This section will be extended at some point, but it seems that there is no operad-free proof of Hochschild cohomology being isomorphic to polyvector fields as Gerstenhaber algebras. That is unfortunate, as we want to avoid operads in this chapter.

1.3.7 Exercises

Exercise 101. Explain remark 80.

Exercise 102. We can now discuss the dependence on the base field in the definition of Hochschild (co)homology. In example 29 we have seen what the Hochschild (co)homology of a field considered as an algebra over itself is. Compare this to $\mathrm{HH}_1(\mathbb{C})$, considered as a $\mathbb{Q}$-algebra.

More generally, explain the relationship between $\Omega^1_{K/k}$ and whether K/k is a finite separable (= étale) extension. A good reference on the role of $\Omega^1_{K/k}$ for field extensions is [23, §25, §26].

1.4 Variations on Hochschild (co)homology

This will be skipped during the course, unless there is time and interest to revisit the noncommutative calculus of Hochschild (co)homology and cyclic homology at the end of the course. The interested reader is invited to use the sources mentioned in the introduction.

1.5 Deformation theory of algebras

We have seen in theorem 27 that the second Hochschild cohomology group $\mathrm{HH}^2(A, M)$ parametrises extensions of A by M. As explained there, we will usually take $M = A$, so that we are effectively describing algebra structures on $A \oplus At$ with t^2 that reduce to the original multiplication on A when t is set to 0. In which case we call this a *(first order) deformation* of A, as is customary in algebraic geometry.

In this section we will discuss the higher-order deformation theory of an associative algebra A, not just up to first-order. It turns out that this is also controlled by the Hochschild cohomology, where we will also use

1. the Gerstenhaber bracket;
2. the third Hochschild cohomology $\mathrm{HH}^3(A)$.

We will also discuss the general formalism of differential graded Lie algebras governing deformation problems, using the Maurer–Cartan equation.

Summarising the results (at least on the infinitesimal level) we can draw the following picture, which gives the interpretation of the first, second and third Hochschild cohomology group in the deformation theory of algebras, together with the role of the Gerstenhaber bracket. Recall that it has degree -1, so we are landing in the appropriate spaces.

infinitesimal automorphisms

(1.150)
$$\begin{tikzcd} \mathrm{HH}^2(A) \arrow[loop above, "{[\mathrm{HH}^1(A),-]}"] \arrow[r, "{\mu\mapsto[\mu,\mu]}"] & \mathrm{HH}^3(A) \end{tikzcd}$$

deformations obstructions

To streamline the discussion we will in this section assume that k is not of characteristic 2. Whenever necessary we will even restrict ourselves to characteristic zero, but this will be mentioned explicitly.

1.5.1 Obstructions and the third Hochschild cohomology

We have seen in theorem 27 that $\mathrm{HH}^2(A)$ corresponds to first-order deformations of A. Given a first-order deformation, the first step to take is to (try to) extend it to a second-order deformation. Sometimes this is possible, sometimes it fails. When it fails it is because of an obstruction. In this section we discuss how to analyse the failure of extending, using $\mathrm{HH}^3(A)$. Once the first-order deformation is extended to a second-order deformation, we can try to extend it further. Again the third Hochschild cohomology and the Gerstenhaber bracket control this behaviour. If the extension is possible at each step we end up with a formal deformation, which is discussed in section 1.5.2.

From first-order to second-order Let $\mu_1\colon A \otimes_k A \to A$ be a Hochschild 2-cocycle, for which $\mu_0 + \mu_1 t$ is a first-order deformation, as in the discussion following theorem 27. We wish to extend this using a $\mu_2\colon A \otimes_k A \to A$, such that

(1.151) $a * b = \mu_0(a \otimes b) + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2$

gives an associative product on $A[t]/(t^3)$ (considered as a module, not as an algebra). The associativity relation

(1.152) $(a * b) * c = a * (b * c)$

breaks up into two[15] conditions: one expression an equality for the coefficients of t and another for t^2. The first condition we have already discussed, and says precisely that μ_1 needs to be a Hochschild 2-cocycle. The associativity condition at t^2 can be written as

(1.153) $\mu_1(\mu_1(a \otimes b) \otimes c) - \mu_1(a \otimes \mu_1(b \otimes c)) = a\mu_2(b \otimes c) - \mu_2(ab \otimes c) + \mu_2(a \otimes bc) - \mu_2(a \otimes b)c.$

The right-hand side of this equality is given by $\mathrm{d}(\mu_2)(a \otimes b \otimes c)$, so the left-hand side, considered as an element of $\mathrm{Hom}_k(A^{\otimes 3}, A)$ must be a coboundary.

But the left-hand side (which only depends on μ_1, which is a cocycle by the first associativity condition) is always a cocycle, because it is equal to $\mu_1 \circ \mu_1 = \frac{1}{2}[\mu_1, \mu_1]$, and the Gerstenhaber bracket of the cocycle μ_1 with itself is again a cocycle. So the equality (1.153) says that the cocycle is actually a coboundary, so it defines $0 \in \mathrm{HH}^3(A)$. In general, when we are only given μ_1 and we are looking for a compatible μ_2, we can define the following.

Definition 103. The class in $\mathrm{HH}^3(A)$ defined by the left-hand side of (1.153) is the *obstruction* to extending μ_1. We call $\mathrm{HH}^3(A)$ the *obstruction space* of A.

If this obstruction class vanishes there exists a $\mu_2 \in \mathrm{Hom}_k(A^{\otimes 2}, A)$ which turns (1.151) into an associative product on $A[t]/(t^3)$, which we will call a *second-order deformation*. So we have proven the following proposition.

Proposition 104. Let $\mu_0 + \mu_1 t$ be a first-order deformation. Then it can be extended to a second-order deformation if and only if $[\mu_1, \mu_1] = 0$ in $\mathrm{HH}^3(A)$.

Definition 105. We call $[\mu_1, \mu_1]$ the *obstruction* to extending the first-order deformation $\mu_0 + \mu_1 t$ to a second-order deformation. If it vanishes we call the deformation *unobstructed*, otherwise we call it an *obstructed* deformation.

In particular, if $\mathrm{HH}^3(A) = 0$ then all obstructions automatically vanish. On the other hand it is possible that $\mathrm{HH}^3(A) \neq 0$ but a first-order deformation still extends to a second-order deformation. For this we can consider the following example, which explains this behaviour.

Example 106. Let $A = k[x, y, z]/(xy - z, x^2, y^2, z^2)$. Then A is a commutative 4-dimensional algebra, which we can also express as $k[x]/(x^2) \otimes_k k[y]/(y^2)$. We will use the basis $(1, x, y, z)$ for A, and the induced basis of tensor products for $A \otimes_k A$. Consider the following two infinitesimal deformations, or 2-cocycles:

1. $f \colon A \otimes_k A \to A$ is the cocycle $f(y \otimes x) = z$, and 0 for other basis vectors;
2. $g \colon A \otimes_k A \to A$ is the cocycle $g(x \otimes x) = y$, and 0 for other basis vectors.

Then one sees that

1. f defines an unobstructed noncommutative first-order deformation of A;
2. g defines an unobstructed commutative first-order deformation of A.

On the other hand, $f + g$ defines another first-order deformation of A, and one can check that

(1.154) $[f + g, f + g] \neq 0$

in $\mathrm{HH}^3(A)$, so this is an obstructed deformation.

[15]Or rather three, but we know that μ_0 is associative. See also remark 109.

Remark 107. Observe that, if the obstruction vanishes, there is actually a choice of μ_2, all of which gives an equivalent deformation. But for the purpose of extending it to higher-order deformations, the choice might matter.

Extending to higher order deformations Let us now generalise the previous discussion to arbitrary order, i.e. given an nth order deformation of A, when can we extend it to an $(n+1)$th order deformation of A? So we start with an assocative multiplication

$$\mu_0 + \mu_1 t + \ldots + \mu_n t^n \tag{1.155}$$

on $A[t]/(t^{n+1})$. We wish to extend it to an associative multiplication on $A[t]/(t^{n+2})$. The analysis as before gives us a hierarchy of associativity conditions at t^i, for $i = 1, \ldots, n+1$, with only the condition at t^{n+1} being new. By isolating the terms involving μ_0 and μ_{n+1} in this expression we obtain the equality

$$\sum_{i=1}^{n} \mu_i\left(\mu_{n+1-i}(a \otimes b) \otimes c\right) - \mu_i\left(a \otimes \mu_{n+1-i}(b \otimes c)\right) = \mathrm{d}(\mu_{n+1})(a \otimes b \otimes c). \tag{1.156}$$

As the left-hand side can be interpreted as the sum over $[\mu_i, \mu_{n+1-i}]$ this is a 3-cocycle. So by the same reasoning as before we get the following, where we denote left-hand side by D_{n+1} we have that

Proposition 108. Let $\mu_0 + \mu_1 t + \ldots + \mu_n t^n$ be an nth order deformation. Then it can be extended to an $(n+1)$th order deformation if and only if $D_{n+1} = 0$ in $\mathrm{HH}^3(A)$.

Observe that, again, there is a choice of μ_{n+1} at this point, and this choice might lead you into trouble when you want to continue this process: for some choices the following step might be obstructed, whilst for others it isn't. See also exercise 130.

Remark 109. The condition for $i = 0$ is just saying that $[\mu_0, \mu_0] = 0$. Recall that the Gerstenhaber bracket did *not* involve the original multiplication on A, and the condition that the bracket of μ_0 with itself vanishes expresses precisely that μ_0 is associative.

Deformation functors To make the link with the formalism used in algebraic geometry we want to consider algebras more general than $k[t]/(t^{n+1})$. These are the *test algebras*, which are commutative artinian local k-algebras with residue field k. In particular, the maximal ideal $\mathfrak{m}$ is nilpotent. We will denote them by R, so that there is no confusion with the algebras that we are deforming (which are denoted A). Then generalising the notion of an nth order deformation we have the following.

Definition 110. Let $(R, \mathfrak{m})$ be a test algebra. An *R-deformation* of A is an associative and R-bilinear multiplication $- * -$ on $A \otimes_k R$ such that modulo $\mathfrak{m}$ it reduces to the multiplication on A.

In other words, the square

$$\begin{tikzcd}
(A \otimes_k R) \otimes_R (A \otimes_k R) \arrow[r] \arrow[d, "-*-"] & A \otimes_k A \arrow[d, "\mu_0"] \\
A \otimes_k R \arrow[r] & A
\end{tikzcd} \tag{1.157}$$

commutes.

Then the equivalence relation for first-order deformations is generalised as follows.

Definition 111. Let $- *_1 -$ and $- *_2 -$ be two R-deformations. We say that they are *(gauge) equivalent* if there exists an automorphism f of the R-module $A \otimes_k R$ which is the identity modulo $\mathfrak{m}$, such that

$$f(a *_1 b) = f(a) *_2 f(b). \tag{1.158}$$

A morphism of test algebras $f\colon R \to S$ allows us to base change a deformation $A \otimes_k R$ to $A \otimes_k R \otimes_R S$, and hence we have a covariant functor

$$\mathrm{Def}(A,-)\colon \mathcal{R} \to \mathrm{Set} : R \mapsto \mathrm{Def}(A,R) \tag{1.159}$$

where $\mathrm{Def}(A,R)$ is the set of R-deformations of A up to equivalence. In particular, $\mathrm{Def}(A,k) = \{A\}$ and $\mathrm{Def}(A,k[t]/(t^2)) = \mathrm{HH}^2(A)$.

Axiomatising the properties that this functor has, we obtain the following definition.

Definition 112. A functor $F\colon \mathcal{R} \to \mathrm{Set}$ is a *deformation functor* if for every cartesian diagram

$$\begin{tikzcd} R' \times_R R'' \arrow[r] \arrow[d] & R'' \arrow[d] \\ R' \arrow[r] & R \end{tikzcd} \tag{1.160}$$

the induced morphism

$$\eta\colon F(R' \times_R R'') \to F(R') \times_{F(R)} F(R'') \tag{1.161}$$

is

1. bijective, if $R \cong k$;
2. surjective, if $R' \to R$ is surjective.

One can check that $\mathrm{Def}(A,-)$ is a deformation functor in this general sense of the word. Later on we will see a way of describing the deformation functor using Hochschild cohomology, and explain the role dg Lie algebras play in describing deformation functors.

We will say that $F(k[t]/(t^2))$ is the *tangent space* to a deformation functor, so we see that the second Hochschild cohomology is the tangent space to the deformation functor $\mathrm{Def}(A,-)$.

1.5.2 Formal deformations

The intuition from deformation theory in algebraic geometry tells us that deformations over

$$k[[t]] = \varprojlim k[t]/(t^{n+1}) \tag{1.162}$$

are supposed to describe deformations in a sufficiently small (indeed: infinitesimally small) open neighbourhood around the algebra that we are interested in. These are precisely the deformations one obtains when taking the limit of the process with the $k[t]/(t^{n+1})$ in the previous section. Similarly we will discuss later in this section how the step from $k[t]/(t^{n+1})$ to local artinian k-algebras has an analogue, going from $k[[t]]$ to complete augmented k-algebras.

One-parameter formal deformations Let A be a k-algebra. We can consider the $k[[t]]$-module $A[[t]]$. We are interested in new algebra structures on the module $A[[t]]$ in the following sense.

Definition 113. A *(one-parameter) formal deformation* of A is an associative and $k[[t]]$-bilinear multiplication

$$- * -\colon A[[t]] \otimes_k A[[t]] \to A[[t]] \tag{1.163}$$

which is continuous in the t-adic topology, such that

$$a * b \equiv ab \mod t \tag{1.164}$$

for all $a, b \in A \subseteq A[[t]]$.

The continuity in the previous definition is expressed by saying that the multiplication takes on the form

$$\left(\sum_{i\geq 0} a_i t^i\right) * \left(\sum_{j\geq 0} b_j t^j\right) = \sum_{k\geq 0} \sum_{i+j=k} (a_i * b_j) t^{i+j}, \tag{1.165}$$

i.e. it is given by the *Cauchy product.* Because of this the multiplication is completely determined by the restriction $- * -\colon A \otimes_k A \to A[[t]]$, see appendix A.2 for more information.

For $a, b \in A$ we will write

$$a * b = \mu_0(a \otimes b) + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \dots \tag{1.166}$$

where $\mu_0(a \otimes b) = \mu(a \otimes b)$ is the original multiplication, as before. So a formal deformation consists of μ_i's such that they define an associative multiplication, which is expressed by considering (1.156) for all n simultaneously.

The relationship between $k[[t]]$ and $k[t]/(t^{n+1})$ is also easily expressed by observing that setting $t^{n+1} = 0$ in a formal deformation results in an nth order deformation.

Remark 114. Although we formally speaking haven't introduced the notation for the left-hand side yet (as this involves gauge equivalence for $k[[t]]$), we have a bijection

$$\mathrm{Def}(A, k[[t]]) = \varprojlim \mathrm{Def}(A, k[t]/(t^{n+1})), \tag{1.167}$$

in other words: a formal deformation consists of a *compatible* family of nth order deformations. It is also important to observe that the obstructions we have discussed earlier tell us that the morphism

$$\mathrm{Def}(A, k[[t]]) \to \mathrm{Def}(A, k[t]/(t^{n+1}) \tag{1.168}$$

is necessarily surjective.

Complete augmented algebras Recall that an augmentation for an algebra is a morphism $R \to k$ of k-algebras, whose kernel is then called the augmentation ideal R_+. As in our setting it will be a maximal ideal we will denote it $\mathfrak{m}$.

Definition 115. Let R be a commutative augmented k-algebra. We say that R is a *complete augmented k-algebra* if R is complete with respect to the $\mathfrak{m}$-adic topology.

In particular they are local, as completions at maximal ideals are local.

Remark 116. The test algebras from before are complete augmented k-algebras: as $\mathfrak{m}$ is nilpotent the topology is trivial, and there is no need for a completion. Just like $k[[t]]$ is the limit of the $k[t]/(t^{n+1})$, a complete augmented algebra is the limit of $R/\mathfrak{m}^n$, which are test algebras.

To deal with the topology, we will need to use the *completed tensor product*:

$$A\widehat{\otimes}_k R := \varprojlim A \otimes_k R/\mathfrak{m}^n. \tag{1.169}$$

Definition 117. Let R be a complete augmented algebra. Then a *formal deformation* of A over R is an R-algebra B such that

1. $B \cong \widehat{\otimes}_k R$ as R-modules;
2. B reduces to A modulo $\mathfrak{m}$, i.e. $B \otimes_R k \cong A$ as k-algebras;
3. the multiplication is continuous, i.e. $B \cong \varprojlim B \otimes_k R/\mathfrak{m}^n$.

We have the following lemma, which is an analogue of the observations we have made before that the multiplication law on a deformation can be described by looking at a small part of the deformed algebra.

Lemma 118. Let R be a complete augmented algebra, and A' a formal deformation of A over R. Then the algebra structure on A' is determined by the restriction to $A \otimes_k A$.

1.5.3 The Maurer–Cartan equation

We will now discuss some aspects of the general formalism of deformation theory using dg Lie algebras. The main example to keep in mind is of course $\mathrm{C}^\bullet(A)[1]$ (and variations constructed using this). Recall that the reason for the shift is to make the Gerstenhaber bracket of degree 0. In this way the relevant cohomology groups will be H^0, H^1 and H^2, but they really are HH^1, HH^2 and HH^3.

We will now assume that k is of characteristic 0.

Definition 119. Let $\mathfrak{g}^\bullet$ be a dg Lie algebra. The *Maurer–Cartan equation* for $\mathfrak{g}^\bullet$ is

$$\mathrm{d}(f) + \frac{1}{2}[f, f] = 0 \tag{1.170}$$

where $f \in \mathfrak{g}^1$. Elements of $\mathfrak{g}^1$ satisfying the Maurer–Cartan are *Maurer–Cartan elements*, and we will denote

$$\mathrm{MC}(\mathfrak{g}) := \left\{ f \in \mathfrak{g}^1 \mid \mathrm{d}(f) + \frac{1}{2}[f, f] = 0 \right\} \subseteq \mathfrak{g}^1 \tag{1.171}$$

the space of Maurer–Cartan elements.

The dg Lie algebras that we will be using are constructed from $\mathrm{C}^\bullet(A)[1]$ as follows: let $(R, \mathfrak{m})$ be a complete augmented k-algebra, then $\mathfrak{m}\widehat{\otimes}_k \mathrm{C}^\bullet(A)[1]$ is again a dg Lie algebra, see exercise 172. As before, the main examples to which we will apply this are $(k[t]/(t^{n+1}), (t))$ and $(k[[t]], (t))$. For the latter we will use the shorthand notation $t\,\mathrm{C}^\bullet(A)[1][[t]]$.

With these definitions we have the following theorem, explaining the importance of the Maurer–Cartan equation in our situation.

Theorem 120. Let A be an associative algebra. Then

$$\{\text{one-parameter formal deformations of } A\} \xleftrightarrow{1:1} \mathrm{MC}(t\,\mathrm{C}^\bullet(A)[1][[t]]). \tag{1.172}$$

Proof. Consider

$$\mu := \sum_{i=1}^{+\infty} \mu_i t^i \in t\,\mathrm{C}^1(A)[1][[t]], \tag{1.173}$$

i.e. $\mu_i \in \mathrm{C}^2(A)$. Then on $A[[t]]$ we can define the multiplication

$$a \otimes b \mapsto ab + \sum_{i=1}^{+\infty} \mu_i(a \otimes b)t^i \tag{1.174}$$

for $a, b \in A$, and then extended bilinearly to all of $A[[t]]$. We only need to check that associativity of this multiplication corresponds to μ satisfying the Maurer–Cartan equation. But this is checked in exactly the same was as for $k[t]/(t^{n+1})$. □

Remark 121. The same proof works for arbitrary formal deformations over arbitrary complete augmented k-algebras, using the dg Lie algebra $\mathfrak{m}\widehat{\otimes}_k \mathrm{C}^\bullet(A)[1]$, only adding some mild notational complexity.

Example 122. As an application of the previous remark, if we consider $R = k[t]/(t^2)$ then there is no need for the completed tensor product, and $\mathrm{MC}(\mathfrak{m} \otimes_k \mathrm{C}^\bullet(A)[1]) = \mathrm{MC}(\mathrm{C}^\bullet(A)[1])$ is the set of infinitesimal deformations (not up to equivalence, yet).

Remark 123. If we take μ_0 into account, then we observe that the Maurer–Cartan equation can be reinterpreted using the equalities

$$\begin{aligned}[][\mu_0 + \sum_{i\geq 1}\mu_i t^i, \mu_0 + \sum_{i\geq 1}\mu_i t^i] &= [\mu_0, \mu_0] + [\mu_0, \sum_{i=\geq 1}\mu_i t^i] + [\sum_{i\geq 1}\mu_i t^i, \mu_0] + [\sum_{i\geq 1}\mu_i t^i, \sum_{i\geq 1}\mu_i t^i] \\ &= 2\,\mathrm{d}\left(\sum_{i\geq 1}\mu_i t^i\right) + [\sum_{i\geq 1}\mu_i t^i, \sum_{i\geq 1}\mu_i t^i]\end{aligned} \tag{1.175}$$

as $[\mu_0, \mu_0] = 0$ by associativity (see also remark 109), and $[\mu, -] = \mathrm{d}$ by remark 46.

Gauge equivalence Observe that in the bijection of theorem 120 we are not considering formal deformations up to gauge equivalence. Likewise, in example 122 we are not taking the equivalence relation on first-order deformations into account. To do this, we need to introduce gauge equivalence for the Maurer–Cartan locus. For convergence reasons, we will assume that $\mathfrak{g}^\bullet$ is of the form $\mathfrak{h}^\bullet\widehat{\otimes}_k\mathfrak{m}$, for some complete augmented k-algebra $(R, \mathfrak{m})$ (as this is the situation we are interested in).

Definition 124. Let $g_1, g_2 \in \mathrm{MC}(\mathfrak{g}^\bullet)$ be Maurer–Cartan elements. We say that they are *gauge equivalent* if there exists an element $h \in \mathfrak{g}^0$ such that

$$g_2 = \exp(\operatorname{ad} h)(g_1) + \frac{1 - \exp(\operatorname{ad} h)}{h}(\mathrm{d}(h)). \tag{1.176}$$

We can now relate the two notions of gauge equivalence. Observe that by the assumption on the characteristic, a gauge equivalence ϕ between formal deformations $- *_1 -$ and $- *_2 -$ on $A[[t]]$ can be written as $\exp(h)$ for some $h \in t\,\mathrm{Hom}_k(A, A)[[t]]$. And recall that $\mathrm{Hom}_k(A, A) = \mathrm{C}^1(A)$.

Proposition 125. The formal deformations $- *_1 -$ and $- *_2 -$ are gauge equivalent if and only if the associated Maurer–Cartan elements g_1 and g_2 (see theorem 120) are gauge equivalent.

Proof. If we denote $\mu_0\colon A\otimes_k A\to A$ the original multiplication, then for $a,b\in A$ we have that

$$
\begin{aligned}
a *_2 b &= \phi\left(\phi^{-1}(a) *_1 \phi^{-1}(b)\right)\\
&= \exp(h)\,(\exp(-h)(a) *_1 \exp(-h)(b))\\
&= \exp(\mathrm{ad}(h))(\mu+g_1)(a\otimes b)\\
&= \left(\mu+\exp(\mathrm{ad}(h))(g_1)+\frac{1-\exp(\mathrm{ad}(h))}{\mathrm{ad}(h)}(\mathrm{d}(h))\right)(a\otimes b)
\end{aligned}
\tag{1.177}
$$

by remark 46, and applying $\mathrm{ad}(h)$ once in all the non-constant applications to μ. So gauge equivalence for $-*_i-$ is expressed through gauge equivalence for g_i and vice versa. □

Deformation functors Without going into any details, one can extend the previous discussion to obtain an isomorphism

$$\mathrm{Def}(A,R)\cong \mathrm{MC}(\mathfrak{m}\widehat{\otimes}_k \mathrm{C}^\bullet(A)[1]). \tag{1.178}$$

At some point this discussion should be extended, but for time reasons we won't go further into this in class.

1.5.4 Kontsevich's formality theorem

Let us (briefly) return to the situation we used for the Hochschild–Kostant–Rosenberg theorem, i.e. we let A be a smooth commutative k-algebra. We will again assume that the characteristic of k is zero. Then we have shown in section 1.3 that

$$\mathrm{HH}^\bullet(A)\cong \bigwedge^\bullet \mathrm{Der}(A). \tag{1.179}$$

Although we haven't proved it in these notes this isomorphism is compatible with the Lie brackets on both sides: the Gerstenhaber bracket on the left, and the Schouten–Nijenhuis bracket on the right.

With the risk of causing some confusion we are going to combine a commutative algebra structure with a Lie algebra structure, but not in the way we have done for a Gerstenhaber algebra structure (where the Lie bracket has degree -1). In an ungraded setting (or with a Lie bracket of degree 0, with a suitable modification of the definition) we get the following notion.

Definition 126. A commutative k-algebra A is a *Poisson algebra* if there exists a Lie bracket $\{-,-\}$ on A (i.e. a skew-symmetric bilinear map satisfying the Jacobi identity) such that the Poisson identity

$$\{ab,c\}=a\{b,c\}+b\{a,c\} \tag{1.180}$$

holds.

We phrased the definition in this way because the algebra A is usually fixed, whilst the Lie bracket $\{-,-\}$ is obtained via some choices. Indeed, an element π of $\bigwedge^2\mathrm{Der}(A)$ can give rise to a Poisson structure via the Lie bracket

$$[a,b]=\pi(\mathrm{d}f\wedge\mathrm{d}g) \tag{1.181}$$

where we used the pairing between $\bigwedge^2\mathrm{Der}(A)$ and Ω_A^2. We said *can* give rise, because the Poisson identity is not automatic: indeed, one can check that the associated bilinear morphism is a Poisson structure if and only if $[\pi,\pi]=0$, where $[-,-]$ now denotes the Schouten–Nijenhuis bracket for the

Lie algebra $\bigwedge^\bullet \mathrm{Der}(A)$. In this case we call π a *Poisson bivector.* Hence under the Hochschild–Kostant–Rosenberg isomorphism we see that Poisson structures on A correspond to *unobstructed* infinitesimal deformations of A.

A *deformation quantisation* of a Poisson algebra is a one-parameter formal deformation $- * -$ of a Poisson algebra A, such that

(1.182) $a * b - b * a \equiv \{a, b\}t \bmod t^2$

for all $a, b \in A$. Using the correspondence between Poisson structures and unobstructed infinitesimal deformations this means that μ_1 corresponds to π. If we equip the graded algebra of polyvector fields with the trivial differential, then

Then Kontsevich proved the following extremely important theorem, whose proof is out of scope of these lecture notes.

Theorem 127 (Kontsevich formality)**.** Let A be a Poisson algebra. Then A can always be quantised, i.e. there always exists a deformation quantisation of this Poisson structure.

In other words, once we know that the first obstruction vanishes for a first-order deformation of A, we can always find a one-parameter formal deformation lifting the first-order deformation.

In this version of the statement it is unclear what exactly the term formality is supposed to mean: the presence of the word formal in the statement is a red herring here. Rather, we have on one hand the dg Lie algebra $\mathrm{C}^\bullet(A)[1]$, and on the other hand we can equip $\bigwedge^\bullet \mathrm{Der}(A)[1]$ with the zero differential to obtain another dg Lie algebra. This latter dg Lie algebra is *formal*: it is quasi-isomorphic to its cohomology *as a dg Lie algebra*[16]

We won't define precisely what a L_∞-quasi-isomorphism is (at least for now), but imagining we have done so, we have the following statement.

Theorem 128. There exists a L_∞-quasi-isomorphism

(1.183) $\bigwedge^\bullet \mathrm{Der}(A)[1] \to \mathrm{C}^\bullet(A)[1].$

In other words, Kontsevich formality says that the Hochschild complex is formal, when A is a smooth commutative k-algebra.

We will probably come back to this at the very end of these lecture notes.

1.5.5 Exercises

Exercise 129. Check the claims in example 106, i.e.

1. f and g are cocycles;
2. f and g are unobstructed;
3. $f + g$ is obstructed.

Also, assuming the Künneth formula for Hochschild cohomology[17]

(1.184) $\mathrm{HH}^\bullet(A_1 \otimes_k A_2) \cong \mathrm{HH}^\bullet(A_1) \otimes_k \mathrm{HH}^\bullet(A_2)$

[16] Because we are working over a field, every cochain complex is formal as a cochain complex. The extra structure matters here.

[17] Which we haven't proven, but which holds in this case by [30, theorem 2.1.2].

describe the Hochschild cohomology of A using example 32.

Exercise 130. Perform the following reality checks.

1. If the trivial deformation has $\mu_i = 0$ for all $i \geq 1$, why can't we choose μ_1 non-zero, and then take $\mu_i = 0$ for all $i \geq 2$?
2. Explain that, if $\mu_1 = 0$, then any cocycle μ_2 gives a second-order deformation.
3. Use exercise 129 to construct an unobstructed second-order deformation, which extends trivially to a third-order deformation (why can we do this?), such that the third-order deformation is obstructed.

Exercise 131. Write out the details in the proof of theorem 120.

The following exercise is a modification [30, exercises 4.1.14 and 4.1.15].

Exercise 132. Consider $A = \mathbb{C}[x, y]$. On $A[[t]]$ we define two multiplications $- *_i -$ as follows:

1. define $- *_i -$ for $x, y \in A$ as

$$\begin{aligned} x *_1 x &= x^2 & x *_1 y &= xy \\ y *_1 y &= y^2 & y *_1 x &= xy + t \end{aligned} \tag{1.185}$$

   resp.

$$\begin{aligned} x *_1 x &= x^2 & x *_1 y &= xy \\ y *_1 y &= y^2 & y *_1 x &= xy \exp(t), \end{aligned} \tag{1.186}$$

   where $\exp(t)$ is expanded as a formal power series in t;
2. extend to all of A by writing monomials of A lexicographically, and defining $x^a y^b *_i x^c y^d$ using the relation $y *_i x$ inductively whenever $b, c \geq 1$.

With this definition

1. Describe (as much as possible of) μ_1 for these products. Are they the same?
2. Describe (as much as possible of) μ_2 for these products. Is $\mu_2 = 0$ for $- *_1 -$?
3. If you like a challenge, describe μ_n completely for all $n \geq 1$.
4. What can be said about the convergence of the star products?

The star product $- *_1 -$ gives rise to the *Weyl algebra* $\mathbb{C}\langle x, y\rangle/(xy - yx - 1)$, by setting $t = 1$.

The star product $- *_2 -$ gives rise to the *skew planes* $\mathbb{C}_q[x, y] := \mathbb{C}\langle x, y\rangle/(xy - qyx)$ where $q \neq 0$, by setting $t = \log q$.

Exercise 133. Let $\mathfrak{g}^\bullet$ be any dg Lie algebra. Prove that the Maurer–Cartan equation for $g \in \mathfrak{g}^1$ is equivalent to the flatness of the connection ∇_g defined as

$$\nabla_g(f) := \mathrm{d}(f) + [g, f] \tag{1.187}$$

for $f \in \mathfrak{g}^0$, i.e. that $\nabla_g^2 = 0$. Flatness of connections was the first instance in which the Maurer–Cartan equation appeared.

Chapter 2

Differential graded categories

Conventions We will continue with our convention that k is a field, sometimes required to be of characteristic 0.

2.1 Differential graded categories

2.1.1 Triangulated categories

To understand why dg categories arise naturally it is necessary to talk about triangulated categories. It is possible to dedicate a whole monograph to their foundations [24], so we will necessarily have to skip over some details. In this section we provide just enough details to

1. understand why a notion of (dg) enhancement is necessary;
2. understand what happens with derived categories of dg categories, and if time permits pretriangulated dg categories.

The first chapter of [15] is an excellent introduction, where more details can be found for the interested reader.

Definition There are different versions of the definition of a triangulated category (mostly with variations in (TR4)), the following definition is the classical one as taken from [13, 24].

Definition 134. A *triangulated category* is an additive category $\mathcal{T}$ together with

1. a *translation functor* (or *shift functor*, or *suspension functor*) $[1]$, which is an automorphism of $\mathcal{T}$;
2. a class of *distinguished triangles* closed under isomorphisms, which are sextuples (X, Y, Z, u, v, w) where $u\colon X \to Y$, $v\colon Y \to Z$, $w\colon Z \to X[1]$ are morphisms in $\mathcal{T}$, and these are also often written as

   (2.1) $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1];$

such that

(TR0) for every object X of $\mathcal{T}$ we have that $(X, X, 0, \mathrm{id}_X, 0, 0)$ is distinguished;

(TR1) for every morphism $u\colon X \to Z$ there exists an object Z called the *mapping cone* of u and morphisms $v\colon Y \to Z$, $w\colon Z \to X[1]$ such that (X, Y, Z, u, v, w) is a distinguished triangle;

(TR2) the triangle (X, Y, Z, u, v, w) is distinguished if and only if $(Y, Z, X[1], -v, -w, -u[1])$ is.

(TR3) if (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') are distinguished triangles, and $f : X \to X'$ and $g\colon Y \to Y'$ are morphisms such that the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd} \tag{2.2}$$

commutes, then there exists a (not necessarily unique) morphism $h\colon Z \to Z'$ making the diagram commute;

(TR4) if (X, Y, Z', u, j, k), (Y, Z, X', v, l, i) and $(X, Z, Y', v \circ u, m, n)$ are distinguished triangles, then there exists a distinguished triangle $(Z', Y', X', f, g, j[1] \circ i)$ such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "j"] \arrow[d, "v"] & Z' \arrow[r, "k"] \arrow[d, dashed, "\exists f"] & X[1] \arrow[d, equal] \\ X \arrow[r, "v\circ u"] & Z \arrow[r, "m"] \arrow[d, "l"] & Y' \arrow[r, "n"] \arrow[d, dashed, "\exists g"] & X[1] \\ & X' \arrow[r, equal] \arrow[d, "i"] & X' \arrow[d, "j[1]\circ i"] & \\ & Y[1] \arrow[r, "j[1]"] & Z'[1] & \end{tikzcd} \tag{2.3}$$

commutes, and moreover $u[1] \circ n = i \circ g$.

One way of interpreting (TR4) is as an analogue of the third isomorphism theorem in an abelian category: given the morphisms $u\colon X \to Y$ and $v\colon Y \to Z$ we can consider their composition and from (TR1) we get three distinguished triangles, where (TR4) asserts that the mapping cones in these distinguished triangles can be made into a distinguished triangle themselves in a way compatible with all other triangles. It is still not clear whether (TR4) follows from the other axioms [24, remark 1.3.15].

Remark 135. Axiom (TR4) is also called the *octahedral axiom*. To understand the origin of this name one has to collapse the identities, remove the shifts and fold the triangles on the left and right so that they meet in a point, resulting in (with some imagination) the octahedron

$$\begin{tikzcd} & Y' \arrow[dr, dashed, "\exists g"] & \\ Z' \arrow[ur, dashed, "\exists f"] \arrow[d] & & X' \arrow[ll, "j[1]\circ i"] \arrow[ddl, "i"] \\ X \arrow[uur] \arrow[rr, "v\circ u"] \arrow[dr, "u"'] & & Z \arrow[uul] \arrow[u] \\ & Y \arrow[uul, "j"] \arrow[ur, "v"'] & \end{tikzcd} \tag{2.4}$$

where the distinguished triangles correspond to four non-adjacent faces, whilst the other four faces correspond to the commutative squares in (2.4) containing an identity morphism.

Example 136. The main examples of triangulated categories are *derived categories* of abelian categories. If A is a k-algebra then we will be interested in $\mathbf{D}(\operatorname{Mod} A)$, and provided A is noetherian, $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. If X is a scheme, we will be interested in $\mathbf{D}(\operatorname{Qcoh} X)$, and provided X is noetherian, $\mathbf{D}^{\mathrm{b}}(\operatorname{coh} X)$.

Non-functoriality of the cone There is an important deficiency in the definition of a triangulated category, which cannot easily be fixed. The mapping cone from (TR1) is only unique up to non-unique isomorphism, so the morphism $h\colon Z \to Z'$ in (TR3) cannot be chosen functorially in general. In other words, we don't have a functor $\mathcal{T}^{[1]} \to \mathcal{T}$ of triangulated categories. Indeed, we have the following result (see [28, proposition 1.2.13]).

Proposition 137. Let $\mathcal{T}$ be a triangulated category which is idempotent complete (i.e. every idempotent splits). If $\mathcal{T}$ admits a functorial cone, then $\mathcal{T}$ is semisimple abelian.

But the derived category of a non-semisimple k-algebra is not semisimple, so cones are not functorial. For a short and self-contained discussion, see `http://www.maths.gla.ac.uk/~gstevenson/no_functorial_cones.pdf`.

Compact objects Crucial in many aspects of the study of triangulated categories are compact objects. This even lead Thomason to say

> "Compact objects are as necessary to triangulated categories as air to breathe."

A whole lot of theory can be developed without compact objects, as is evident from [24] but the notion of compact objects does come with important consequences, some of which were discussed in class (not reproduced here). Let us at least define them formally, as we will need them later on.

Definition 138. Let $\mathcal{T}$ be a triangulated category. We say that an object $T \in \mathcal{T}$ is *compact* if for all families $\{T_i \mid i \in I\}$ the natural morphism

$$\bigoplus_{i\in I} \operatorname{Hom}_{\mathcal{T}}(T, T_i) \to \operatorname{Hom}_{\mathcal{T}}\left(T, \bigoplus_{i\in I}\right) \tag{2.5}$$

is an isomorphism. The full subcategory of compact objects will be denoted $\mathcal{T}^{\mathrm{c}}$.

2.1.2 Differential graded categories

Motivation For a long time, when triangulated categories were mostly used as technical tools in homological algebra, the non-functoriality of the cone was not really a problem. But once triangulated categories were studied as objects on their own, this caused some issues. In the context of these notes, recall that in corollary 16 we saw that Hochschild (co)homology only depended on the abelian category of modules. If one generalises Morita theory for module categories to Morita theory for derived categories of modules it turns out that Hochschild (co)homology only depends on the derived category. But it is impossible to define it *intrinsically*, which is what one would like to do in such a situation. This is a somewhat obscure motivation for the need of enhancements, the usual invariant to be considered in this situation is higher algebraic K-theory.

Another take on why the axioms for triangulated categories are lacking is that it is impossible to define new triangulated categories out of old. For instance, considering exact functors between triangulated categories, it is usually not possible to equip this category with a triangulated structure. Likewise for descent problems, for which a good explanation can be found in [27, §2.2(d)]. Summing up: triangulated categories forget too much information, and we need to enhance their construction a bit to keep track of this.

There are many solutions to this, but we will use dg categories. They are the closest to the homological algebra we have been using all along.

There currently does not exist a textbook dedicated to dg categories, although Gustavo Jasso is writing one. The main references are the original article by Keller [18] and his ICM address [19]. One can also use [26, tag 09JD].

Definitions We will now define the notion of a dg category, and some associated constructions. Implicitly in all the constructions in this section is a straightforward check of the axioms, left to the reader.

Definition 139. A *differential graded category*, or *dg category* is a category $\mathcal{C}$ such that

1. $\mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet$ is a cochain complex of k-vector spaces for all $x,y\in\mathrm{Obj}(\mathcal{C})$;
2. we have a morphism $1_x\colon k\to\mathrm{Hom}_{\mathcal{C}}(x,x)$ of cochain complexes[1] for all $x\in\mathrm{Obj}(\mathcal{C})$;
3. the composition

   $$\mu=\mu_{x,y,z}\colon \mathrm{Hom}_{\mathcal{C}}(y,z)^\bullet\otimes_k\mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet\to\mathrm{Hom}_{\mathcal{C}}(x,z)^\bullet \tag{2.6}$$

   is a morphism of cochain complexes for all $x,y,z\in\mathrm{Obj}(\mathcal{C})$;

such that

1. the composition has a unit, i.e. the diagrams

   $$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet\otimes_k k \arrow[r, "\mathrm{id}\otimes 1_x"] \arrow[dr, equal] & \mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet\otimes_k\mathrm{Hom}_{\mathcal{C}}(x,x)^\bullet \arrow[d, "\mu_{x,x,y}"] \\ & \mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet \end{tikzcd} \tag{2.7}$$

   and

   $$\begin{tikzcd} k\otimes_k\mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet \arrow[r, "1_y\otimes\mathrm{id}"] \arrow[dr, equal] & \mathrm{Hom}_{\mathcal{C}}(y,y)^\bullet\otimes_k\mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet \arrow[d, "\mu_{x,y,y}"] \\ & \mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet \end{tikzcd} \tag{2.8}$$

   commute for all $x,y\in\mathrm{Obj}(\mathcal{C})$;
2. the composition is associative, i.e. the diagram

   $$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(y,z)^\bullet\otimes_k\mathrm{Hom}_{\mathcal{C}}(x,y)^\bullet\otimes_k\mathrm{Hom}_{\mathcal{C}}(w,x)^\bullet \arrow[r, "\mathrm{id}\otimes\mu_{w,x,y}"] \arrow[d, "\mu_{x,y,z}\otimes\mathrm{id}"] & \mathrm{Hom}_{\mathcal{C}}(y,z)^\bullet\otimes_k\mathrm{Hom}_{\mathcal{C}}(w,y)^\bullet \arrow[d, "\mu_{w,y,z}"] \\ \mathrm{Hom}_{\mathcal{C}}(x,z)^\bullet\otimes_k\mathrm{Hom}_{\mathcal{C}}(w,z)^\bullet \arrow[r, "\mu_{w,x,z}"] & \mathrm{Hom}_{\mathcal{C}}(w,z) \end{tikzcd} \tag{2.9}$$

   commutes for all $w,x,y,z\in\mathrm{Obj}(\mathcal{C})$.

[1]In particular we have that the image of $1\in k$ under 1_x is closed.

This is just saying that it is a category enriched over the monoidal category of cochain complexes of k-vector spaces.

Besides every (dg) algebra considered as a dg category with a single object, the following is an essential example.

Example 140. We can turn the abelian category $\mathrm{Ch}(k)$ of cochain complexes of k-vector spaces into a dg category as follows. Recall that a morphism $f\colon M^\bullet \to N^\bullet$ in $\mathrm{Ch}(k)$ is of the form

$$\begin{tikzcd} \dots \arrow[r] & M^{i-1} \arrow[r, "\mathrm{d}_M^{i-1}"] \arrow[d, "f^{i-1}"] & M^i \arrow[r, "\mathrm{d}_M^i"] \arrow[d, "f^i"] & M^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \dots \\ \dots \arrow[r] & N^{i-1} \arrow[r, "\mathrm{d}_N^{i-1}"] & N^i \arrow[r, "\mathrm{d}_N^i"] & N^{i+1} \arrow[r] & \dots \end{tikzcd} \tag{2.10}$$

where every square commutes.

We will define the cochain complex $\mathrm{Hom}_{\mathrm{Ch_{dg}}(k)}(M^\bullet, N^\bullet)^\bullet$ by setting the degree n component the morphisms of degree n, i.e. we consider morphisms

$$\begin{tikzcd} \dots \arrow[r] & M^{i-1} \arrow[r, "\mathrm{d}_M^{i-1}"] \arrow[d, "f^{i-1}"] & M^i \arrow[r, "\mathrm{d}_M^i"] \arrow[d, "f^i"] & M^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \dots \\ \dots \arrow[r] & N^{i-1+n} \arrow[r, "\mathrm{d}_N^{i-1+n}"] & N^{i+n} \arrow[r, "\mathrm{d}_N^{i+n}"] & N^{i+1+n} \arrow[r] & \dots \end{tikzcd} \tag{2.11}$$

where (and this is important) we do not impose that the diagrams commute. The differential of a morphism $f\colon M^\bullet \to N^{\bullet+n}$ of degree n is then given by setting

$$\mathrm{d}(f^i) \coloneqq \mathrm{d}_{N^\bullet} \circ f^i - (-1)^n f^i \circ \mathrm{d}_{M^\bullet}\colon M^i \to N^{i+1+n} \tag{2.12}$$

for its component in position i.

If we are given a dg category, there are a few things we might want to do with it.

Definition 141. Let $\mathcal{C}$ be a dg category. Its *underlying category* $\mathrm{Z}^0(\mathcal{C})$ has the same objects as $\mathcal{C}$, and

$$\mathrm{Hom}_{\mathrm{Z}^0(\mathcal{C})}(x,y) \coloneqq \mathrm{Z}^0(\mathrm{Hom}_{\mathcal{C}}(x,y)). \tag{2.13}$$

A closely related construction is the following.

Definition 142. Let $\mathcal{C}$ be a dg category. Its *homotopy category* $\mathrm{H}^0(\mathcal{C})$ has the same objects as $\mathcal{C}$, and

$$\mathrm{Hom}_{\mathrm{H}^0(\mathcal{C})}(x,y) \coloneqq \mathrm{H}^0(\mathrm{Hom}_{\mathcal{C}}(x,y)). \tag{2.14}$$

The underlying category and homotopy categories are k-linear categories. The following two examples explain their roles.

Example 143. Consider the dg category $\mathrm{Ch_{dg}}(k)$. Then

$$\mathrm{Z}^0(\mathrm{Ch_{dg}}(k)) \cong \mathrm{Ch}(k) \tag{2.15}$$

where on the right we have the usual abelian category of cochain complexes over k, and

$$\mathrm{H}^0(\mathrm{Ch_{dg}}(k)) \cong \mathbf{K}(k) \tag{2.16}$$

where on the right we have the usual (triangulated) homotopy category of k. Remark that in this case we have an equivalence $\mathbf{K}(k) \cong \mathbf{D}(k)$, as k is semisimple.

Remark 144. One can do the construction of example 140 for an arbitrary k-algebra, and the resulting dg category will be denoted $\mathrm{Ch}_{\mathrm{dg}}(A)$. We obtain

$$
(2.17)\quad \begin{aligned} \mathrm{Z}^0(\mathrm{Ch}_{\mathrm{dg}}(A)) &\cong \mathrm{Ch}(A) \\ \mathrm{H}^0(\mathrm{Ch}_{\mathrm{dg}}(A)) &\cong \mathbf{K}(A). \end{aligned}
$$

To write $\mathbf{D}(A)$ as the homotopy category of a dg category (which is an important thing to do!) we need to find a *dg enhancement*. In this case it can be given by the full subcategory of $\mathrm{Ch}_{\mathrm{dg}}(A)$ provided by the *homotopy-injective* or *K-injective*, which are complexes $I^\bullet$ such that $\mathrm{Hom}_{\mathbf{K}(A)}(M^\bullet, I^\bullet) = 0$ whenever $M^\bullet$ is acyclic.

Another construction for dg categories, generalising a known construction for algebras, is that of the tensor product.

Definition 145. Let $\mathcal{C}$ and $\mathcal{D}$ be dg categories. Their *tensor product* $\mathcal{C} \otimes_k \mathcal{D}$ is the dg category $\mathcal{C} \otimes_k \mathcal{D}$, with

1. $\mathrm{Obj}(\mathcal{C} \otimes_k \mathcal{D}) = \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$;
2. $(\mathcal{C} \otimes_k \mathcal{D})((x,y),(x',y')) \coloneqq \mathcal{C}(x,x') \otimes_k \mathcal{D}(y,y')$ for $(x,y),(x',y') \in \mathrm{Obj}(\mathcal{C} \otimes_k \mathcal{D})$;
3. $1_{(x,y)} \coloneqq 1_x \otimes 1_y$ for $(x,y) \in \mathrm{Obj}(\mathcal{C} \otimes_k \mathcal{D})$;
4. the composition law is defined pointwise.

It is clear that the algebra k considered as a dg category is the unit for this tensor product.

2.1.3 Derived categories of dg categories

The main example of a triangulated category we have seen in section 2.1.1 is the derived category of an abelian category. But dg categories also have a notion of derived category, and one of the original reasons to introduce dg categories was to provide more flexibility in studying derived categories [18].

Most of the constructions here are the straightforward generalisations of notions for algebras to those of dg categories, taking into account that everything is actually a cochain complex, and that dg categories are like dg algebras with multiple objects. So many notions will be defined pointwise.

We will not be concerned with issues of set-theoretical or foundational nature[2]: they can be taken care of, as in [18]. And as before with triangulated categories, we won't be needing many abstract results in any case.

Differential graded functors Of course we want to consider functors between dg categories, and we want them to be compatible with the enrichment over cochain complexes. This yields the following notion.

Definition 146. Let $\mathcal{C}$ and $\mathcal{D}$ be dg categories. A *dg functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a functor such that

$$
(2.18)\quad F_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{D}(F(x),F(y))
$$

is a morphism of cochain complexes, and

[2] If you want to see some example of what can go wrong, consider [26, tag 07JS].

1. it is compatible with composition, i.e. the diagram

$$\begin{tikzcd} \mathcal{C}(x,y) \otimes_k \mathcal{C}(x,y) \arrow[r] \arrow[d, "F_{y,z}\otimes F_{x,y}"'] & \mathcal{C}(x,z) \arrow[d, "F_{x,z}"] \\ \mathcal{D}(F(y),F(z)) \otimes_k \mathcal{D}(F(x),F(y)) \arrow[r] & \mathcal{D}(F(x),F(z)) \end{tikzcd} \tag{2.19}$$

commutes for all $x, y, z \in \operatorname{Obj} \mathcal{C}$);

2. it is compatible with the unit morphism, i.e. the diagram

$$\begin{tikzcd} k \arrow[r, "1_x"] \arrow[dr, "1_{F(x)}"'] & \mathcal{C}(x,x) \arrow[d, "F_{x,x}"] \\ & \mathcal{D}(F(x),F(x)) \end{tikzcd} \tag{2.20}$$

commutes for all $x \in \operatorname{Obj}(\mathcal{C})$.

There is also a notion of dg natural transformation between dg functors, the details of which we won't spell out for now.

The following is an important example of a dg functor, as it makes an appearance in the dg Yoneda lemma.

Example 147. Let $\mathcal{C}$ be a dg category, and $x \in \operatorname{Obj}(\mathcal{C})$. Then the *representable dg functor* $\mathcal{C}(-,x)$ is the dg functor

(2.21) $\mathcal{C}(-,x) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}_{\mathrm{dg}}(k)$.

It sends $y \in \operatorname{Obj}(\mathcal{C})$ to $\mathcal{C}(y,x)$, and the morphisms

(2.22) $\mathcal{C}(y,y') \to \mathrm{Ch}_{\mathrm{dg}}(k)\,(\mathcal{C}(y,x), \mathcal{C}(y',x))$

for $y, y' \in \operatorname{Obj}(\mathcal{C})$ are defined using the composition in $\mathcal{C}$.

This example is just one instance of the following notion, generalising the notion of module over an algebra to the setting of dg categories.

Definition 148. Let $\mathcal{C}$ be a dg category. A *(right) dg $\mathcal{C}$-module* is a dg functor $M\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}_{\mathrm{dg}}(k)$. A morphism between (right) dg modules is a dg natural transformation, and the category of such dg modules is denoted $\mathrm{dgMod}_{\mathcal{C}}$. This is itself a dg category.

There is a minor clash of notation one needs to be aware of, as the dg category $\mathrm{Ch}_{\mathrm{dg}}(k)$ is the same as dgMod_k.

We will also need the notion of a dg bimodule. As an A-B-bimodule (where A and B are k-algebras) is also a right $A \otimes_k B^{\mathrm{op}}$-module, we will say that a $\mathcal{C}$-$\mathcal{D}$-bimodule is a $\mathcal{C} \otimes_k \mathcal{D}^{\mathrm{op}}$-module. If the need for notation arises, we will use $\mathrm{dgMod}_{\mathcal{C}}^{\mathcal{D}}$ or $\mathrm{dgMod}_{\mathcal{C}\otimes_k\mathcal{D}^{\mathrm{op}}}$.

Example 149. An important example of bimodule for us in the context of Hochschild (co)homology, is that of the *diagonal bimodule.* If $\mathcal{C}$ is our dg category, then it is the bimodule

(2.23) $\mathcal{C}^{\mathrm{op}} \otimes_k \mathcal{C} \to \mathrm{Ch}_{\mathrm{dg}}(k) : (x,y) \mapsto \mathcal{C}(x,y)$

for $x, y \in \operatorname{Obj}(\mathcal{C})$, which will be denoted $\mathcal{C}$. In the setting of ordinary algebras, this is nothing but considering A as an A-bimodule.

More generally, for any functor $F\colon \mathcal{C} \to \mathcal{D}$ we define the $\mathcal{C}$-$\mathcal{D}$-bimodule

$$\mathcal{D}_F\colon \mathcal{C}^{\mathrm{op}} \otimes_k \mathcal{D} \mapsto \mathrm{Ch}_{\mathrm{dg}}(k) : (x,y) \mapsto \mathcal{D}(F(x),y). \tag{2.24}$$

The diagonal bimodule is thus the bimodule associated to the identity functor.

The following proposition is an easy but important result [18, §6.1].

Proposition 150. Let $\mathcal{C}, \mathcal{D}$ be dg categories. Let M be a $\mathcal{C}$-$\mathcal{D}$-bimodule. Then we have an adjunction

$$- \otimes_{\mathcal{C}} M \dashv \mathrm{Hom}_{\mathcal{D}}(M,-)\colon\ \mathrm{dgMod}_{\mathcal{C}} \rightleftarrows \mathrm{dgMod}_{\mathcal{D}} \tag{2.25}$$

where

1. for a dg $\mathcal{C}$-module N we consider the dg functor given by[3]

   $$N \otimes_{\mathcal{C}} M(x) \coloneqq \mathrm{coker}\left(\bigoplus_{y,y' \in \mathrm{Obj}(\mathcal{D})} N(y') \otimes_k \mathcal{D}(y,y') \otimes_k M(x,y) \xrightarrow{\nu} \bigoplus_{y \in \mathrm{Obj}(\mathcal{D})} N(y) \otimes_k M(x,y) \right) \tag{2.26}$$

   for $x \in \mathrm{Obj}(\mathcal{C})$, where the morphism ν is defined as

   $$\nu(n \otimes f \otimes m) \coloneqq N(n)(f) \otimes m - n \otimes M(x,f)(m); \tag{2.27}$$

2. for a dg $\mathcal{D}$-module N' we consider the dg functor given by

   $$\mathrm{Hom}_{\mathcal{D}}(M,N')(y) \coloneqq \mathrm{dgMod}_{\mathcal{C}}(M(-,y),N') \tag{2.28}$$

   for $y \in \mathrm{Obj}(\mathcal{D})$.

This is nothing but an intimidating looking version of the adjunction

$$- \otimes_A M \dashv \mathrm{Hom}_B(M,-)\colon\ \mathrm{Mod}\,A \to \mathrm{Mod}\,B, \tag{2.29}$$

when A and B are ordinary k-algebras, and M is an A-B-bimodule. This is an important source of functors in noncommutative algebra. And when we are studying Fourier–Mukai transforms in algebraic geometry (as we will in chapter 3) we are doing exactly the same thing.

We will come back to $- \otimes_{\mathcal{C}} M$ after having introduced derived categories, as we will be interested in the derived functor $- \otimes^{\mathbf{L}}_{\mathcal{C}} M$.

Derived categories We now come to a very important construction for dg categories. We will assume that the reader is familiar with the construction of the derived category of an abelian category, for which [15, §2] and [29, §10] are good references. As they work in settings with suitable boundedness assumptions, but we work in the unbounded setting, one can find the appropriate generalisations in [26, Tag 05QI].

The following two definitions are just pointwise generalisations of the familiar notions for cochain complexes over an abelian category.

Definition 151. Let M be a dg $\mathcal{C}$-module. We say that it is *acyclic* if for all $x \in \mathrm{Obj}(\mathcal{C})$ the cochain complex $M(x)$ is acyclic, i.e. $\mathrm{H}^n(M(x)) = 0$ for all $n \geq 0$.

[3] When we work with dg categories with a single object this reduces to the usual definition of the tensor product as the quotient of the free module, modulo the relations encoding bilinearity.

Similarly we have the following.

Definition 152. Let $f\colon M \to N$ be a morphism in $\mathrm{Z}^0(\mathrm{dgMod}_{\mathcal{C}})$. We say that it is a *quasi-isomorphism* if $f_x\colon M(x) \to N(x)$ is a quasi-isomorphism for all $x \in \mathrm{Obj}(\mathcal{C})$.

Without paying any attention to set-theoretical questions we can then define the following.

Definition 153. Let $\mathcal{C}$ be a dg category. Its derived category $\mathbf{D}(\mathcal{C})$ is the localisation of $\mathrm{Z}^0(\mathrm{dgMod}_{\mathcal{C}})$ with respect to quasi-isomorphisms.

Equivalently, we can define it as the Verdier quotient $\mathrm{H}^0(\mathrm{dgMod}_{\mathcal{C}})/\,\mathrm{H}^0(\mathrm{Acyc}_{\mathcal{C}})$, where $\mathrm{Acyc}_{\mathcal{C}}$ is the full dg subcategory of acyclic dg modules.

As is the case for derived categories of abelian categories, we can also localise $\mathrm{H}^0(\mathrm{dgMod}_{\mathcal{C}})$ at quasi-isomorphisms.

An important subcategory of $\mathbf{D}(\mathcal{C})$ is that of its compact objects.

Definition 154. Let $\mathcal{C}$ be a dg category. Then $\mathrm{Perf}\,\mathcal{C} := \mathbf{D}(\mathcal{C})^{\mathrm{c}}$ is the full subcategory of compact objects in $\mathbf{D}(\mathcal{C})$.

Remark 155. Alternatively, we could look at the set of representable dg modules. Then $\mathrm{Perf}\,\mathcal{C}$ is the thick closure of this set of objects, i.e. it is the smallest strict triangulated subcategory of $\mathbf{D}(\mathcal{C})$, closed under taking direct summands of objects. In other words, we take the closure under shifts, cones and summands.

Doing homological algebra in derived categories of dg categories In the context of derived categories of abelian categories, the main notion for doing homological algebra is that of a resolution. We need to generalise this to derived categories of dg categories, in order to understand how statements like theorem 15 can be made to work in this more general setting. We won't attempt to be exhaustive here, we only aim to introduce some important notions. As mentioned before, because everything is by default unbounded, we need to make sure we deal with this properly.

Definition 156. Let $\mathcal{C}$ be a dg category. A dg $\mathcal{C}$-module P is *homotopy-projective* if for every acyclic dg $\mathcal{C}$-module $N^\bullet$ we have

(2.30) $$\mathrm{H}^0(\mathrm{dgMod}_{\mathcal{C}})(P, N) \cong 0.$$

This is inspired by the notion of homotopy-projective (or K-projective) cochain complexes, introduced by Spaltenstein [**MR0932640**]. For Grothendieck abelian categories we will use the dual notion of a homotopy-injective cochain complex. By the existence of homotopy-injective resolutions [26, tag 079P] we get the following proposition.

Proposition 157. Let $\mathcal{A}$ be a Grothendieck abelian category. Then

(2.31) $$\mathbf{D}(\mathcal{A}) \cong \mathbf{K}(\mathrm{hInj}\,\mathcal{A}),$$

where $\mathrm{hInj}\,\mathcal{A}$ is the full subcategory of $\mathrm{Ch}(\mathcal{A})$ on the homotopy-injective cochain complexes.

So computing left exact functors in this setting is easy: one takes a homotopy-injective resolution of a cochain complex, and apply the functor to it.

In the setting of dg categories we will mostly be working with the analogues of right exact functors, namely the tensor product introduced in proposition 150. So we will mostly be concerned with homotopy-projective resolutions. These exist, by [18, §3.1] or [7, appendix C]. In particular, we have the following description of the derived category of a dg category.

Theorem 158. Let $\mathcal{C}$ be a dg category. Then

$$\mathbf{D}(\mathcal{C}) \cong \mathrm{H}^0(\mathrm{hProj}\,\mathcal{C}), \tag{2.32}$$

where $\mathrm{hProj}\,\mathcal{C}$ is the full dg subcategory of $\mathrm{dgMod}_{\mathcal{C}}$ on the homotopy-projective dg modules.

The results cited for this description construct not just homotopy-projective resolutions, rather they construct an even stronger notion. This is analogous to what one does for the construction of projective resolutions of a module: these are usually *free* resolutions. But for dg modules we cannot use free objects as freely[4], like we do when working with projective resolutions of modules. See exercise 165 for an easy exercise explaining this. Rather we will use the following notion.

Definition 159. Let M be a dg module. It is *free* if it is the direct sum of shifts of representable dg modules.

It is *semifree* if there exists a filtration

$$0 = M_{-1} \subseteq M_0 \subseteq \ldots M_i \subseteq M_{i+1} \subseteq \ldots \subseteq M \tag{2.33}$$

which

1. is exhaustive, i.e. $M = \bigcup_{i\geq 0} M_i$;
2. the quotients M_{i+1}/M_i are free for all $i \geq 0$.

This is a special case of a homotopy-projective dg module.

Lemma 160. Let M be a semi-free dg module. Then it is homotopy-projective.

Proof. We will prove the result by induction. We have that M_0 is a free module, and by the Yoneda lemma these are homotopy-projective.

Assume that M_i is homotopy-projective for some $i \geq 0$, then we can consider the triangle

$$M_i \xrightarrow{j_i} M_{i+1} \to M_{i+1}/M_i \xrightarrow{+1}. \tag{2.34}$$

Applying the cohomological functor $\mathrm{H}^0(\mathrm{dgMod}_{\mathcal{C}}(-, N)$, where N is an acyclic dg module shows that M_{i+1} is homotopy-projective.

Because the inclusions in the filtrations are split on the graded level, and split exact sequences provide the triangles in the homotopy category, we can describe M using the triangle

$$\bigoplus_{i\geq 0} M_i \xrightarrow{\Phi} \bigoplus_{i\geq 0} M_i \to M \xrightarrow{+1} \tag{2.35}$$

in $\mathrm{H}^0(\mathrm{dgMod}_{\mathcal{C}}$, where Φ is defined as

$$M_i \xrightarrow{\begin{pmatrix}\mathrm{id}\\ -j_i\end{pmatrix}} M_i \oplus M \tag{2.36}$$

and $j_i\colon M_i \hookrightarrow M_{i+1}$ is the inclusion.

Now we just apply the cohomological functor $\mathrm{H}^0(\mathrm{dgMod}_{\mathcal{C}}(-, N)$ to (2.35), and use the same reasoning to conclude. □

When defining Hochschild cohomology we will use the following lemma.

[4] Pun very much intended.

Lemma 161. Let M be a dg module, together with a filtration

$$0 = M_{-1} \subseteq M_0 \subseteq \dots M_i \subseteq M_{i+1} \subseteq \dots \subseteq M \tag{2.37}$$

which is exhaustive, such that M_{i+1}/M_i is semifree for all $i \geq 0$. Then M itself is semifree.

2.1.4 Exercises

Exercise 162. Prove that example 140 is an example of a dg category.

Exercise 163. Prove the two equivalences from example 143.

Exercise 164. Prove the claim in remark 155.

Exercise 165. Explain what goes wrong in the following example. Consider k as a dg algebra. Let M be the dg module (i.e. cochain complex) which is

$$\dots \to k \xrightarrow{\mathrm{id}} k \to \dots \tag{2.38}$$

concentrated in degrees 0 and 1. It is generated by 1 in degrees 0 and 1. Why is the map

$$k \oplus k[1] \to M \tag{2.39}$$

which sends the generator of k to the corresponding generator of M not the start of a free resolution?

Exercise 166. Prove lemma 161.

2.2 Hochschild (co)homology for differential graded categories

2.3 Limited functoriality for Hochschild cohomology

2.4 Derived categories of smooth projective varieties

Chapter 3

Schemes

Conventions We will continue with our convention that k is a field, sometimes required to be of characteristic 0.

3.1 Hochschild (co)homology for schemes

3.2 Polyvector fields

3.3 The Hochschild–Kostant–Rosenberg decomposition

3.4 Riemann–Roch versus Hochschild homology

3.5 Căldăraru's conjecture

Appendix A

Preliminaries

A.1 Differential graded (Lie) algebras

Definition 167. A *differential graded algebra* $A^\bullet$ is a graded algebra $A^\bullet$ together with the structure of a cochain complex $\mathrm{d}\colon A^\bullet \to A^{\bullet+1}$ satisfying the *graded Leibniz rule*, i.e. for all homogeneous $a, b \in A^\bullet$

(A.1) $\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|}a\,\mathrm{d}(b)$.

We will abbreviate differential graded algebra to *dg algebra*.

Observe that the graded Leibniz rule implies the following.

Proposition 168. Let $A^\bullet$ be a dg algebra. Then $\mathrm{H}^\bullet(A^\bullet)$ is a graded algebra.

Definition 169. A *differential graded Lie algebra* $L^\bullet$ is a graded Lie algebra together with the structure of a cochain complex $\mathrm{d}\colon L^\bullet \to L^{\bullet+1}$ satisfying the *graded Leibniz rule*, i.e. for all homogeneous $l, m \in L^\bullet$

(A.2) $\mathrm{d}([l,m]) = [\mathrm{d}(l), m] + (-1)^{|l|}[l, \mathrm{d}(m)]$.

We will abbreviate differential graded Lie algebra to *dg Lie algebra*, or even *dgla*.

Remark 170. For graded algebras the axioms do not pick up any signs. For graded Lie algebras there are non-trivial signs involved in the axioms: for all homogeneous $l, m \in L^\bullet$ the graded skew-symmetry is

(A.3) $[l,m] = (-1)^{|l||m|}[m,l]$

whilst the graded Jacobi identity is

(A.4) $[l,[m,n]] = [[l,m],n] + (-1)^{|l||m|}[m,[l,n]]$.

Observe that the graded Leibniz rule implies the following.

Proposition 171. Let $L^\bullet$ be a dg Lie algebra. Then $\mathrm{H}^\bullet(L^\bullet)$ is a graded Lie algebra.

A.1.1 Exercises

Exercise 172. Let $L^\bullet$ be a dg Lie algebra, and R a (not necessarily unital) commutative k-algebra. Then $L^\bullet \otimes_k R$ is again a dg Lie algebra, when equipped with the natural structure

1. $(L^\bullet \otimes_k R)^i = L^i \otimes_k R$;
2. $\mathrm{d}_{L^\bullet \otimes_k R} = \mathrm{d}_{L^\bullet} \otimes_k \mathrm{id}_R$;
3. $[l_1 \otimes r_1, l_2 \otimes r_2] = [l_1, l_2] \otimes r_1 r_2$.

This works similarly for $-\widehat{\otimes}_k-$.

A.2 Completion and topologies

The Artin–Rees lemma can be used to show the following.

Lemma 173. Let V be a finite-dimensional vector space. Then

$$R\widehat{\otimes}_k V \cong R \otimes_k V \tag{A.5}$$

as R-modules.

We have the following universal property.

Proposition 174. Let U, V be vector spaces. Let R be a complete augmented k-algebra. There exists an isomorphism

$$\mathrm{Hom}_k(U, R\widehat{\otimes}_k V) \cong \mathrm{Hom}_R^{\mathrm{cont}}(R\widehat{\otimes}_k U, R\widehat{\otimes}_k V). \tag{A.6}$$

The multiplication on $R\widehat{\otimes}_k A$ is continuous and R-bilinear. This means it is given as a morphism

$$(R\widehat{\otimes}_k A)\widehat{\otimes}_R(R\widehat{\otimes}_k A) \to R\widehat{\otimes}_k A \tag{A.7}$$

where $-\widehat{\otimes}_R-$ denotes the completed tensor product of complete R-modules (see [26, tag 0AMU]).

One of the properties of the completed tensor product that we will need is the following.

Lemma 175. Let U, V be vector spaces. Let R be a complete augmented k-algebra. Then

$$(R\widehat{\otimes}_k U)\widehat{\otimes}_R(R\widehat{\otimes}_k V) \cong R\widehat{\otimes}_k(U \otimes_k V). \tag{A.8}$$

Proposition 176. Let R be a complete augmented k-algebra. Let $(R\widehat{\otimes}_k A, \mu)$ be a formal deformation of A over R. Then the multiplication is determined completely by its values on

$$A \otimes_k A \subseteq R\widehat{\otimes}_k(A \otimes_k A) \cong R\widehat{\otimes}_k(A \otimes_k A). \tag{A.9}$$

This explains why the Cauchy product appeared in (1.165).

A.2.1 Exercises

Exercise 177. Is $k[[x]] \otimes_k k[[y]] \cong k[[x, y]]$? Is $k[[x]]\widehat{\otimes}_k k[[y]] \cong k[[x, y]]$?

Using this, explain why continuity of a multiplication $- * -$ on $A[[t]]$ does not follow automatically from bilinearity, except when A is finite-dimensional.

A.3 Chevalley–Eilenberg cohomology

Appendix B

Additional topics

B.1 Kontsevich's formality theorems

B.2 Calabi–Yau algebras and Poincaré–Van den Bergh duality

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